

On the strongly starlikeness of starlike functions

Mamoru Nunokawa

Abstract

It is the purpose of the present paper to prove that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $|z| < 1$ and if

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \frac{1+\alpha}{2} \quad \text{in } |z| < 1$$

where $0 \leq \alpha < 1$.

Then $f(z)$ is strongly starlike of order $(1 - \alpha)$ or

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}(1 - \alpha) \quad \text{in } |z| < 1.$$

1 Introduction

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be convex in $\mathbb{E} = \{z \mid |z| < 1\}$, that is, $f(z)$ is analytic in $\mathbb{E}$ and maps $\mathbb{E}$ univalently onto a convex domain. It is well known that the necessary and sufficient condition for the convexity of $f(z)$ is

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } \mathbb{E}.$$

On the other hand, let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be starlike in $\mathbb{E}$, that is, $f(z)$ is analytic in $\mathbb{E}$ and maps $\mathbb{E}$ univalently onto a starlike domain which is starshaped with respect to the origin. It is well known that the necessary and sufficient condition for the starlikeness of $f(z)$ is

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } \mathbb{E}.$$

Sheil-Small [2] proved the following theorem.

Theorem A Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be starlike in $\mathbb{E}$, let $C(r, \theta) = \{f(te^{i\theta}) \mid 0 \leq t \leq r\}$ and let $T(r, \theta)$ be the total variation of $\arg f(te^{i\theta})$ on $C(r, \theta)$, so that

$$T(r, \theta) = \int_0^r \left| \frac{\partial}{\partial t} \arg f(te^{i\theta}) \right| dt.$$

Then we have

$$T(r, \theta) < \pi.$$

2 Theorem

Theorem 1 Let $f(z)$ be convex in $\mathbb{E}$ and suppose that

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \frac{1+\alpha}{2} \quad \text{in } \mathbb{E},$$

where $0 \leq \alpha < 1$.

Then $f(z)$ is starlike in $\mathbb{E}$ and strongly starlike of order $(1-\alpha)$ or

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}(1-\alpha) \quad \text{in } \mathbb{E}.$$

Proof. Let us put

$$(1) \quad 1 + \frac{zf''(z)}{f'(z)} = \left(\frac{1+\alpha}{2} \right) + \left(1 - \frac{1+\alpha}{2} \right) \frac{zg'(z)}{g(z)}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic in $\mathbb{E}$.

Then, from the hypothesis of the theorem, $g(z)$ is starlike in $\mathbb{E}$. From (1), we have

$$(2) \quad \frac{f''(z)}{f'(z)} = \left(\frac{1-\alpha}{2} \right) \left(\frac{g'(z)}{g(z)} - \frac{1}{z} \right)$$

and integrating (2) along the straight line from the origin to z , then it follows that

$$\int_0^z \frac{f''(\zeta)}{f'(\zeta)} d\zeta = \int_0^z \left(\frac{1-\alpha}{2} \right) \left(\frac{g'(\zeta)}{g(\zeta)} - \frac{1}{\zeta} \right) d\zeta$$

and so on

$$\log f'(z) = \left(\frac{1-\alpha}{2} \right) \log \frac{g(z)}{z}$$

or

$$f'(z) = \left(\frac{g(z)}{z} \right)^{\frac{1-\alpha}{2}}.$$

Then we have

$$(3) \quad \begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{z \left(\frac{g(z)}{z} \right)^{\frac{1-\alpha}{2}}}{\int_0^z \left(\frac{g(\zeta)}{\zeta} \right)^{\frac{1-\alpha}{2}} d\zeta} \\ &= \frac{1}{\int_0^z \left(\frac{z}{\zeta} \right)^{\frac{1-\alpha}{2}} \left(\frac{g(\zeta)}{g(z)} \right)^{\frac{1-\alpha}{2}} \frac{d\zeta}{z}} \end{aligned}$$

where $z = re^{i\theta}$, $\zeta = te^{i\theta}$ and $0 \leq t \leq r$.

From (3), it follows that

$$(4) \quad \frac{zf'(z)}{f(z)} = \left(\int_0^1 t^{\frac{\alpha-1}{2}} \left(\frac{g(tz)}{g(z)} \right)^{\frac{1-\alpha}{2}} dt \right)^{-1}$$

Then, from Theorem A, we have

$$(5) \quad -\pi < \arg \left(\frac{g(tz)}{g(z)} \right) < \pi$$

where $0 \leq t \leq 1$.

Putting

$$s = t^{\frac{\alpha-1}{2}} \left(\frac{g(tz)}{g(z)} \right)^{\frac{1-\alpha}{2}},$$

then we have

$$(6) \quad \arg s = - \left(\frac{1-\alpha}{2} \right) \arg \left(\frac{g(tz)}{g(z)} \right)$$

and from (4), we have

$$(7) \quad \arg \frac{zf'(z)}{f(z)} = - \arg \left(\int_0^1 s \, dt \right).$$

Then from (5) and (6), we have

$$(8) \quad |\arg s| < \frac{\pi}{2}(1-\alpha) \quad \text{in } \mathbb{E}.$$

From the property of integral mean (see e.g. [1, Lemma 1]) and from (8), we have

$$(9) \quad \left| \arg \left(\int_0^1 s \, dt \right) \right| < \frac{\pi}{2}(1-\alpha) \quad \text{in } \mathbb{E}.$$

Then, from (7) and (9), we have

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}(1-\alpha) \quad \text{in } \mathbb{E}.$$

This completes the proof of Theorem 1. □

References

- [1] Ch. Pommerenke, *On close-to-convex functions*, Trans. Amer. Math. Soc., **114** (1965), 176-186.
- [2] T. Sheil-Small, *Some conformal mapping inequalities for starlike and convex functions*, J. London Math. Soc., **1** (1969), 577-587.

Mamoru Nunokawa
Emeritus Professor of University of Gunma
Hoshikuki-cho 798-8, Chuou-Ward,
Chiba city 260-0808
Japan
e-mail: mamoru_nuno@doctor.nifty.jp