

Regular subgraphs of a regular graph (*)

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1. Introduction

We consider a finite graph, which may have multiple edges but has no loops. A graph without multiple edges is called a simple graph. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For a vertex v of a subgraph H , we write $d_H(v)$ for the degree of v in H . A graph G is called an r -regular graph if $d_G(x)=r$ for all $x \in V(G)$, a spanning r -regular subgraph is called an r -factor. Notation and definitions not given in this paper can be found in [1].

We shall consider the following problem. For given integers k and r , $0 < k < r$, does an r -regular graph contain a k -regular subgraph? Let us begin with some known results on this problem. The first theorem is well-known as Petersen's 2-factorization theorem.

Theorem A.(Petersen [9]) Every $2r$ -regular graph has a $2k$ -factor for every integer k , $0 < k < r$.

Theorem B.(Taskinov[10]) Every $(2r+1)$ -regular graph has a $(2k+1)$ -regular subgraph for every integer k , $0 < k < r$.

Taskinov [10] also proved that every 4-regular simple graph contains a 3-regular subgraph, which was conjectured by C. Berge. The next conjecture given by Chvatal et al [3] is still open.

Conjecture Every 4-regular graph of even order contains a 3-regular subgraph.

In this paper we shall prove the following two theorems.

(*) 文献 [10] のタイトルと一見同じであるが違う点に注意された!!

Theorem 1. Let r be an odd integer and k be an even integer such that $2 \leq k \leq 2r/3$. Then every r -regular graph contains a k -regular subgraph.

Theorem 2. Let r be an even integer and k be an odd integer such that $1 \leq k \leq r/2$. Then every r -regular graph of even order contains a k -regular subgraph. Moreover, every r -regular graph of odd order with edge-connectivity two contains a k -regular subgraph.

Note that for every integer r , $r \geq 6$, there exists an r -regular simple graph which has no $(r-1)$ -regular subgraphs ([7],[10]). Furthermore, for every odd integer $r > 0$, there exists a $2r$ -regular graph of odd order that has no $(r+2)$ -regular subgraphs. For example, such a graph is obtained from the complete graph K_3 of order 3 by replacing each edge of K_3 by r -multiple edges.

2. Proofs of Theorems

We begin by introducing some definitions. Let G be a graph, and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a (g,f) -factor if $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. A (g,f) -factor satisfying $g(x) = f(x)$ for all $x \in V(G)$ is briefly called an f -factor. For a vertex subset X of G , we write $G-X$ for the subgraph of G induced by $V(G) \setminus X$. For an edge subset Y of G , $G-Y$ denotes the subgraph of G obtained from G by deleting all the edges in Y . For two disjoint subsets S and T of $V(G)$, we denote by $e_G(S,T)$ the number of edges of G joining S and T . For a set $\{a,b,\dots,d\}$ of integers, a graph G is called an $\{a,b,\dots,d\}$ -graph if $d_G(x) \in \{a,b,\dots,d\}$ for all $x \in V(G)$. A spanning $\{a,b,\dots,d\}$ -subgraph is called an $\{a,b,\dots,d\}$ -factor.

Lemma 1.(Lovasz [8]) Let G be a graph, and g and f be integer-valued functions defined on $V(G)$ satisfying $g(x) \leq f(x)$ for all $x \in V(G)$.

Then G has a (g,f) -factor if and only if

$$\delta(S,T) = \sum_{t \in T} (d_G(t) - g(t)) + \sum_{s \in S} f(s) - e_G(S,T) - h(S,T) \geq 0$$

for all disjoint subsets S and T of $V(G)$, where $h(S,T)$ denotes the number of components C of $G - (S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$, and

$$\sum_{x \in V(C)} f(x) + e_G(T, V(C)) \equiv 1 \pmod{2}.$$

Lemma 2. ([5], [6]) Let G be an n -edge-connected graph ($n \geq 1$), f be an integer-valued function defined on $V(G)$, θ be a real number such that $0 \leq \theta \leq 1$, and A and B be disjoint subsets of $E(G)$. If (1), (2) and one of $\{(3a), (3b)\}$ hold, then G has an f -factor F such that $F \supset A$ and $F \cap B = \emptyset$.

$$(1) \sum_{x \in V(G)} f(x) \equiv 0 \pmod{2}.$$

$$(2) \varepsilon = \sum_{x \in V(G)} |f(x) - \theta d_G(x)| + 2\theta|A| + 2(1-\theta)|B| < 2.$$

$$(3a) \quad n\theta \geq 1 \quad \text{and} \quad n(1-\theta) \geq 1$$

$$(3b) \quad \{f(x) \mid x \in V(G)\} \text{ consists of even numbers and } m(1-\theta) \geq 1, \text{ where } m \in \{n, n+1\} \text{ and } m \equiv 1 \pmod{2}.$$

Lemma 3. [2] Let k be an even integer and r be an odd integer such that $2 \leq k \leq 2r/3$. Then every 2-edge-connected r -regular graph has a k -factor.

Note that this lemma can be obtained from Lemma 2 with $A=B=\emptyset$, $m=3$, $\theta = k/r$ and $f(x)=k$ for all $x \in V(G)$.

Proof of Theorem 1. Let G be a connected r -regular graph. By Lemma 3, we may assume G has bridges. Then G has a block H which is a 2-edge-connected $\{r-1, r\}$ -subgraph having exactly one vertex of degree $r-1$. Put $\theta = k/r$, $A=B=\emptyset$ and $n=2$, and define a function f on $V(H)$ by $f(x)=k$ for all $x \in V(H)$. We can show that H, θ and f satisfy conditions (1), (2) and (3b) of Lemma 2 since

$$\varepsilon = |k - (k/r)(r-1)| < 2 \quad \text{and} \quad m(1-\theta) = 3(1-k/r) \geq 1.$$

Therefore H has an f -factor, which is a desired k -regular subgraph of G .

Lemma 4. [4] Let k be an odd integer and r be an even integer such that $1 \leq r/4 \leq k \leq 3r/4$. Then every 4-edge-connected r -regular graph of even order has a k -factor.

Note that the above lemma can be proved by Lemma 2 with $A=B=\emptyset$, $n=4$, $\theta=k/r$ and $f(x)=k$ for all $x \in V(G)$.

Lemma 5. Let k be an odd integer and r be an even integer such that $1 \leq r/4 \leq k \leq 3r/4$. Let H be a 3-edge-connected graph of even order. If H is an $\{r-1, r\}$ -graph having exactly two vertices of degree $r-1$, or an $\{r-2, r\}$ -graph having exactly one vertex of degree $r-2$, then H has a k -factor.

Proof Suppose that H is a 3-edge-connected $\{r-1, r\}$ -graph with exactly two vertices u and v of degree $r-1$. We construct an r -regular graph H^* from H by adding a new edge uv . Then H^* is a 4-edge-connected r -regular graph of even order. Put $A=\emptyset$, $B=\{uv\}$, $\theta=k/r$ and $n=4$, and define a function f by $f(x)=k$ for all $x \in V(H^*)$. Then (1), (2) and (3a) of Lemma 2 hold since $\varepsilon=2(1-\theta)|B|=2(1-k/r)<2$, $n\theta=4k/r \geq 1$ and $n(1-\theta)=4(1-k/r) \geq 1$. Hence H^* has a k -factor F which does not contain the edge uv . Therefore, F is a desired k -factor of H . If H is a $\{r-2, r\}$ -graph with one vertex of degree $r-2$, then H is 4-edge-connected, and so we can prove the theorem by Lemma 2.

Lemma 6. Let k be an odd integer greater than 1, and let H be a 3-edge-connected graph of odd order. If H is $\{2k-1, 2k\}$ -graph having exactly two vertices of degree $2k-1$, or a $\{2k-2, 2k\}$ -graph having exactly one vertex of degree $2k-2$, then H has a $\{k-1, k\}$ -factor F such that $d_F(w)=k-1$ and $d_F(x)=k$ for all $x \in V(G) \setminus w$ for any given vertex w of degree $2k-1$ or $2k-2$.

Proof Set $\theta=1/2$, $n=3$ and define a function f on $V(G)$ by $f(w)=k-1$ and $f(x)=k$ for all $x \in V(G) \setminus w$. Then (1), (2) and (3a) of Lemma 2 hold. Hence H has an f -factor, which is a $\{k-1, k\}$ -factor with the required property.

Lemma 7. Let k be an odd integer greater than or equal to three, and G be a connected $\{2, 2k\}$ -graph with at least one vertex of degree 2. Then G has a $\{0, 1, k\}$ -factor F such that

$$d_F(x) = \begin{cases} 0 \text{ or } 1 & \text{if } d_G(x)=2 \\ k & \text{if } d_G(x)=2k. \end{cases}$$

Proof Define two functions g and f on $V(G)$ as

$$g(x) = \begin{cases} 0 & \text{if } d_G(x)=2 \\ k & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } d_G(x)=2 \\ k & \text{otherwise.} \end{cases}$$

We shall prove that G has a (g, f) -factor, which is a desired factor.

Suppose that G has no (g, f) -factor. By Lemma 1, there exist disjoint subsets S and T of $V(G)$ such that

$$\delta(S, T) = \sum_{t \in T} (d_G(t) - g(t)) + \sum_{s \in S} f(s) - e_G(S, T) - h(S, T) < 0$$

Among all the pairs (S', T') of disjoint subsets of $V(G)$ such that $\delta(S', T')$ is minimum, choose a pair (S, T) so that $|S \cup T|$ is minimum. Since G has a vertex of degree 2, $\delta(\emptyset, \emptyset) = -h(\emptyset, \emptyset) = 0$. Hence $S \cup T \neq \emptyset$. We first show that T does not contain any vertex of degree 2. Suppose that T contains a vertex u of degree 2. Let $e_1 = uw_1$ and $e_2 = uw_2$ be the edges of G incident with u , where we allow $w_1 = w_2$. We can easily obtain

$$\delta(S, T \setminus u) = \delta(S, T) - 2 + e_G(S, u) + h(S, T) - h(S, T \setminus u)$$

We consider three cases. If $w_1, w_2 \in S$, then $e_G(S, u) = 2$ and $h(S, T) = h(S, T \setminus u)$, and so $\delta(S, T \setminus u) = \delta(S, T)$, which contradicts the choice of S and T . If $w_1 \in S$ and $w_2 \notin S$, then $e_G(S, u) = 1$ and $h(S, T) - h(S, T \setminus u) \leq 1$, and so $\delta(S, T \setminus u) \leq \delta(S, T)$, a contradiction. If $w_1, w_2 \notin S$, then $e_G(S, u) = 0$ and $h(S, T \setminus u) - h(S, T) \leq 2$, and so $\delta(S, T \setminus u) \leq \delta(S, T)$, a contradiction. Therefore, H has no vertex of degree 2.

Let $C_1, C_2, \dots, C_m$ be the components of $G - (S \cup T)$ which satisfy the

conditions on $h(S,T)$ in Lemma 1, in particular, $m=h(S,T)$. Since $d_G(x)=2k$ for all $x \in V(C_i)$ and

$$\sum_{x \in V(C_i)} d_G(x) = 2|E(C_i)| + e_G(SuT, V(C_i)),$$

we have $e_G(SuT, V(C_i)) \equiv 0 \pmod{2}$. Hence

$$e_G(SuT, V(C_i)) \geq 2 \quad \text{for all } i, 1 \leq i \leq m. \quad (1)$$

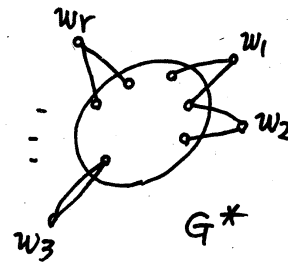
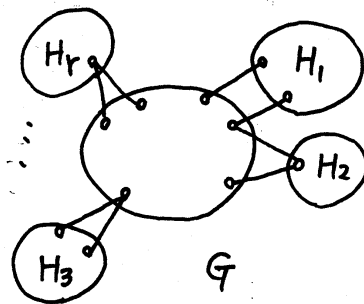
It follows that

$$\begin{aligned} \delta(S,T) &= \frac{1}{2} \sum_{t \in T} d_G(t) + \frac{1}{2} \sum_{s \in S} d_G(s) - e_G(S,T) - h(S,T) \\ &\geq \frac{1}{2} \{e_G(T,S) + \sum_{i=1}^m e_G(T, V(C_i))\} + \frac{1}{2} \{e_G(S,T) + \sum_{i=1}^m e_G(S, V(C_i))\} \\ &\quad - e_G(S,T) - m \\ &= \sum_{i=1}^m \left\{ \frac{1}{2} e_G(SuT, V(C_i)) - 1 \right\} \geq 0 \quad (\text{by (1)}), \end{aligned}$$

which is a required contradiction. Consequently, G has a (g,f) -factor.

Proof of Theorem 2.

Let k be an odd integer greater than or equal to three, and let G be a connected $2k$ -regular graph. If G is 4-edge-connected and of even order, then G has a k -factor by Lemma 4. Hence we may assume that the edge-connectivity of G is two. Note that the order of G may be odd. Choose a cut $\{e,f\}$ of G ($e,f \in E(G)$) so that a component of $G - \{e,f\}$ is minimal. If $\{c,d\}$ is a cut of H , then $\{c,d\}$ is not a cut of G and $\{e,c,d\}$ is a cut of G , which contradicts the fact that G is eulerian. Thus H is 3-edge-connected. If H is of even order, then H contains a k -factor by Lemma 5. Hence we may assume that H is of odd order. Let $\{e_1, f_1\}, \{e_2, f_2\}, \dots, \{e_r, f_r\}$ be the cuts of G such that a component of H_1 of $G - \{e_1, f_1\}$ is a 3-edge-connected. We obtain a $\{2, 2k\}$ -graph G from G by contracting each H_i to one vertex w_i (see Figure). By Lemma 6, G has a $\{0, 1, k\}$ -factor F such that $d_F(w_i) = 0$ or 1 , and $d_F(x) = k$ for all $x \in V(G) \setminus \{w_1, \dots, w_r\}$. Without loss of generality, we may assume that $d_F(w_i) = 1$



for each i , $1 \leq i \leq t$, and $d_F(w_j) = 0$ for each j , $t < j \leq r$. Let $u_1 w_1, \dots, u_t w_t$ be edges of F , and let $u_1 v_1, \dots, u_t v_t$ be edges of G such that $v_i \in V(H_i)$ for every i , $1 \leq i \leq t$. For each i , $1 \leq i \leq t$, we can take a $\{k-1, k\}$ -factor $I(i)$ of H_i such that $d_{I(i)}(v_i) = k-1$ and $d_{I(i)}(x) = k$ for all $x \in V(H_i) \setminus v_i$ by Lemma 6. Consequently, we obtain a k -regular subgraph $F \cup I(1) \cup \dots \cup I(t)$.

The theorem follows from the above result and Theorems A and B.

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or Math. Notes 36, 3 (1984) ?

A list of results related to this paper can be found in the following survey article.

J. Akiyama and M. Kano, Factors and factorizations of graphs — A survey, J. of Graph Theory 9 (1985) 1 - 42

On factors of regular graphs, the reader should refer to
M. Kano, Factors of regular graphs, J. Combinatorial Theory (B), to appear.