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Boundedness of γ-Cesàro means ($\gamma>0$) of operators

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In this talk I would like to report some recent results on the boundedness properties of γ-Cesàro means of operators, where $\gamma>0$. The results are taken from joint works with Jeng-Chung Chen, Yuan-Chuan Li, and Sen-Yen Shaw (cf. [1], [3]).

1. The discrete case. Let $T:X\to X$ be a bounded linear operator on a Banach space X. The Cesàro means of order γ (or γ-Cesàro means) of T, where $\gamma\geq 0$, are defined by

$$C_n^\gamma = C_n^\gamma(T) := \frac{1}{\sigma_n^\gamma}\sum_{k=0}^{n}\sigma_{n-k}^{\gamma-1}T^k, \qquad n=0,\,1,\,2,\ldots,$$

where $\sigma_n^\beta=\binom{\beta+n}{n}$ for $n\geq 1$, and $\sigma_0^\beta=1$ (see Zygmund [5, Chapter 3]). Among them are the following two particular means: $C_n^0=C_n^0(T)=T^n$, and $C_n^1=C_n^1(T)=(n+1)^{-1}\sum_{k=0}^{n}T^k$.

The Abel means of T are the operators $A_r=A_r(T):=(1-r)\sum_{k=0}^{\infty}r^kT^k$, defined for $0<r<1/r(T)$, where $r(T)=\lim_{n\to\infty}\|T^n\|^{1/n}$ denotes the spectral radius of T. Clearly, $r(T)\leq 1$ if and only if A_r exists for all $0<r<1$. (Moreover, in this cse, we have $A_r=(1-r)(I-rT)^{-1}$ for each $0<r<1$.) The following is well-known (cf. [5]):

If $0<\gamma<\beta<\infty$, then

$$\text{(1)} \qquad \sup_{0<r<1}\|A_r\|\leq\sup_{n\geq 0}\|C_n^\beta\|\leq\sup_{n\geq 0}\|C_n^\gamma\|\leq\sup_{n\geq 0}\|C_n^0\|\ (=\sup_{n\geq 0}\|T^n\|);$$

in particular, if T is a positive linear operator on a Banach lattice X, then

$$\text{(2)} \qquad \sup_{0<r<1}\|A_r\|<\infty \iff \sup_{n\geq 0}\|C_n^1\|<\infty \qquad \text{(cf. Emilion [2]).}$$

In connection with these relations, two questions come up naturally:

(A) *If T is positive, then does the implication* "$\sup_{0<r<1} \|A_r\| < \infty \Rightarrow \sup_{n\geq 1} \|C_n^\gamma\| < \infty$" *holds for a certain constant γ, with $0 < \gamma < 1$?*

(B) *If T is not assumed to be positive, then does the implication* "$\sup_{0<r<1} \|A_r\| < \infty \Rightarrow \sup_{n\geq 1} \|C_n^\gamma\| < \infty$" *hold for a certain constant γ, with $\gamma \geq 1$?*

Our answers are as follows.

Theorem 1. *For any γ, with $0 < \gamma < 1$, there exists a positive linear operator T on an L_1-space such that $\sup_{n\geq 1} \|C_n^\beta\| < \infty$ for all $\beta > \gamma$, but $\sup_{n\geq 1} \|C_n^\gamma\| = \infty$.*

Theorem 2. *There exists a positive linear operator T on an L_1-space such that $\sup_{n\geq 1} \|C_n^\beta\| < \infty$ for all $\beta > 0$, but $\sup_{n\geq 1} \|T^n\| = \infty$.*

Theorem 3. *Let $\dim X = \infty$. Then the following hold:*

(i) *For any integer $k \geq 1$, there exists a bounded linear operator T on X such that $\sup_{n\geq 1} \|C_n^k\| < \infty$, but $\sup_{n\geq 1} \|C_n^\beta\| = \infty$ for all β, with $0 \leq \beta < k$.*

(ii) *There exists a bounded linear operator T on X, with $r(T) = 1$, such that $\sup_{0<r<1} \|A_r\| < \infty$, but $\sup_{n\geq 1} \|C_n^\beta\| = \infty$ for all $\beta \geq 0$.*

2. The continuous case. Let $T(\cdot)$ be a C_0-semigroup of bounded linear operators on a Banach space X. The γ-th Cesàro means of $T(\cdot)$, where $\gamma \geq 0$, are defined as $C_0^\gamma = C_0^\gamma(T(\cdot)) := T(0)$ and, for $t > 0$,

$$C_t^\gamma = C_t^\gamma(T(\cdot)) := \begin{cases} T(t) & \text{if } \gamma = 0, \\ \gamma t^{-\gamma} \int_0^t (t-u)^{\gamma-1} T(u)\, du & \text{if } \gamma > 0. \end{cases}$$

The Abel means of $T(\cdot)$ are the operators

$$A_\lambda = A_\lambda(T(\cdot)) := \lambda \int_0^\infty e^{-\lambda u} T(u)\, du = \lim_{t\to} \lambda \int_0^t e^{-\lambda u} T(u)\, du,$$

defined for $\lambda > 0$ if the limit exists. As in the discrete case, we have (cf. [4]):

If $0 < \gamma < \beta < \infty$, then

$$\sup_{0<\lambda<\infty} \|A_\lambda\| \leq \sup_{t>0} \|C_t^\beta\| \leq \sup_{t>0} \|C_t^\gamma\| \leq \sup_{t>0} \|C_t^0\| \ (= \sup_{t>0} \|T(t)\|); \tag{3}$$

inparticular, if $T(\cdot)$ is a positive C_0-semigroup on a Banach lattice X, then

$$(4)\qquad \sup_{0<\lambda<\infty} \|A_\lambda\| < \infty \iff \sup_{t>0} \|C_t^1\| < \infty \qquad \text{(cf. [2])}.$$

The following are the continuous case results:

Theorem 1′. *For any γ, with $0 < \gamma < 1$, there exists a positive C_0-semigroup $T(\cdot)$ on an L_1-space such that* $\sup_{t>0} \|C_t^\beta\| < \infty$ *for all $\beta > \gamma$, but* $\sup_{t>0} \|C_t^\gamma\| = \infty$.

Theorem 2′. *There exists a positive C_0-semigroup $T(\cdot)$ on an L_1-space such that* $\sup_{t>0} \|C_t^\beta\| < \infty$ *for all $\beta > 0$, but* $\sup_{t>0} \|T(t)\| = \infty$.

Theorem 3′. *Let* $\dim X = \infty$. *Then the following hold:*

(i) *For any integer $k > 1$, there exists a C_0-semigroup $T(\cdot)$ on X such that* $\sup_{t>0} \|C_t^k\| < \infty$, *but* $\sup_{t>0} \|C_t^\beta\| = \infty$ *for all β, with $0 \le \beta < k$.*

(ii) *There exists a C_0-semigroup $T(\cdot)$ on X such that* $\sup_{0<\lambda<\infty} \|A_\lambda\| < \infty$, *but* $\sup_{t>0} \|C_t^\beta\| = \infty$ *for all $\beta \ge 0$.*

References

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