

ON THE RUMIN COMPLEX

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As is well known, Rumin introduced his complex for contact manifolds. Obviously, his method is applicable to the strongly pseudo convex boundary case(see [Ru], and also [A-M1]). Namely, let $(M, {}^0T'')$ be a strongly pseudo convex CR manifold in a complex manifold N , then by Rumin we have a differential complex on M ,

$$\begin{aligned} \wedge^{n-1}({}^0T')^* &\xrightarrow{D} \theta \wedge \wedge^{n-2}({}^0T')^* \wedge^1({}^0T'')^* \xrightarrow{\bar{\partial}_b^{(1)}} \theta \wedge \wedge^{n-2}({}^0T')^* \wedge^2({}^0T'')^* \\ &\xrightarrow{\bar{\partial}_b^{(p-1)}} \theta \wedge \wedge^{n-2}({}^0T')^* \wedge^p({}^0T'')^* \xrightarrow{\bar{\partial}_b^{(p)}} \theta \wedge \wedge^{n-2}({}^0T')^* \wedge^{p+1}({}^0T'')^*, \end{aligned}$$

which recovers the Kohn-Rossi cohomology.

$$\begin{aligned} \text{for } p = 1, \text{ Ker } \bar{\partial}_b^{(1)} / \text{Im } D &\simeq H_b^{(1)}(M, \wedge^{(n-1)}(T')^*), \\ \text{for } p > 1, \text{ Ker } \bar{\partial}_b^{(p)} / \text{Im } \bar{\partial}_b^{(p-1)} &\simeq H_b^{(p)}(M, \wedge^{(n-1)}(T')^*), \end{aligned}$$

where D is a second order differential operator which is introduced by Rumin, and ${}^0T' = \overline{{}^0T''}$ and θ means a contact form(this is a part of Rumin's result but a typical one. More precisely, see Sect.1 in this paper).

However this differential complex has an essential weak point. Namely, $\bar{\partial}_b^{(p)}$ has a natural extension to an ambient space N , but it is not clear if the above second order differential operator D has a natural extension to an operator on N or not. By this reason, we propose a new complex, based on Rumin complex, which is applicable to several complex variables(see Sect.2).

Sect.1. CR structures and Rumin complex

Let $(M, {}^0T'')$ be a CR structure. This means that: M is a real $2n-1$ dimensional C^∞ manifold and ${}^0T''$ is a complex subbundle of the complexified tangent bundle satisfying:

- 1) ${}^0T'' \cap {}^0T' = 0$, $\dim_C(C \otimes TM / ({}^0T'' + {}^0T')) = 1$
- 2) $[\Gamma(M, {}^0T''), \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T'')$,

where ${}^0T' = \overline{{}^0T''}$. In this paper we assume more. Namely, we assume that there is a real global C^∞ vector bundle ξ satisfying:

- 3) $\xi_p \notin {}^0T''_p + {}^0T'_p$ for every point p of M ,

For brevity, we use the following notation.

$$4) T' = C \otimes \xi + {}^0T'.$$

Now we set a real one form θ by

$$\begin{aligned} \theta(\xi) &= 1, \\ \theta|_{{}^0T'' + {}^0T'} &= 0. \end{aligned}$$

Let

$$\Omega = d\theta.$$

If this 2-form is positive or negative definite, then our CR-structure $(M, {}^0T'')$ is called strongly pseudo convex. From now on, we assume that our CR is strongly pseudo convex. By using these notations, we can introduce a C^∞ vector bundle decomposition:

$$\begin{aligned} 5) \wedge^k (C \otimes TM)^* &= \sum_{p+q=k-1, p, q \geq 0} \theta \wedge \wedge^p ({}^0T')^* \wedge \wedge^q ({}^0T'')^* \\ &\quad \sum_{r+s=k, r, s \geq 0} \wedge^r ({}^0T')^* \wedge \wedge^s ({}^0T'')^* \end{aligned}$$

We fix this decomposition. And we would like to consider a double complex (for the precise definition, see [A4]). For $u \in \Gamma(M, \theta \wedge \wedge^{p-1} ({}^0T')^* \wedge \wedge^{q-1} ({}^0T'')^*)$, we set an element of $\Gamma(M, \wedge^p ({}^0T')^* \wedge \wedge^q ({}^0T'')^*)$ by

$$u \rightarrow (du)_{\wedge^p ({}^0T')^* \wedge \wedge^q ({}^0T'')^*}$$

Proposition 1.1. *This map is a bundle map.*

(The proof is a direct computation, and more precisely, see Sect.3.)

Proposition 1.2. *If $p + q \geq n$, then this map is surjective and especially, if $p + q = n$, then by comparing dimensions, this is isomorphic.*

(For the proof, see lemma 3.3 in [A4])

We use the notation κ^p for this isomorphic map from $\theta \wedge \wedge^{p-1}({}^0T'') \wedge \wedge^{q-1}({}^0T'')$ to $\wedge^p({}^0T')^* \wedge \wedge^q({}^0T'')^*$, where $q = n - p$. By using this κ^p , we set an element ψ_u of $\Gamma(M, \theta \wedge \wedge^{p-1}({}^0T'')^* \wedge \wedge^{q-1}({}^0T'')^*)$ by

$$\psi_u = (\kappa^p)^{-1}((\bar{\partial}_b u)_{\wedge^p({}^0T')^* \wedge \wedge^{n-p}({}^0T'')^*})$$

where $(\bar{\partial}_b u)_{\wedge^p({}^0T')^* \wedge \wedge^{n-p}({}^0T'')^*}$ means the projection of $\bar{\partial}_b u$ to $\wedge^p({}^0T')^* \wedge \wedge^{n-p}({}^0T'')^*$ according to (5). So by the definition of ψ_u , our ψ_u includes the first derivative of u .

Sect.2. New complex

Now we introduce a new complex. For a simplicity, we discuss only in the case $p = n - 2$, which is quite related to deformation theory. We set

$$H^0 = \{u : u \in \Gamma(M, \wedge^{n-1}({}^0T')^*), (\partial_b u)_{\wedge^{n-1}({}^0T')^* \wedge \wedge({}^0T')^*} = 0\}$$

$$H^1 = \{u : u \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^* \wedge \wedge({}^0T'')^*), (\bar{\partial}_b^{(1)} u)_{\wedge^{n-1}({}^0T')^* \wedge \wedge^2({}^0T')^*} = 0\}$$

$$H^2 = \{u : u \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^* \wedge \wedge^2({}^0T'')^*), (\bar{\partial}_b^{(2)} u)_{\wedge^{n-1}({}^0T')^* \wedge \wedge^3({}^0T')^*} = 0\}$$

Then by definition our $(H^i, \bar{\partial}_b)$ is a differential complex

$$H^0 \xrightarrow{\bar{\partial}_b} H^1 \xrightarrow{\bar{\partial}_b^{(1)}} H^2$$

Of course H^1 (resp. H^2) is nothing but our $F^{n-2,1}$ (resp. $F^{n-2,2}$) (see [A-M1], [A4]). And we note that our H^0 doesn't come from C^∞ sections of any C^∞ vector bundle on M . This is purely a vector space of some $\wedge^{n-1}({}^0T')^*$ -valued C^∞ sections. We put an L^2 norm on these spaces and discuss the Kodaira Hodge type decomposition theorem on H^1 . For this, we have to show an a priori estimate. And a difficult problem is to compute the adjoint operator of $\bar{\partial}_b$. By the definition of H^0 , H^0 is a subspace of

$$\Gamma(M, \wedge^{n-1}({}^0T')^*) = \Gamma(M, \wedge^{n-1}({}^0T')^*) + \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T'')^*).$$

And in this canonical decomposition of $\Gamma(M, \wedge^{n-1}({}^0T')^*)$, our H^0 can be regarded as a graph of the following map.

$$\text{For } u \in \Gamma(M, \wedge^{n-1}({}^0T')^*),$$

we set

$$\psi_u \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^*),$$

where ψ_u is introduced in Sect.1 in this paper. So

$$H^0 = \{v : v = u + \psi_u, u \in \Gamma(M, \wedge^{n-1}({}^0T')^*)\} \subset \Gamma(M, \wedge^{n-1}(T')^*).$$

For this correspondence, we call the *graph map* i . On $\Gamma(M, \wedge^{n-1}(T')^*)$, and $\Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^* \wedge \wedge^p({}^0T'')^*)$, $p = 1, 2, \dots$, we put L^2 norm and consider the Kodaira Hodge decomposition theorem on $(H^p, \bar{\partial}_b)$. The problem is to show an a priori estimate. In proving an a priori estimate, we have to compute, explicitly "the adjoint operator" on H^p spaces (namely we have to write down the term of "the adjoint operator", otherwise, it is impossible to obtain an a priori estimate. We discuss this in the next section.

Sect.3. The adjoint operators on H^p spaces

We consider the projection of $\Gamma_2(M, \wedge^{n-1}(T')^*)$ to $\tilde{H}^0$, where $\Gamma_2(M, \wedge^{n-1}(T')^*)$ means the L_2 - completion of $\Gamma(M, \wedge^{n-1}(T')^*)$, and $\tilde{H}^0$ means the L_2 closure of H^0 in $\Gamma_2(M, \wedge^{n-1}(T')^*)$. We recall the graph map i .

$$\Gamma(M, \wedge^{n-1}({}^0T')^*) \xrightarrow{\text{graph map } i} H^0$$

We use the notation A for the composition map of this graph map i and the inclusion map of H^1 to $\Gamma(M, \wedge^{n-1}(T')^*)$. So A is a map from $\Gamma(M, \wedge^{n-1}({}^0T')^*)$ to $\Gamma(M, \wedge^{n-1}(T')^*)$. By the way, if we put a L_2 norm on $\Gamma(M, \wedge^{n-1}({}^0T'')^*)$ by : for $v \in \Gamma(M, \wedge^{n-1}({}^0T'')^*)$,

$$\|v\|_{\Gamma(M, \wedge^{n-1}({}^0T'')^*)}^2 = \|v\|^2 + \|\psi_v\|^2 \quad (\text{a graph norm}),$$

our i is a norm preserving map (almost tautology). Therefore

$$\begin{aligned} (i^*iv, w)_{\Gamma(M, \wedge^{n-1}({}^0T'')^*)} &= (iv, iw) \\ &= (v, w) + (\psi_v, \psi_w) \\ &= (v, w)_{\Gamma(M, \wedge^{n-1}({}^0T')^*)} \end{aligned}$$

Namely

$$i^*i = \text{identity on } \Gamma(M, F)$$

Especially,

$$i^*(v + \psi_v) = v, \text{ for } v \in \Gamma(M, \wedge^{n-1}({}^0T'')^*).$$

Here i^* means the adjoint operator of i with respect to this graph norm (on $\Gamma(M, \wedge^{n-1}({}^0T')^*)$, we use the graph norm defined by i , and on $\Gamma(M, \wedge^{n-1}(T')^*)$, the standard L_2 is used). Then, our main theorem is

Main Theorem. *The projection map $= i \cdot A^*$ on $\Gamma(M, T')$.*

Proof. For this, it suffices to show that:

- (1) *for $w = w_1 + w_2$, which is orthogonal to H^0 , we have $i \cdot A^*w = 0$,*
- (2) *for $u \in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T')^*)$, we have $i \cdot A^*(u + \psi_u) = u + \psi_u$.*

where

$$\begin{aligned} w &\in \Gamma(M, \wedge^{n-1}(T')^*), \\ w_1 &\in \Gamma(M, \wedge^{n-1}({}^0T')^*), \\ w_2 &\in \Gamma(M, \theta \wedge \wedge^{n-2}({}^0T'')^*). \end{aligned}$$

For the proof of (1), by the definition of $w = w_1 + w_2$,

$$(w_1, v) + (w_2, \psi_v) = 0 \text{ for } v \in \Gamma(M, \wedge^{n-1}({}^0T')^*),$$

But this means

$$(w, Av) = 0 \text{ for } v \in \Gamma(M, \wedge^{n-1}({}^0T')^*).$$

So

$$(A^*w, v) = 0 \text{ for } v \in \Gamma(M, \wedge^{n-1}({}^0T')^*).$$

So we have (1).

For the proof of (2),

$$\begin{aligned} (i \cdot A^*(u + \psi_u) - (u + \psi_u), v + \psi_v) &= ((u + \psi_u), Ai^* \cdot (v + \psi_v)) - (u + \psi_u, u + \psi_u), \\ &= (u + \psi_u, Av) - (u + \psi_u, u + \psi_u) \end{aligned}$$

for $u, v \in \Gamma(M, \wedge^{n-1}({}^0T')^*)$. This becomes

$$\begin{aligned} (A^*(u + \psi_u) - u, v) &= (u + \psi_u, v + \psi_v) - (u + \psi_u, v + \psi_v) \\ &= 0. \end{aligned}$$

Sect.4. Kodaira – Hodge type decomposition theorem

In order to establish a Kodaira-Hodge type decomposition theorem on H^1 , we have to show the following two conditions.

- 1) $H^1 \cap \text{Dom } S$ is dense in $\tilde{H}^1$.
- 2) The following type a priori estimate.

$$\|S\phi\| + \|\partial_b^{(1)}\phi\| + \|\phi\| \geq C\|\phi\|_{1/2}$$

for $\phi \in H^1 \cap \text{Dom } S$, where S means the adjoint operator of ∂_b in (H^p, ∂_b) complex, and C is a positive constant. While, by the result in Sect.2, we have

$S = \text{the composition of } \bar{\partial}_b^* \text{ and "the projection operator of } \Gamma_2(M, \wedge^{n-1}(T')^*) \text{ to } \tilde{H}^0",$

where ∂_b^* means the adjoint operator ∂_b in the standard complex. Because our complex is a subcomplex, this result follows from functional analysis. Namely, S is nothing but the adjoint operator of D , which Rumin finds. So, by his estimate (1) is now obvious), we have a Kodaira-Hodge type decomposition theorem over H^1 .

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