

Toward a Path-Integral Description of Inequivalent Quantizations

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ABSTRACT

A possible direction of constructing a path-integral description of inequivalent quantizations on coset spaces is discussed.

The subject of quantizations on manifolds (isomorphic to coset spaces G/H) was pioneered by Mackey [1] in 60's and further developed by others (see [2]). However, it renewed its interest rather recently when Landsman and Linden [3] discovered that the inequivalent quantizations found by Mackey entail a background Yang-Mills potential on the coset space when dynamics is taken into account. Essentially the same conclusion was drawn more recently by Ohnuki and Kitakado [4] in an independent attempt to quantizing on manifolds (see also [5]). All of these studies are based on group theoretical ideas which, roughly speaking, generalize the conventional canonical quantization procedure so that the new approach can accommodate those manifolds of consideration. However, as mentioned in [2], the obvious drawback of this approach is that extension to field theories is extremely difficult, given that the groups underlying those cases are infinite dimensional and their representation theory hardly known.

To remedy this drawback, it is desirable (and perhaps natural) to develop a path-integral formalism of the inequivalent quantizations in place of the 'operator formalism' mentioned above. In view of this, in [6] a path-integral description was argued intending to reproduce the results of the operator formalism and at the same time admit an immediate extension to field theories. The essence of the argument in [6] is that, if one is to get the many inequivalent quantizations on the coset space from Dirac's constraint point of view, one has to allow the constraints to be modified in character from first class to a mixture of first and second class. Once this 'anomalous' behaviour of the constraints is accepted, one can readily re-derive the inequivalent quantum theories where the induced Yang-Mills potentials appear.

It is also easy to see that the set of integers labelling the many quantizations (which corresponds to the character of the irreducible representations of the subgroup H) arises as a consequence of a consistency requirement in the path-integral, a familiar device to derive parameter quantizations in the path-integral.

The status quo is not satisfactory, though. In fact, this path-integral argument has not quite achieved our intended goal, because we have not understood what causes the modification of the constraints and, more fundamentally, why we need to regard the system as constrained in the first place. Both of these stem from the fact that we lack a path-integral account of the inequivalent quantizations from the first principle: *sum over all possible paths*. The demand that the successful path-integral must fulfill is that it allows to be constructed under a fairly simple and intuitive manner based on the principle (as the conventional path-integral does). It must also allow to have some intrinsic ambiguity of order $\hbar$ that leads to inequivalent quantizations when the configuration space Q is non-trivial. More explicitly, we expect the (configuration space) path-integral to take the form,

$$Z = \int d\mu(Q) \exp \frac{i}{\hbar} (S_{\text{cl}} + \hbar S_{\text{amb}}) , \quad (1)$$

where $d\mu(Q)$ is the measure assigned to the configuration space, S_{cl} stands for the classical action while S_{amb} for the action containing ambiguous terms. But what is the guiding principle for determining S_{amb} ?

At this point we wish to recall that, given a classical system, there is no unique prescription for constructing a quantum system from it — quantization is a deformation of classical theory, which is a non-unique process although the opposite, the contraction, is. This does not imply that the ambiguity action S_{amb} is uncontrollable. It will not be unnatural to require that S_{amb} be ‘intrinsically’ of order $\hbar^0$, that is, a mere number without $\hbar$, because then the path-integral (1) would reproduce the classical theory in the limit $\hbar \rightarrow 0$. As far as we know, the only means to ensure this in the path-integral is either that S_{amb} is made of topological terms, or that there is some gauge invariance hidden so that a consistency of the path-integral provides the property needed — in any case some topological quantization mechanism is crucial. This seems to be the clue to the path-integral description we are searching for. Indeed, when Q is a coset space G/H , it is not difficult to convince oneself that the terms allowed for S_{amb} are precisely the ones we find in the actual path-integral expression [6].

The other question concerns with the definition of the measure $d\mu(Q)$. This

however seems highly non-trivial, because the sum over all passible paths involves considerations of the topological structure of the configuration space Q ; for example, we need to take into account paths winding around a hole several times if the first homotopy group π_1 of Q is non-trivial. In Schulman's book [7] this problem has been solved for $Q \simeq S^1$ and further discussed for more general cases, but there seems no complete prescription for giving the measure for a generic Q . In this respect, it may be worth stressing that if Q is a manifold of a compact Lie group then the semi-classical approximation of the path-integral becomes exact [7], a result suggesting that group manifolds form a distinguished class for which possibly the measure is well-defined or manageable; in fact for this case one could define it from the Haar measure of the group. Thus we may speculate that by this reason one should first go to a group manifold G with which the given Q becomes a coset G/H with a proper subgroup H . Then one can define the quantum theory in the path-integral regarding the theory constrained, that is, one performs the reduction with respect to the H . This is actually what we did in [6] for Q isomorphic to a coset space.

In conclusion, we discussed a possible direction of the path-integral for quantizing on non-trivial manifolds, which, although still hypothetical, appears to be working in the case $Q \simeq G/H$. We hope to report on this subject again with more definite ideas in the near future.

References

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