

Normal form theorem of natural deduction for modal logic S4

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1 Introduction

Gentzen's Hauptsatz [2], which has been stated in the systems of sequent calculi, was reconstructed by Prawitz [5] as normalization theorem in the systems of natural deduction. Concerning the modal logic S4, Prawitz introduced three formulations in natural deduction, and noticed that the third one enjoys normalization theorem. Later, Medeiros [3] mentioned that Prawitz's proof does not work, and gave a new proof of normalization theorem with another formulation of S4 in natural deduction. But her proof also contains gaps, as we have seen in our previous technical report [1].

In this paper, we prove the *normal form theorem* of natural deduction for S4 in the formulation of Medeiros. Notice that it is not the *normalization theorem* of the system in a narrow sense. It means that we can show the existence of a normal derivation for any given derivation, but we do not define non-trivial normalization procedure in the system. Our proof depends on the cut-elimination theorem of sequent calculus for S4 proved by Ohnishi and Matsumoto [4].

First, we recall the formulation of Medeiros, and give the definition of the maximal formula, the redex of a derivation. Second, we define the transformation of a given cut-free derivation in sequent calculus for S4 to a normal derivation in natural deduction for the same logic.

2 The system NS4

In [3], Medeiros introduced a new formalization of the system in natural deduction for classical propositional modal logic S4, called **NS4**. It has $\wedge, \vee, \supset, \perp, \Box$ as logical constants, and the inference rules for introduction and elimination of $\wedge, \vee, \supset$ are defined as usual. The rules for introduction and elimination of the modal operator $\Box$ are defined as below.

- Introduction rule for $\Box$ is:

$$\frac{\Box B_1 \dots \Box B_n \quad \frac{[\Box B_1] \dots [\Box B_n] \quad A}{(\Box I)} \quad \Box A}{\Box A}$$

where A depends on no assumptions other than $\Box B_1, \dots, \Box B_n$, and these assumptions are all discharged at the rule $(\Box I)$. For the assumption (as formula occurrence) $\Box B_i$ discharged at the rule, we call the premiss (as formula occurrence) $\Box B_i$ the *corresponding premiss* of the assumption.

Notice that one assumption $\Box B_i$ has exactly one corresponding premiss, and that one premiss $\Box B_i$ may have arbitrary many (possibly zero) assumptions of the form $\Box B_i$.

- Elimination rule for $\Box$ is:

$$\frac{\Box A}{A} (\Box E).$$

Further, the system has the inference rule so called classical absurdity rule:

$$\frac{[A \supset \perp]}{\perp} (\perp_e).$$

We define the notion of maximal formula as below. Our definition is slightly different from that of Medeiros', but we can see that if a derivation has no maximal formula in our sense, it has no maximal segment (therefore no maximal formula) in the meaning of Medeiros' definition.

Definition (Maximal formula). A formula-occurrence A in a derivation Π is called a *maximal formula* in Π , if A satisfies one of the following conditions (a) or (b):

- (a) A is the conclusion of an introduction rule, a $\vee$ -elimination rule, or a classical absurdity rule. Moreover, A is the major premiss of an elimination rule.
- (b) A is an assumption discharged at a $\Box$ -introduction rule and it is also the major premiss of an elimination rule. Moreover, the corresponding premiss of A is the conclusion of an introduction rule, a $\vee$ -elimination rule, or a classical absurdity rule.

Definition (Normal derivation). A derivation Π is called *normal* if it has no maximal formula.

3 The normal form theorem

We show the normal form theorem of the system NS4, by using the cut-elimination theorem of the sequent calculus for the same propositional modal logic S4, which have been proved by Ohnishi and Matsumoto [4]. The inference rules of the sequent calculus for S4, which we shall call **LS4** after this, are those of propositional fragment of Gentzen's LK [2], and the rules for modal operator $\Box$ as follows:

- $\Box$ -left rule is:

$$\frac{A, \Gamma \rightarrow \Theta}{\Box A, \Gamma \rightarrow \Theta} (\Box L)$$

- $\Box$ -right rule is:

$$\frac{\Box \Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A} (\Box R)$$

For the sake of normal form theorem of NS4, we introduce another system of sequent calculus called **LS4'**. It is obtained from NS4 by adding the logical constant $\perp$ and the axiom:

$$\perp \rightarrow$$

Fact 1. If a NS4-derivation Π of a formula A from assumptions Γ is given, we can construct a LS4'-derivation Π' of the sequent $\Gamma \rightarrow A$.

Proof. By induction on the length of Π . ■

Theorem (Cut-elimination theorem of LS4'). *If a LS4'-derivation is given, we can construct a cut-free LS4'-derivation of the same end-sequent.*

Proof. We can prove the theorem similarly to the proof in [4] of the cut-elimination theorem for LS4, which is an extension of Gentzen's Hauptsatz [2]. ■

Fact 2. If a cut-free LS4'-derivation Π of a sequent $\Gamma \rightarrow \Theta$ is given, we can construct a normal NS4-derivation Π' such that:

- (a) In the case that Θ is empty, Π' is a derivation of the formula $\perp$ from assumptions Γ .
- (b) Otherwise, here we suppose Θ is the concatenation of Θ^- and C where Θ^- is a (possibly empty) sequence of formulae and C is a formula, Π' is a derivation of the formula C from assumptions Γ and $\neg\Theta^-$, where the formulae in $\neg\Theta^-$ are obtained from the formulae in Θ^- by applying the negation.

Proof. By induction on the length of Π . Notice that the condition of the dependence of the upper formula of the introduction rule for $\Box$ are preserved at each induction step. ■

Theorem (Normal form theorem of NS4). *If a NS4-derivation Π of a formula A from assumptions Γ is given, we can construct a normal NS4-derivation Π' of A from Γ .*

Proof. By the cut-elimination theorem of LS4' and the above two facts. ■

References

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