

Scaling limit for determinantal point processes on spheres

Dedicated to the memory of Professor Yoichiro Takahashi

By

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Abstract

The unitary group with the Haar probability measure is called Circular Unitary Ensemble. All the eigenvalues lie on the unit circle in the complex plane and they can be regarded as a determinantal point process on $\mathbb{S}^1$. It is also known that the scaled point processes converge weakly to the determinantal point process associated with the so-called sine kernel as the size of matrices tends to ∞ . We extend this result to the case of high-dimensional spheres and show that the scaling limit processes are determinantal point processes associated with the kernels expressed by the Bessel functions of the first kind.

§ 1. Introduction

Circular Unitary Ensemble (CUE) is the group $U(n)$ of $n \times n$ unitary matrices with the normalized Haar measure. It is well-known that the joint probability distribution of eigenvalues $\{e^{\sqrt{-1}\theta_j}\}_{j=1}^n$ is given by

$$p_{\text{CUE}}(\theta_1, \theta_2, \dots, \theta_n) = \frac{1}{(2\pi)^n n!} \prod_{1 \leq j < k \leq n} |e^{\sqrt{-1}\theta_j} - e^{\sqrt{-1}\theta_k}|^2$$

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and they form a determinantal point process on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with $\lambda(d\theta) = d\theta/(2\pi)$ on $\mathbb{T}$ and correlation kernel K_n given by

$$K_n(\theta, \theta') = \sum_{k=0}^{n-1} e^{\sqrt{-1}k\theta} \overline{e^{\sqrt{-1}k\theta'}} = v(\theta) \underbrace{\frac{\sin[n(\theta - \theta')/2]}{\sin[(\theta - \theta')/2]}}_{:=\tilde{K}_n(\theta, \theta')} \overline{v(\theta')},$$

where $v(\theta) = e^{\sqrt{-1}(n-1)\theta/2}$. Since determinantal point processes (DPPs, for short in what follows) are invariant under a gauge transformation $K(\theta, \theta') \mapsto \tilde{K}(\theta, \theta') := u(\theta)K(\theta, \theta')\overline{u(\theta')}$ with $u : \mathbb{T} \rightarrow U(1)$, the kernel

$$\tilde{K}_n(\theta, \theta') = \frac{\sin[n(\theta - \theta')/2]}{\sin[(\theta - \theta')/2]}$$

defines the same DPP (see Section 2.1 for the definition and some invariance properties of DPPs). The empirical distribution $\frac{1}{n} \sum_{i=1}^n \delta_{\theta_i}$ of points converges to the uniform distribution on $\mathbb{T}$ almost surely. On the other hand, scaling as $x_i = n\theta_i \in \mathbb{R}$ yields

$$\frac{1}{n} \tilde{K}_n\left(\frac{x}{n}, \frac{x'}{n}\right) = \frac{1}{n} \frac{\sin[(x - x')/2]}{\sin[(x/n - x'/n)/2]} \rightarrow \frac{\sin[(x - x')/2]}{(x - x')/2} =: K_{\text{sinc}}(x, x').$$

Here we used the term *sinc* instead of *sine*. From this observation, we can see that, as $n \rightarrow \infty$,

$$n\text{-point DPP on } \mathbb{T} \Rightarrow \text{the DPP on } \mathbb{R}^1 \text{ with } K_{\text{sinc}},$$

where $\Rightarrow$ means convergence in law. In this article, we discuss a generalization of this fact (see Theorem 5.2): under suitable scaling,

$$n\text{-point DPP on } \mathbb{S}^d \Rightarrow \text{the DPP on } \mathbb{R}^d \text{ with } K^{(d)}(x, y),$$

where

$$K^{(d)}(x, y) = \frac{1}{(2\pi|x - y|)^{d/2}} J_{d/2}(|x - y|).$$

Here $J_\alpha(x)$ is the Bessel function of the first kind defined by

$$(1.1) \quad J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \alpha + 1)} \left(\frac{x}{2}\right)^{2k + \alpha}.$$

For $d = 1$, the correlation kernel $K^{(1)}$ coincides with K_{sinc} up to constant multiple. We also remark that the kernel $K^{(d)}$ is the reproducing kernel for the so-called generalized Paley-Wiener space, which is the space of functions that are the Fourier transforms of L^2 -functions whose support is contained in the unit ball centered at the origin in $\mathbb{R}^d$.

The organization of this paper is as follows: In Section 2, we discuss two DPPs on $\mathbb{S}^2$ and show limit theorems for these DPPs. In Section 3, we review spherical harmonics

on $\mathbb{S}^d$ and their properties. In Section 4, we introduce determinantal point processes on $\mathbb{S}^d$ by using spherical harmonics and in Section 5 we show their scaled processes converge to the determinantal point processes associated with the Bessel functions of the first kind.

§ 2. Two determinantal point processes on $\mathbb{S}^2$

In this section, we consider two generalizations of CUE on $\mathbb{T} \simeq \mathbb{S}^1$ to the sphere $\mathbb{S}^2$. First, we regard the joint distribution of CUE eigenvalues as

$$p_{\text{CUE}}(z_1, z_2, \dots, z_n) = \frac{1}{(2\pi)^n n!} \prod_{1 \leq j < k \leq n} \|z_j - z_k\|_{\mathbb{R}^2}^2 \quad (z_j \in \mathbb{S}^1 \subset \mathbb{R}^2)$$

by identifying $e^{\sqrt{-1}\theta_j} \in \mathbb{C}$ with $z_j = e^{\sqrt{-1}\theta_j} \in \mathbb{S}^1 \subset \mathbb{R}^2$. By replacing $\mathbb{S}^1 \subset \mathbb{R}^2$ with $\mathbb{S}^2 \subset \mathbb{R}^3$ on the right-hand side, we obtain a generalization of CUE eigenvalues, which turns out to be a DPP on $\mathbb{S}^2$.

Second, as before, we regard the eigenvalues of CUE as the DPP with kernel

$$K_n(\theta, \phi) = \sum_{k=0}^{n-1} e^{\sqrt{-1}k\theta} \overline{e^{\sqrt{-1}k\phi}}$$

and $\lambda(d\theta) = d\theta/(2\pi)$ on $\mathbb{S}^1$. Here $e^{\sqrt{-1}k\theta}$ is an eigenfunction of the Laplacian $-\Delta_{\mathbb{S}^1} = -d^2/d\theta^2$ corresponding to the eigenvalue k^2 , i.e.,

$$-\Delta_{\mathbb{S}^1} e^{\sqrt{-1}k\theta} = k^2 e^{\sqrt{-1}k\theta}.$$

The kernel K_n defines the projection onto the subspace spanned by $\{e^{\sqrt{-1}k\theta}\}_{k=0}^{n-1}$ in $L^2(\mathbb{S}^1)$. We can generalize CUE from this point of view by considering the Laplace-Beltrami operator on $\mathbb{S}^2$ and its eigenspaces spanned by the spherical harmonics.

In Sections 2.2 and 2.3, we will see that those two point processes converge to different DPPs in the bulk.

§ 2.1. Determinantal point processes

In this subsection, we briefly review determinantal point processes from the viewpoint of transformations. See, for instance, [8] for the theory of point processes and [11, 13, 15, 16, 17] for details of the basic properties of determinantal point processes.

Let S be a locally compact Hausdorff space with countable basis and λ be a Radon measure on S . We denote by $\text{Conf}(S)$ the configuration space over S , the set of non-negative integer valued Radon measures, which is equipped with the topological Borel σ -field with respect to the vague topology. A point process on S is a $\text{Conf}(S)$ -valued

random variable $\xi = \xi(\omega)$ on a probability space $(\Omega, \mathcal{F}, P)$ or its probability distribution on $\text{Conf}(S)$. For a locally trace class integral operator $K : L^2(S, \lambda) \rightarrow L^2(S, \lambda)$ with $0 \leq K \leq I$, there exists uniquely a probability measure on $\text{Conf}(S)$ whose correlation function is given by

$$\rho_n(s_1, \dots, s_n) = \det(K(s_i, s_j))_{i,j=1}^n,$$

where $K(x, y)$ is called a correlation kernel. This point process is called a determinantal point process associated with K and λ (or just associated with K). The measure λ is sometimes called a background measure. This situation arises in the reproducing kernel Hilbert space setting. A Hilbert space H of functions on S is called a reproducing kernel Hilbert space (RKHS) if there is a hermitian kernel $K : S \times S \rightarrow \mathbb{C}$ such that (i) $K(\cdot, s') \in H$ for any $s' \in S$ and (ii) $f(s') = \langle f, K(\cdot, s') \rangle_H$ for any $f \in H$. When H is realized as a closed subspace of $L^2(S, \lambda)$, the kernel $K(s, s')$ defines a projection integral operator K from $L^2(S, \lambda)$ onto H so that one can define for such H (and K) a DPP on S . In this sense, for example, the Ginibre point process is the DPP associated with the Bargmann-Fock space, which is the space of square-integrable analytic functions on $\mathbb{C}$ with respect to the complex Gaussian measure. The theory of reproducing kernel Hilbert spaces can be found in [3] and several DPPs are discussed in this context (cf. [1, 5, 14]).

A pair of kernel and background measure of a DPP is not unique. For example, the following changes of kernels and measures do not change the law of DPP.

(i) (Gauge transformation). For a non-vanishing measurable function $u : S \rightarrow \mathbb{C}$,

$$(2.1) \quad K(s, s') \mapsto \tilde{K}(s, s') := u(s)K(s, s')u(s')^{-1}.$$

(ii) For a measurable function $g : S \rightarrow [0, \infty)$,

$$(2.2) \quad (K(s, s'), g(s)\lambda(ds)) \mapsto (\sqrt{g(s)}K(s, s')\sqrt{g(s')}, \lambda(ds)).$$

For example, in the case of $S = \mathbb{C}$, the corresponding DPP associated with $K(z, w) = e^{z\bar{w}}$ and $\lambda(dz) = \pi^{-1}e^{-|z|^2}dz$ is called a Ginibre point process. From (2.2), this DPP is also expressed as a DPP associated with $\tilde{K}(z, w) = e^{z\bar{w} - (|z|^2 + |w|^2)/2}$ and $\tilde{\lambda}(dz) = \pi^{-1}dz$.

For a one-to-one measurable mapping $h : X \rightarrow Y$ and a DPP $\xi = \sum_j \delta_{x_j}$ on X associated with K and λ , the transformed point process $\tilde{\xi} = \sum_j \delta_{h(x_j)}$ turns out to be the DPP on Y associated with

$$(2.3) \quad \tilde{K}(y, y') = K(h^{-1}(y), h^{-1}(y')), \quad \tilde{\lambda}(dy) = (\lambda \circ h^{-1})(dy).$$

By using this formula, we can easily see that if $\xi = \sum_j \delta_{x_j}$ is the DPP on $X = \mathbb{R}^d$ associated with K and dx , then the scaled point process $S_c(\xi) := \sum_j \delta_{cx_j}$ ($c > 0$) is the DPP associated with

$$(2.4) \quad K_c(y, y') := \frac{1}{c^d} K\left(\frac{y}{c}, \frac{y'}{c}\right), \quad \lambda(dy) = dy.$$

Here we used (2.3) with $h(x) = cx$ and then (2.2) with $g(x) \equiv c^{-d}$.

We often use the fact that the measure $K(s, s)\lambda(ds)$ is the mean measure of the DPP on S associated with K and λ , which describes the density of points of the DPP on S .

§ 2.2. Spherical ensemble

The Ginibre ensemble is defined as a random matrix $G = (G_{ij})_{i,j=1}^N$ whose elements are i.i.d. and $G_{ij} \sim N_{\mathbb{C}}(0, 1)$, which is the complex standard normal distribution. We write it

$$G \sim \text{Ginibre}(N) \iff \{G_{ij}\}_{i,j=1}^N \text{ are i.i.d. and } G_{ij} \sim N_{\mathbb{C}}(0, 1).$$

Let $A, B \sim \text{Ginibre}(N)$ be independent. Krishnapur [12] showed that the eigenvalues of $A^{-1}B$ form a DPP on $\mathbb{C}$ with the following kernel and background measure

$$K_N(z, w) = (1 + z\bar{w})^{N-1}, \quad \lambda(dz) = \frac{N}{\pi(1 + |z|^2)^{N+1}} dz.$$

The corresponding reproducing kernel Hilbert space is the space of polynomials of degree $\leq N - 1$:

$$H_{K_N} = \text{Span}\{z^k : k = 0, 1, \dots, N - 1\}$$

in $L^2(\mathbb{C}, \lambda)$. The mean measure of the DPP is $K_N(z, z)\lambda(dz) = \frac{N}{\pi(1 + |z|^2)^2} dz$, which is the push-forward of N -times the uniform measure on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to $\mathbb{C}$. This fact suggests that it is natural to view this DPP as a point process on $\hat{\mathbb{C}} \simeq \mathbb{S}^2$ through the stereographic projection. The joint distribution with respect to the surface measure is given by

$$(\text{const.}) \prod_{1 \leq j < k \leq N} \|u_j - u_k\|_{\mathbb{R}^3}^2 \quad \text{on } \hat{\mathbb{C}} \simeq \mathbb{S}^2.$$

This point process is again a DPP on $\mathbb{S}^2$ and is called the *spherical ensemble* [2, 12]. This DPP is clearly $O(3)$ -invariant, and uniformly distributed with density $N/4\pi$. This may be considered as a spherical version of CUE eigenvalues. The correlation kernel as DPP is given by

$$\begin{aligned} K_N(u, u') &= K_N((\theta, \phi), (\theta', \phi')) \\ (2.5) \quad &= \frac{N}{4\pi} \left(e^{\sqrt{-1}(\phi - \phi')} \sin(\theta/2) \sin(\theta'/2) + \cos(\theta/2) \cos(\theta'/2) \right)^{N-1} \end{aligned}$$

with respect to the surface measure $\sigma(d\theta d\phi) = \sin \theta d\theta d\phi$ on $\mathbb{S}^2$, where (θ, ϕ) is the polar coordinates of $\mathbb{S}^2$. As $N \rightarrow \infty$, the empirical measure $\frac{1}{N} \sum_{i=1}^N \delta_{u_i}$ converges weakly to the uniform measure on $\mathbb{S}^2$ almost surely. We define a point process on the tangent

space $T_{e_3}(\mathbb{S}^2)$ at the north pole $e_3 = (0, 0, 1)$ as the pullback of points on the sphere by the exponential map $\exp : T_{e_3}(\mathbb{S}^2) \rightarrow \mathbb{S}^2$, i.e., using the polar coordinates (θ, ϕ) ,

$$T_{e_3}(\mathbb{S}^2) \ni (\theta \cos \phi, \theta \sin \phi) \mapsto (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \in \mathbb{S}^2.$$

We also identify $(\theta \cos \phi, \theta \sin \phi) \in T_{e_3}(\mathbb{S}^2)$ with $\theta e^{\sqrt{-1}\phi} \in \mathbb{C}$.

Let ξ_N be the spherical ensemble, which is an N -point DPP on $\mathbb{S}^2$. Since this DPP is rotation invariant, it suffices to look at the north pole $e_3 = (0, 0, 1)$. Let $T_{e_3}(\mathbb{S}^2)$ be the tangent space at e_3 . For fixed $\epsilon > 0$, we consider the pullback of points in ξ_N on $\mathbb{S}^2 \cap B_\epsilon(e_3)$ by the exponential map $\exp : T_{e_3}(\mathbb{S}^2) \rightarrow \mathbb{S}^2$ and denote it by $\eta_N^{(\epsilon)}$, i.e.,

$$\eta_N^{(\epsilon)} := \sum_{u \in \xi_N : u \in B_\epsilon(e_3)} \delta_{\exp^{-1}(u)},$$

where $u \in \xi$ means $\xi(\{u\}) = 1$ and $B_\epsilon(e_3)$ is the ϵ -neighborhood of e_3 . For a configuration $\eta = \sum_j \delta_{x_j} \in \text{Conf}(\mathbb{R}^d)$, we define a scaling map $S_c : \text{Conf}(\mathbb{R}^d) \rightarrow \text{Conf}(\mathbb{R}^d)$ by

$$S_c(\eta) = \sum_j \delta_{cx_j} \quad (c > 0).$$

In the rest of Section 2, we use this scaling map with $d = 2$. It follows from (2.4) that the scaled point process $S_c(\eta_N^{(\epsilon)})$ is the DPP associated with the kernel

$$K_{N,c,\epsilon}(x, y) = \frac{1}{c^2} K_N(u_c, v_c) \mathbf{1}_{B_\epsilon}(u_c) \mathbf{1}_{B_\epsilon}(v_c) \quad x, y \in T_{e_3}(\mathbb{S}^2),$$

where $u_c = \exp(x/c)$ and $v_c = \exp(y/c)$, $B_\epsilon = B_\epsilon(e_3)$ and $\mathbf{1}_{B_\epsilon}$ is its indicator function. For weak convergence of DPPs as $N, c \rightarrow \infty$, it suffices to show the uniform convergence of the kernel $K_{N,c,\epsilon}$ on each compact set $C \subset (T_{e_3}(\mathbb{S}^2))^2$ (cf. Proposition 3.10 in [16]). The function $\mathbf{1}_{B_\epsilon}(u_c)$ is eventually 1 on every compact C if c is large enough so that we only need to check the convergence of $K_{N,c}(x, y) := c^{-2} K_N(u_c, v_c)$.

Then we have the following proposition.

Proposition 2.1. *The scaled point process $S_{\sqrt{N}}(\eta_N^{(\epsilon)})$ converges weakly to the Ginibre DPP with density $1/4\pi$. Here the Ginibre DPP with density ρ is the DPP on $\mathbb{C}$ associated with the kernel and background measure*

$$K(z, w) = e^{\pi \rho z \bar{w}}, \quad \lambda(dz) = \rho e^{-\pi \rho |z|^2} dz.$$

Proof of Proposition 2.1. We put $\theta = rN^{-1/2}$ and $\theta' = r'N^{-1/2}$. As $N \rightarrow \infty$, we see that

$$\begin{aligned} & e^{\sqrt{-1}(\phi - \phi')} \sin(\theta/2) \sin(\theta'/2) + \cos(\theta/2) \cos(\theta'/2) \\ & \sim e^{\sqrt{-1}(\phi - \phi')} \frac{1}{4} \theta \theta' + 1 - \frac{1}{8} (\theta^2 + (\theta')^2) \\ & = 1 + \frac{1}{4N} \{z\bar{w} - \frac{1}{2}(|z|^2 + |w|^2)\}, \end{aligned}$$

where $z = re^{\sqrt{-1}\phi}$ and $w = r'e^{\sqrt{-1}\phi'}$. Therefore, from (2.5), we have

$$\begin{aligned} K_{N,\sqrt{N}}(z, w) &= \frac{1}{N} K_N(\exp(\frac{z}{\sqrt{N}}), \exp(\frac{w}{\sqrt{N}})) \\ &\sim \frac{1}{4\pi} \left(1 + \frac{1}{4N} \{z\bar{w} - \frac{1}{2}(|z|^2 + |w|^2)\} \right)^{N-1} \\ &\rightarrow \frac{1}{4\pi} e^{\frac{1}{4}\{z\bar{w} - \frac{1}{2}(|z|^2 + |w|^2)\}}. \end{aligned}$$

This together with (2.2) completes the proof. $\square$

Remark 2.2. Spherical ensemble has also been studied as the $2D$ one-component plasma (OCP) or $2D$ Coulomb gas (log-gas) on $\mathbb{S}^2$ (see, for instance, [6] and Section 15.6 in [7]). Proposition 2.1 was observed in [6] in the context of $2D$ OCP on a sphere, where the assertion is stated as follows: the thermodynamic limit of the OCP on the sphere is identical to that of the OCP on the plane.

§ 2.3. DPP associated with the spherical harmonics on $\mathbb{S}^2$

Let us consider the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^2}$ on $L^2(\mathbb{S}^2)$. The spectrum consists of the discrete eigenvalue $\ell(\ell+1)$ with multiplicity $2\ell+1$ for $\ell = 0, 1, \dots$. We denote by E_ℓ the eigenspace corresponding to the eigenvalue $\ell(\ell+1)$ and then there is a spectral decomposition of $L^2(\mathbb{S}^2)$ as

$$L^2(\mathbb{S}^2) \simeq \bigoplus_{\ell=0}^{\infty} E_\ell.$$

Each eigenspace E_ℓ is spanned by the so-called spherical harmonics given by

$$(2.6) \quad Y_m^\ell(\theta, \phi) := \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_m^\ell(\cos \theta) e^{\sqrt{-1}m\phi} \quad (-\ell \leq m \leq \ell),$$

where $P_m^\ell(x)$ is the associated Legendre polynomial of degree m , i.e.,

$$E_\ell = \text{Span}\{Y_m^\ell : m = -\ell, -\ell+1, \dots, \ell\}.$$

Now we consider the subspace corresponding to the eigenvalues up to $n(n+1)$, that is $E_{\leq n} := \bigoplus_{\ell=0}^n E_\ell$. The reproducing kernel for $E_{\leq n}$ is given by

$$K_n(x, y) = \sum_{\ell=0}^n \sum_{m=-\ell}^{\ell} Y_m^\ell(x) \overline{Y_m^\ell(y)} =: \sum_{\ell=0}^n Z_\ell(x, y),$$

where $Z_\ell(x, y)$ is the reproducing kernel for E_ℓ .

We consider the DPP ξ_n on $\mathbb{S}^2$ associated with $K_n(x, y)$ and the Lebesgue measure on $\mathbb{S}^2$. The number of points of DPP ξ_n on $\mathbb{S}^2$ is $(n+1)^2$. As $n \rightarrow \infty$, the empirical measure converges weakly to the uniform measure on $\mathbb{S}^2$. For this DPP, we consider the same problem as in the previous subsection. For fixed $\epsilon > 0$, let $\eta_n^{(\epsilon)}$ be the pullback of ξ_n on $\mathbb{S}^2 \cap B_\epsilon(e_3)$ by the exponential map $\exp : T_{e_3}(\mathbb{S}^2) \rightarrow \mathbb{S}^2$. Here we have $(n+1)^2$ -points and hence the scaling S_n makes the density of points on $T_{e_3}(\mathbb{S}^2)$ of $O(1)$.

Theorem 2.3. *The scaled point process $S_n(\eta_n^{(\epsilon)})$ converges weakly to the DPP on $T_{e_3}(\mathbb{S}^2) \simeq \mathbb{R}^2$ associated with the kernel and background measure*

$$K(x, y) = \frac{1}{2\pi|x-y|} J_1(|x-y|), \quad \lambda(dx) = dx,$$

where $J_1(r)$ is the Bessel function of the first kind (1.1) with $\alpha = 1$.

The proof is given in Section 5 in higher dimensional setting.

§ 3. Spherical harmonics on $\mathbb{S}^d$

In this section, we recall basic properties of spherical harmonics on $\mathbb{S}^d$ (cf. [9]).

For $d \in \mathbb{N}$, let $\mathcal{P} := \mathcal{P}(\mathbb{R}^{d+1})$ be a vector space of all complex-valued polynomials on $\mathbb{R}^{d+1}$, and $\mathcal{P}_\ell, \ell \in \mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$, be its subspaces consisting of homogeneous polynomials of degree ℓ ; $p(x) = \sum_{|\alpha|=\ell} c_\alpha x^\alpha$, $c_\alpha \in \mathbb{C}, x \in \mathbb{R}^{d+1}$. The vector space of all harmonic functions in $\mathcal{P}$ is denoted by $\mathcal{H} := \{p \in \mathcal{P} : \Delta p = 0\}$ and let $\mathcal{H}_\ell := \mathcal{H} \cap \mathcal{P}_\ell$ for $\ell \in \mathbb{Z}_{\geq 0}$.

Now we consider a unit sphere in $\mathbb{R}^{d+1}$ denoted by $\mathbb{S}^d$, in which we use the polar coordinates for $u = (u_1, \dots, u_{d+1}) \in \mathbb{S}^d$,

$$\begin{aligned} u_1 &= \sin \theta_d \cdots \sin \theta_2 \sin \theta_1, \\ u_k &= \sin \theta_d \cdots \sin \theta_k \cos \theta_{k-1}, \\ u_{d+1} &= \cos \theta_d, \\ (3.1) \quad &\theta_1 \in [0, 2\pi), \quad \theta_k \in [0, \pi], \quad k = 2, \dots, d. \end{aligned}$$

Note that $|u|^2 := \sum_{j=1}^{d+1} u_j^2 = 1$. The standard measure on $\mathbb{S}^d$ is then given as

$$(3.2) \quad \sigma_d(du) := \sin^{d-1} \theta_d \sin^{d-2} \theta_{d-1} \cdots \sin \theta_2 d\theta_1 \cdots d\theta_d, \quad u \in \mathbb{S}^d.$$

The total measure of $\mathbb{S}^d$ is calculated as

$$(3.3) \quad \omega_d := \sigma_d(\mathbb{S}^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}.$$

We write the restriction of harmonic polynomials in $\mathcal{H}_\ell$ on $\mathbb{S}^d$ as

$$\mathcal{Y}_\ell := \{h|_{\mathbb{S}^d} : h \in \mathcal{H}_\ell\}, \quad \ell \in \mathbb{Z}_{\geq 0}.$$

It is known that

$$(3.4) \quad d_\ell := \dim \mathcal{Y}_\ell = \frac{2\ell + d - 1}{d - 1} \binom{\ell + d - 2}{\ell}.$$

The reproducing kernel $Z_\ell(u, v)$, $u, v \in \mathbb{S}^d$ is defined as a unique function in $\mathcal{Y}_\ell$ such that the following reproducing properties hold,

$$Y(u) = \int_{\mathbb{S}^d} Z_\ell(u, v) Y(v) \mu_d(dv) \quad \forall Y \in \mathcal{Y}_\ell,$$

where $\mu_d = \omega_d^{-1} \sigma_d$ is normalized to be a probability measure on $\mathbb{S}^d$. Consider an orthonormal basis $\{Y_j^\ell\}_{j=1}^{d_\ell}$ of $\mathcal{Y}_\ell$ with respect to μ_d ;

$$(3.5) \quad \int_{\mathbb{S}^d} Y_n^\ell(u) \overline{Y_m^\ell(u)} \mu_d(du) = \delta_{nm}, \quad n, m \in \mathbb{Z}_{\geq 0}.$$

We note that this normalization is different from (2.6) up to constant multiple for $d = 2$. The reproducing kernel Z_ℓ of $\mathcal{Y}_\ell$ in $L^2(\mathbb{S}^d, \mu_d)$ is expanded as

$$(3.6) \quad Z_\ell(u, v) = \sum_{j=1}^{d_\ell} Y_j^\ell(u) \overline{Y_j^\ell(v)}.$$

For $\lambda > -1/2$, we define the ultraspherical polynomial by

$$(3.7) \quad P_\ell^\lambda(s) := {}_2F_1 \left(-\ell, \ell + 2\lambda; \lambda + \frac{1}{2}; \frac{1-s}{2} \right),$$

where ${}_2F_1$ denotes the Gauss hypergeometric function,

$${}_2F_1(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!},$$

with $(\alpha)_n := \alpha(\alpha+1) \cdots (\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$ and $(\alpha)_0 := 1$. Then the following equality is established,

$$(3.8) \quad Z_\ell(u, v) = d_\ell P_\ell^{(d-1)/2}(u \cdot v), \quad u, v \in \mathbb{S}^d,$$

where d_ℓ is given by (3.4). It is clear that Z_ℓ is $O(d+1, \mathbb{R})$ -invariant in the sense that

$$(3.9) \quad Z_\ell(gu, gv) = Z_\ell(u, v) \quad \forall g \in O(d+1, \mathbb{R}), \quad \forall u, v \in \mathbb{S}^d.$$

If we define the stabilizer subgroup L_0 of $SO(d+1, \mathbb{R})$ at e_{d+1} as

$$L_0 = \left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} : A \in SO(d, \mathbb{R}) \right\},$$

and let

$$\mathcal{Y}_\ell^{L_0} := \{Y \in \mathcal{Y}_\ell : Y(gu) = Y(u), \quad \forall g \in L_0, \quad \forall u \in \mathbb{S}^d\},$$

then $Z_\ell(\cdot, e_{d+1}) \in \mathcal{Y}_\ell^{L_0}$. The space $\mathcal{Y}_\ell^{L_0}$ is the one-dimensional vector space spanned by $P_\ell^{(d-1)/2}(u \cdot e_{d+1})$. In general, any L_0 -invariant function is a constant on each L_0 -orbit $\mathcal{O}_s := \{u \in \mathbb{S}^d : u \cdot e_{d+1} = s\}$ for $s \in [-1, 1]$, and hence functions in $\mathcal{Y}_\ell^{L_0}$ is called the zonal harmonics of degree ℓ . In the next section, we also use the following Gegenbauer polynomial of degree ℓ defined by

$$(3.10) \quad C_\ell^\lambda(s) := \binom{\ell + 2\lambda - 1}{\ell} P_\ell^\lambda(s).$$

The generating function of $C_\ell^\lambda(s)$ is given by

$$(1 - 2sz + z^2)^{-\lambda} = \sum_{\ell=0}^{\infty} C_\ell^\lambda(s) z^\ell.$$

§ 4. Harmonic ensembles on $\mathbb{S}^d$

In Section 2, we discussed two DPPs on $\mathbb{S}^2$ as generalizations of CUE eigenvalues on $\mathbb{S}^1$ and showed limit theorems for those DPPs. In this section, we consider DPPs on $\mathbb{S}^d$ sometimes called *harmonic ensembles* (cf. [4]), which generalize the DPP treated in Section 2.3

For $d \in \mathbb{N}$, $n \in \mathbb{Z}_{\geq 0}$ and $u, v \in \mathbb{S}^d$, we set

$$(4.1) \quad \begin{aligned} \mathcal{K}_n^{(d)}(u, v) &:= \sum_{\ell=0}^n Z_\ell(u, v) = \sum_{\ell=0}^n d_\ell P_\ell^{(d-1)/2}(u \cdot v) \\ &= \sum_{\ell=0}^n \frac{2\ell + d - 1}{d - 1} C_\ell^{(d-1)/2}(u \cdot v). \end{aligned}$$

Since $\mathcal{K}_n^{(d)}(u, v)$ is the reproducing kernel for $\mathcal{Y}_{\leq n} := \oplus_{\ell=0}^n \mathcal{Y}_\ell$ whose dimension is $N_n^{(d)} := \sum_{\ell=0}^n d_\ell$ with respect to the background measure μ_d , we have a DPP ξ_n on $\mathbb{S}^d$ such that the correlation kernel is given by (4.1), which is called the harmonic ensemble on $\mathbb{S}^d$. The harmonic ensemble is rotation invariant and thus the density is uniform on $\mathbb{S}^d$ given by

$$\mathcal{K}_n^{(d)}(u, u) = \sum_{\ell=0}^n d_\ell P_\ell^{(d-1)/2}(1) = N_n^{(d)},$$

where we have used the fact that $P_\ell^\lambda(1) = 1$ for $\lambda > -1/2$. It is easy to see that

$$(4.2) \quad N_n^{(d)} = \frac{d+2n}{d} \binom{d+n-1}{n} = \frac{2}{d!} n^d + o(n^d).$$

We note that the number of points in ξ_n is equal to $N_n^{(d)}$ almost surely.

If we introduce the Jacobi polynomials defined as

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}),$$

and

$$(4.3) \quad Q_n^{(\alpha, \beta)}(x) = \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)},$$

then we have the following.

Lemma 4.1. For $n \in \mathbb{Z}_{\geq 0}$ and $\lambda > -1/2$,

$$(4.4) \quad \sum_{\ell=0}^n \frac{\ell+\lambda}{\lambda} C_\ell^\lambda(x) = \frac{2n+2\lambda+1}{2\lambda+1} \binom{2\lambda+n}{n} Q_n^{(\lambda+1/2, \lambda-1/2)}(x).$$

Proof. By using the contiguous relation (Eq.(18.9.5) in [10])

$$(2n+\alpha+\beta+1)P_n^{(\alpha, \beta)}(x) = (n+\alpha+\beta+1)P_n^{(\alpha, \beta+1)}(x) + (n+\alpha)P_{n-1}^{(\alpha, \beta+1)}(x)$$

together with $P_n^{(\alpha, \beta+1)}(1)/P_{n-1}^{(\alpha, \beta+1)}(1) = (n+\alpha)/n$, we have

$$(4.5) \quad (2n+\alpha+\beta+1)Q_n^{(\alpha, \beta)}(x) = (n+\alpha+\beta+1)Q_n^{(\alpha, \beta+1)}(x) + nQ_{n-1}^{(\alpha, \beta+1)}(x).$$

From the recurrence relation of the Gegenbauer polynomials (see, Eq.(18.9.7) in [10]) and the definition,

$$(n+\lambda)C_n^\lambda(x) = \lambda\{C_n^{\lambda+1}(x) - C_{n-2}^{\lambda+1}(x)\} \quad (n \geq 2), \quad C_0^\lambda(x) = 1, \quad C_1^\lambda(x) = 2\lambda x,$$

we see that

$$\begin{aligned} \sum_{\ell=0}^n \frac{\ell+\lambda}{\lambda} C_\ell^\lambda(x) &= C_0^\lambda(x) + \frac{1+\lambda}{\lambda} C_1^\lambda(x) + \sum_{\ell=2}^n \left\{ C_\ell^{\lambda+1}(x) - C_{\ell-2}^{\lambda+1}(x) \right\} \\ &= C_n^{\lambda+1}(x) + C_{n-1}^{\lambda+1}(x). \end{aligned}$$

The Gegenbauer polynomial can also be written as

$$C_n^{\alpha+1/2}(x) = \binom{n+2\alpha}{n} Q_n^{(\alpha, \alpha)}(x).$$

From (4.5) with $\alpha = \beta + 1$, we obtain

$$\begin{aligned} C_n^{\alpha+1/2}(x) + C_{n-1}^{\alpha+1/2}(x) &= \binom{n+2\alpha}{n} Q_n^{(\alpha,\alpha)}(x) + \binom{n+2\alpha-1}{n-1} Q_{n-1}^{(\alpha,\alpha)}(x) \\ &= \binom{n+2\alpha-1}{n} \left\{ \frac{n+2\alpha}{2\alpha} Q_n^{(\alpha,\alpha)}(x) + \frac{n}{2\alpha} Q_{n-1}^{(\alpha,\alpha)}(x) \right\} \\ &= \binom{n+2\alpha-1}{n} \frac{n+\alpha}{\alpha} Q_n^{(\alpha,\alpha-1)}(x). \end{aligned}$$

Putting $\alpha = \lambda + 1/2$, we obtain the desired formula (4.4). $\square$

The correlation kernel $\mathcal{K}_n^{(d)}$ has the following simple form.

Proposition 4.2. *For $u, v \in \mathbb{S}^d$, $d \in \mathbb{N}$, and $n \in \mathbb{N}$,*

$$(4.6) \quad \mathcal{K}_n^{(d)}(u, v) = N_n^{(d)} Q_n^{(d/2, d/2-1)}(u \cdot v).$$

Proof. Setting $\lambda = (d-1)/2$ in the formula (4.4), we obtain (4.6). $\square$

In the next section, we use the following Mehler-Heine type asymptotic formula for Jacobi polynomials.

Lemma 4.3 (Theorem 8.1.1.[18]). *For $\alpha, \beta \in \mathbb{R}$ and $z \in \mathbb{C}$,*

$$\lim_{n \rightarrow \infty} n^{-\alpha} P_n^{(\alpha, \beta)}\left(\cos \frac{z}{n}\right) = (z/2)^{-\alpha} J_{\alpha}(z),$$

and hence,

$$\lim_{n \rightarrow \infty} Q_n^{(\alpha, \beta)}\left(\cos \frac{z}{n}\right) = \Gamma(\alpha + 1)(z/2)^{-\alpha} J_{\alpha}(z).$$

The convergence takes place uniformly on every compact set in $\mathbb{C}$.

§ 5. Scaling limits of DPPs on $\mathbb{S}^d$

We consider the harmonic ensemble on $\mathbb{S}^d$ and then define a DPP on the tangent space $T_p(\mathbb{S}^d)$ at a point $p \in \mathbb{S}^d$ as the pullback of points by the exponential map $\exp : T_p(\mathbb{S}^d) \rightarrow \mathbb{S}^d$. By rotation invariance, it suffices to look at the neighborhood of the ‘north pole’ e_{d+1} on $\mathbb{S}^d$ as before. For $u \in \mathbb{S}^d$, we use the polar coordinates $(\theta_1, \theta_2, \dots, \theta_d)$ in (3.1) to write $\Omega(u) := (\Omega_1, \dots, \Omega_d) \in \mathbb{S}^{d-1}$ by setting

$$\begin{aligned} \Omega_1 &= \sin \theta_{d-1} \cdots \sin \theta_2 \sin \theta_1, \\ \Omega_k &= \sin \theta_{d-1} \cdots \sin \theta_k \cos \theta_{k-1}, \\ \Omega_d &= \cos \theta_{d-1}, \end{aligned} \quad (5.1) \quad \theta_1 \in [0, 2\pi), \quad \theta_k \in [0, \pi], \quad k = 2, \dots, d-1.$$

The tangent space $T_{e_{d+1}}(\mathbb{S}^d)$ is thus identified with

$$(5.2) \quad T_{e_{d+1}}(\mathbb{S}^d) \simeq \{\theta_d \Omega(u) \in \mathbb{R}^d : \theta_d \geq 0, u \in \mathbb{S}^d\},$$

and then the exponential map is given by

$$(5.3) \quad T_{e_{d+1}}(\mathbb{S}^d) \ni (\theta_d \Omega_1, \dots, \theta_d \Omega_d) \mapsto ((\sin \theta_d) \Omega_1, \dots, (\sin \theta_d) \Omega_d, \cos \theta_d) \in \mathbb{S}^d.$$

Let ξ_n be the DPP on $\mathbb{S}^d$ associated with the kernel $\mathcal{K}_n^{(d)}(u, v)$ defined in the previous section. The number of points in ξ_n is equal to $N_n^{(d)}$ almost surely. For this DPP and fixed $\epsilon > 0$, we define $\eta_n^{(\epsilon)}$ as the pullback of ξ_n restricted to $\mathbb{S}^d \cap B_\epsilon(e_{d+1})$ by the exponential map $\exp : T_{e_{d+1}}(\mathbb{S}^d) \rightarrow \mathbb{S}^d$. We note that the scaling S_n makes the density of points on $T_{e_{d+1}}(\mathbb{S}^d)$ of $O(1)$. For $u, v \in \mathbb{S}^d$, we write $x = \theta_d \Omega(u), y = \vartheta_d \Omega(v) \in T_{e_{d+1}}(\mathbb{S}^d)$ as in (5.2). Since the pullback of μ_d on $\mathbb{S}^d$ is locally equal to ω_d^{-1} times the Lebesgue measure dx on $T_{e_{d+1}}(\mathbb{S}^d) \simeq \mathbb{R}^d$, it follows from (2.3) and (2.4) that the kernel of the scaled DPP $S_n(\eta_n^{(\epsilon)})$ with respect to the Lebesgue measure dx is

$$(5.4) \quad K_{n,\epsilon}^{(d)}(x, y) = \frac{1}{n^d \omega_d} \mathcal{K}_n^{(d)}(u_n, v_n) \mathbf{1}_{B_\epsilon}(u_n) \mathbf{1}_{B_\epsilon}(v_n),$$

where $u_n = \exp(x/n)$ and $v_n = \exp(y/n)$.

In this setting, we have the following limit formula for correlation kernels.

Proposition 5.1. *Assume that $(u^{(n)}, v^{(n)})_{n \in \mathbb{N}}$ is a sequence of pairs of points in $\mathbb{S}^d$ such that*

$$u^{(n)} \cdot v^{(n)} = \cos \frac{r}{n} \quad \text{with } 0 < r < \infty.$$

Then the limit

$$(5.5) \quad k^{(d)}(r) = \lim_{n \rightarrow \infty} \frac{1}{n^d \omega_d} \mathcal{K}_n^{(d)}(u^{(n)}, v^{(n)})$$

exists and have the following expressions,

$$(5.6) \quad k^{(d)}(r) = \frac{J_{d/2}(r)}{(2\pi r)^{d/2}},$$

$$(5.7) \quad = \frac{1}{(2\pi)^{d/2} r^{(d-2)/2}} \int_0^1 s^{d/2} J_{(d-2)/2}(rs) ds.$$

Proof. By (3.3), Proposition 4.2 and Lemma 4.3, we see that

$$\lim_{n \rightarrow \infty} \frac{1}{n^d \omega_d} \mathcal{K}_n^{(d)}(u^{(n)}, v^{(n)}) = \frac{2}{d!} \frac{\Gamma((d+1)/2)}{2\pi^{(d+1)/2}} \Gamma\left(\frac{d}{2}+1\right) \left(\frac{2}{r}\right)^{d/2} J_{d/2}(r) = \left(\frac{1}{2\pi r}\right)^{d/2} J_{d/2}(r).$$

Here we used Legendre's duplication formula for the Gamma function (Eq.(5.5.5) in [10])

$$\Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z).$$

The convergence takes place uniformly in any compacts in $\mathbb{C}$. Hence (5.6) is proved. If we use the integral formula (see, for instance, Eq.(10.22.1) in [10]),

$$\int z^{\nu+1} J_\nu(z) dz = z^{\nu+1} J_{\nu+1}(z),$$

we can derive (5.7) from (5.6), since

$$\frac{1}{r^{(d+2)/2}} \int_0^r u^{(d-2)/2+1} J_{(d-2)/2}(u) du = \int_0^1 s^{d/2} J_{(d-2)/2}(rs) ds.$$

Thus the proof is complete. $\square$

This result implies that for each $d \in \mathbb{N}$ we obtain a limiting DPP on $\mathbb{R}^d$ such that it is uniform on $\mathbb{R}^d$ and the correlation kernel is given by

$$(5.8) \quad K^{(d)}(x, y) = k^{(d)}(|x - y|), \quad x, y \in \mathbb{R}^d,$$

where $k^{(d)}$ is defined by (5.6). See also the formula (5.11) below for odd d .

Putting it all together, we have the following.

Theorem 5.2. *The scaled point process $S_n(\eta_n^{(\epsilon)})$ converges weakly to the DPP on $T_{e_{d+1}}(\mathbb{S}^d) \simeq \mathbb{R}^d$ associated with the kernel*

$$(5.9) \quad K^{(d)}(x, y) = \frac{1}{(2\pi|x - y|)^{d/2}} J_{d/2}(|x - y|)$$

and background measure dx . The kernel is also expressed in terms of Fourier transform as follows:

$$(5.10) \quad K^{(d)}(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{1}_{B_1^{(d)}}(u) e^{\sqrt{-1}u \cdot (x-y)} du,$$

where $B_1^{(d)}$ is the unit ball centered at the origin in $\mathbb{R}^d$.

Proof. First we assume that both $x, y \in \mathbb{R}^d$ are within a ball of radius less than the injectivity radius at e_{d+1} . Let $u_n = \exp(x/n)$ and $v_n = \exp(y/n)$. Then, by (5.2) and (5.3), we see that

$$\begin{aligned} u_n \cdot v_n &= \sin \frac{\theta_d}{n} \sin \frac{\vartheta_d}{n} \Omega(u_1) \cdot \Omega(v_1) + \cos \frac{\theta_d}{n} \cos \frac{\vartheta_d}{n} \\ &= \frac{\theta_d \vartheta_d}{n^2} \Omega(u_1) \cdot \Omega(v_1) + 1 - \frac{1}{2} \frac{\theta_d^2 + \vartheta_d^2}{n^2} + o\left(\frac{1}{n^2}\right) \\ &= 1 - \frac{1}{2n^2} |x - y|^2 + o\left(\frac{1}{n^2}\right) \\ &= \cos \frac{|x - y|}{n} + o\left(\frac{1}{n^2}\right). \end{aligned}$$

For general $x, y \in \mathbb{R}^d$, if necessary, by taking sufficiently large M and considering x/M and y/M instead of x and y themselves, we can show the same equality above. Therefore, from (4.1), (5.4) and Proposition 5.1, we have (5.9). The weak convergence follows from the uniform convergence of the kernel on any compacts as mentioned in the proof of Proposition 5.1. $\square$

The limiting DPP associated with $K^{(d)}(x, y)$ is motion invariant and the density of points is equal to $V_d/(2\pi)^d$, where V_d is the volume of the unit ball in $\mathbb{R}^d$. The kernel $K^{(d)}(x, y)$ is sometimes called the reproducing kernel for a generalized Paley-Wiener space. This kernel also appears in an asymptotic formula for Szegő kernel in the study of universality in [19].

Remark 5.3. While the scaling was $S_{\sqrt{N}}$ in Section 2.2, here is S_n . The symbol N in Section 2.2 was used for the number of points on the sphere $\mathbb{S}^2$. If we use the number of points $N_n^{(d)}$ defined in (4.2) as parameter, then S_n can also be written as $S_{(N_n^{(d)})^{1/d}}$.

Remark 5.4. The Bessel function $J_{m+1/2}(x)$ with $m \in \mathbb{N}$ can be expressed as a linear combination of $\sin x$ and $\cos x$. The spherical Bessel function of the first kind is as follows:

$$j_m(x) := \sqrt{\frac{\pi}{2x}} J_{m+1/2}(x).$$

Rayleigh's formula (Equation (10.51.3) in [10]) states that

$$j_m(x) = (-x)^m \left(\frac{1}{x} \frac{d}{dx} \right)^m \frac{\sin x}{x},$$

from which it is easy to see that

$$(5.11) \quad k^{(d)}(r) = \left(\frac{-1}{2\pi r} \frac{d}{dr} \right)^{(d-1)/2} \frac{\sin r}{\pi r} \quad \text{for odd } d.$$

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References

- [1] Abreu, L. D., Gröchenig, K. and Romero, J. L., Harmonic analysis in phase space and finite Weyl-Heisenberg ensembles, *J. Stat. Phys.* **174** (2019), 1104–1136.

- [2] Alishahi, K. and Zamani, M., The spherical ensemble and uniform distribution of points on the sphere, *Electron. J. Probab.* **20** (2015), no. 23, 1–27.
- [3] Aronszajn, N., Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337–404.
- [4] Beltrán, C., Marzo, J. and Ortega-Cerdà., Energy and discrepancy of rotationally invariant determinantal point processes in high dimensional spheres, *J. Complexity* **37** (2016), 76–109.
- [5] Bufetov, A. I. and Qiu, Y., Determinantal point processes associated with Hilbert spaces of holomorphic functions, *Commun. Math. Phys.* **351** (2017) 1–44.
- [6] Caillol, J. M., Exact results for a two-dimensional one-component plasma on a sphere, *Journal de Physique Lettres* **42** (1981), 245–247.
- [7] Forrester, P. J., *Log-gases and Random Matrices*, London Math. Soc. Monographs, Princeton University Press, Princeton (2010).
- [8] Kallenberg, O., *Random measures*, Fourth ed. Akademie-Verlag, Berlin; Academic Press, Inc., London, 1986.
- [9] Nomura, T., *Spherical Harmonics and Group Representations*. Nihon-Hyoronsya, 2018 (Japanese).
- [10] Olver, F. W. J., Lozier, D. W., Boisvert, R. F. and Clark, C. W.(ed): *NIST Handbook of Mathematical Functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC, Cambridge: Cambridge University Press, 2010; available at <http://dlmf.nist.gov>
- [11] Hough, J. B., Krishnapur, M., Peres, Y. and Virág, B., *Zeros of Gaussian analytic functions and determinantal point processes*. University Lecture Series, 51. American Mathematical Society, Providence, RI, 2009.
- [12] Krishnapur, M. , From random matrices to random analytic functions. *Ann. Probab.* **37** (2009), 314–346.
- [13] Macchi, O., The coincidence approach to stochastic point processes. *Adv. Appl. Prob.* **7** (1975), 83–122.
- [14] Shirai, T., Ginibre-type point processes and their asymptotic behavior. *Journal of the Mathematical Society of Japan*, **67** (2015), 763–787.
- [15] Shirai, T. and Takahashi, Y., Fermion process and Fredholm determinant, *Proceedings of the Second ISAAC Congress*, Vol. 1 (Fukuoka, 1999), 15–23, Kluwer Acad. Publ., 2000.
- [16] Shirai, T. and Takahashi, Y., Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes, *J. Funct. Anal.* **205** (2003), 414–463.
- [17] Soshnikov, A., Determinantal random point fields. *Translation in Russian Math. Surveys* **55** (2000), 923–975.
- [18] Szegő, G., *Orthogonal Polynomials*. American Mathematical Society, Colloquium Publications **23**, 1939.
- [19] Zelditch, S., From random polynomials to symplectic geometry, *XIIIth International Congress on Mathematical Physics (London, 2000)*, 367–376, Int. Press, Boston, MA, 2001.