

INFINITESIMAL DEFORMATION OF PRINCIPAL BUNDLES, DETERMINANT BUNDLES AND AFFINE LIE ALGEBRAS

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In the present notes we shall show that affine Lie algebras appear as infinitesimal automorphism groups of the determinant bundles of a family of associated vector bundles of the versal family of principal bundles over a Riemann surface R . More natural but sophisticated approach can be found in [BS] and [T]. In the following we fix a Riemann surface R .

§1 Infinitesimal deformations of principal bundles.

The arguments in this section are valid for all complex manifolds. Let G be a simply connected complex simple algebraic group realized as a closed subgroup of $GL(N, \mathbb{C})$ for a sufficiently large integer N . Let $\pi : P \rightarrow R$ be a holomorphic principal G -bundle. Let $R = \cup_{\lambda \in \Lambda} U_{\lambda}$ be an open covering of the Riemann surface such that the principal bundle $\pi : P \rightarrow R$ is trivialized on each U_{λ} . Then the principal bundle can be determined by transition functions $\{g_{\lambda\mu}\}$ with $g_{\lambda\mu} \in \Gamma(U_{\lambda\mu}, \mathcal{G})$ where $\mathcal{G}$ is the sheaf of germs of holomorphic sections of R to G . The transition functions $g_{\lambda\mu}$ satisfy the relation

$$g_{\lambda\mu}g_{\mu\nu} = g_{\lambda\nu} \quad \text{on } U_{\lambda\mu\nu}.$$

Let ϵ be the dual number, that is $\epsilon \equiv z \pmod{(z^2)}$ in $\mathbb{C}[z]/(z^2)$. To change the structure of the principal bundle $\pi : P \rightarrow R$ infinitesimally put

$$\hat{g}_{\lambda\mu} := g_{\lambda\mu}(I + \epsilon h_{\lambda\mu})$$

where I is the identity matrix and $h_{\lambda\mu} \in \Gamma(U_{\lambda\mu}, \underline{\mathfrak{g}})$. Here, $\mathfrak{g}$ is the Lie algebra of the Lie group G realized as a Lie subalgebra of the $N \times N$ matrix algebra $M(N, \mathbb{C})$ and $\underline{\mathfrak{g}}$ is the sheaf of germs of holomorphic sections of R to $\mathfrak{g}$. These

new transition functions satisfy the compatibility condition

$$\hat{g}_{\lambda\mu}\hat{g}_{\mu\nu} = \hat{g}_{\lambda\nu} \quad \text{on } U_{\lambda\mu\nu}.$$

The condition can be rewritten in the form

$$(1) \quad h_{\lambda\nu} = g_{\mu\nu}^{-1} h_{\lambda\mu} g_{\mu\nu} + h_{\mu\nu}$$

on $U_{\lambda\mu\nu}$. Let $\underline{ad}(P)$ be the associated vector bundle (adjoint bundle) $P \times_G \mathfrak{g}$ associated with the adjoint representation of G . Then, the condition (1) means that a Chech cocycle $\{h_{\lambda\mu}\}$ defines an element in $H^1(R, \underline{ad}(P))$.

Theorem 1. *There is a one to one correspondence between the set of infinitesimal deformations of the principal bundle $\pi : P \rightarrow R$ and $H^1(R, \underline{ad}(P))$.*

§2 Principal G -bundles with trivializations.

Let us choose a point Q of the Riemann surface R and a local coordinate ξ of R with center Q . In the following we fix the data $(R; Q; \xi)$. We let $(P; \eta^{(k)})$ be a holomorphic principal G -bundle with k -th infinitesimal trivialization at the point Q :

$$\eta^{(k)} : \mathcal{O}_R(P) \otimes \mathcal{O}_{R,Q} / \mathfrak{m}_Q^{k+1} \simeq G(C[\xi] / (\xi^{k+1})).$$

For $k \rightarrow +\infty$ we have a formal trivialization at Q :

$$\hat{\eta} : \mathcal{O}_R(P) \otimes \hat{\mathcal{O}}_{R,Q} \simeq G(C[[\xi]]).$$

Theorem 1 can be generalized in the following form.

Theorem 2. *For each positive integer k there is a one to one correspondence between the set of infinitesimal deformations of the data $(P; \eta^{(k)})$ and by $H^1(R, \underline{ad}(P)(-(k+1)Q))$.*

Let $\mathcal{M}_R^{(k)}(G)$ be the coarse moduli scheme of stable pairs $(P; \eta^{(k)})$. At a point $\mathfrak{X} = (P; \eta^{(k)})$ of $\mathcal{M}_R^{(k)}(G)$ we have a canonical isomorphism of the tangent space at $\mathfrak{X}$ to the first cohomology group:

$$T_{\mathfrak{X}} \mathcal{M}_R^{(k)}(G) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Let us consider an exact sequence

$$0 \rightarrow \underline{ad}(P)(-(k+1)Q) \rightarrow \underline{ad}(P)((m-(k+1))Q) \rightarrow \bigoplus_{\ell=-m+k+1}^k \mathfrak{g} \otimes \xi^\ell \rightarrow 0.$$

If $m \gg 0$, we have

$$H^1(R, \underline{ad}(P)((m-(k+1))Q)) = 0.$$

Hence, there is an isomorphism

$$\bigoplus_{\ell=-m+k+1}^k \mathfrak{g} \otimes \xi^\ell / H^0(R, \underline{ad}(P)((m-(k+1))Q)) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Taking $m \rightarrow \infty$, we have

$$(2) \quad \mathfrak{g} \otimes (\mathbb{C}[\xi, \xi^{-1}] / (\xi^{k+1})) / H^0(R, \underline{ad}(P)(*Q)) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Note that for every principal G -bundle P over R , there is a positive integer ℓ such that $(P; \eta^{(k)})$, $k \geq \ell$ is always stable. Therefore, if we take $k \rightarrow \infty$, the coarse moduli scheme $\mathfrak{M}_R(G)$ of pairs $(P; \hat{\eta})$ of principal G -bundle with formal trivialization at the point Q contains all the pair $(P; \hat{\eta})$ of principal G -bundle with formal trivialization at Q . Moreover, the coarse moduli scheme is fine and there is a universal family $\varpi : \mathcal{P} \rightarrow R \times \mathfrak{M}_R(G)$ of principal G -bundles with formal trivialization.

Now by virtue of (2), the tangent space of $\mathfrak{M}_R(G)$ at a point $\hat{\mathfrak{X}} = (P; \hat{\eta})$ is given by

$$\mathfrak{g} \otimes \mathbb{C}((\xi)) / H^0(R, \underline{ad}(P)(*Q)).$$

This means that the affine Lie algebra $\mathfrak{g} \otimes \mathbb{C}((\xi))$ without centre operates on $\mathfrak{M}_R(G)$ infinitesimally and the action is infinitesimally homogeneous.

§3 Determinant bundles.

Let V be a G -module and $\rho : G \rightarrow \text{Aut}(V)$ be the corresponding representation. Let $\hat{\varpi} : \mathcal{P} \times_G V \rightarrow R \times \mathfrak{M}_R(G)$ be the associated family of vector bundles

with the universal family $\varpi : \mathcal{V} = \mathcal{P} \rightarrow R \times \mathfrak{M}_R(G)$ of principal G -bundles with formal trivializations. For each principal G -bundle on R put

$$V(P) := P \times_G V.$$

For the second projection $q : R \times \mathfrak{M}_R(G) \rightarrow \mathfrak{M}_R(G)$ we let $\det \mathbb{R}q_* \mathcal{V}$ be the determinant bundle of the family of vector bundles $\mathcal{V}$. For a point $\hat{\mathfrak{X}} = (P; \hat{\eta}) \in \mathfrak{M}_R(G)$ the fibre of the determinant bundle $\det \mathbb{R}q_* \mathcal{V}$ at $\hat{\mathfrak{X}}$ is given by

$$\left(\bigwedge^{\max} H^0(R, V(P)) \right) \otimes \left(\bigwedge^{\max} H^1(R, V(P)) \right)^{-1}.$$

The determinant bundle can be easily described by using the universal Grassmann manifold (UGM) due to Sato and the fermion Fock space. (See, for example [KNTY].) At a point $\hat{\mathfrak{X}} = (P; \hat{\eta})$, by taking the Laurent expansion at the point Q , we have a natural inclusion

$$t : H^1(R, V(P)(*Q)) \hookrightarrow V \otimes_{\mathbb{C}} \mathbb{C}((\xi)).$$

This embedding determines a point of $UGM(V)$ and gives an embedding

$$\tau : \mathfrak{M}_R(G) \hookrightarrow UGM(V).$$

Now $UGM(V)$ can be embedded into $P(\mathcal{F})$ by the Plücker embedding where we may regard $\mathcal{F}$ to be a fermion Fock space. Thus we have a projective embedding

$$\hat{\tau} : \mathfrak{M}_R(G) \hookrightarrow P(\mathcal{F}).$$

Then, the pull-back of the dual of hyperplane bundle of $P(\mathcal{F})$ to $\mathfrak{M}_R(G)$ is nothing but the determinant bundle $\det \mathbb{R}q_* \mathcal{V}$.

The projective embedding can be described in the following way. Let us choose and fix a basis $\{e_1, e_2, \dots, e_n\}$ of the vector space V . Put

$$\begin{aligned} V_j &= \langle e_1, e_2, \dots, e_j \rangle_{\mathbb{C}} \\ H_{jk} &:= t^{-1}(V_j \otimes \mathbb{C}((\xi))). \end{aligned}$$

Then, $\{H_{jk}\}$ is an increasing filtration and we choose a normalized basis $\{h_1, h_2, \dots\}$ of $H^0(R, V(P)(*Q))$ by lexicographic ordering with respect to the filtration with

normalization at the coefficient of the first leading term. Then, the infinite exterior product

$$h_1 \wedge h_2 \wedge \cdots$$

gives the point $\hat{\tau}(\hat{x})$ in $P(\mathcal{F})$.

Now let us consider the action of $\otimes C((\xi))$ on $\det \mathbb{R}_{q*} \mathcal{V}$ which is the lift of the one on $\mathfrak{M}_R(G)$. For an element A of $\mathfrak{g} \otimes C((\xi))$ this is very easy, since A acts on $\mathfrak{M}_R(G)$ as infinitesimal change of formal trivializations. That is, $A(h_j)$ is well-defined and the infinite product

$$A(h_1) \wedge A(h_2) \wedge \cdots$$

is also well-defined. This gives the desired action. Let us define the action of an element A of $\mathfrak{g} \otimes C((\xi))$. Let $R = \cup_{\lambda \in \Lambda} U_\lambda$ be a small open covering of R such that a principal G -bundle is given by transition functions $\{g_{\lambda\mu}\}$. The section h_j is given by V -valued holomorphic functions f_λ on U_λ 's with

$$f_\lambda = \rho(g_{\lambda\mu}) f_\mu.$$

We define the action of A on h_j in such a way that

$$A(f_\lambda) = f_\lambda + \epsilon \eta_\lambda$$

for each λ . By the isomorphism (2), the element A defines an element $\{h_{\lambda\mu}\} \in H^1(R, \underline{ad}(P)(-(k+1)Q))$ for a suitable k . Then, we need to have

$$A(f_\lambda) = \rho(g_{\lambda\mu} + \epsilon_{\lambda\mu}) A(f_\mu).$$

This is equivalent to saying that

$$(3) \quad \eta_{\lambda\mu} = \rho(h_{\lambda\mu}) f_\mu + \rho(g_{\lambda\mu}) \eta_\mu.$$

Since $\rho(h_{\lambda\mu}) f_\mu$ defines an element of $H^1(R, V(P)(mQ))$ for a certain integer m and we have

$$H^1(R, V(P)(*Q)) = 0$$

we can always find $\{\eta_\lambda\} \in \Gamma(U_\lambda, V(P)(*Q))$ which satisfy (3). $\{\eta_\lambda\}$ is uniquely determined up to the addition of an element in $H^0(R, V(P)(*Q))$. Therefore, we may choose $\{\eta_\lambda\}$ in such a way that

$$\eta_\lambda \in \Gamma(U_\lambda, V(P)(\ell Q))$$

with

$\ell \gg$ order of pole of h_j at Q .

Then, the infinite wedge product

$$A(h_1) \wedge A(h_2) \wedge \cdots$$

is well-defined. Since the above argument does not determine $A(h_j)$ uniquely, the action of A does not necessarily defines the action of $\mathfrak{g} \otimes C((\xi))$ on the determinant bundle $\det \mathbb{R}q_* \mathcal{V}$. There is a canonical way to define the action of $\mathfrak{g} \otimes C((\xi))$ on the determinant bundle $\det \mathbb{R}q_* \mathcal{V}$ by using the second quantization (or renormalization) of operators acting on the fermion Fock space $\mathcal{F}$. (See [KNTY].) The process shows that we need to take a central extension

$$\mathfrak{g} \otimes C((\xi)) \oplus \mathbb{C} \cdot c$$

of the Lie algebra $\mathfrak{g} \otimes C((\xi))$ to lift the operation of $\mathfrak{g} \otimes C((\xi))$ on $\mathcal{M}_R(G)$ to $\det \mathbb{R}q_* \mathcal{V}$.

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