

ON THE ARGUMENT INEQUALITY OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $p(z)$ be analytic in $|z| < 1$, $p(0) = 1$, $p(z) \neq 0$ in $|z| < 1$ and $|\arg p'(z)| < \pi(\alpha - 1)/4$ in $|z| < 1$ where $1 < \alpha < 2$. Then we have

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{in } |z| < 1.$$

1. INTRODUCTION.

Let $\mathcal{N}$ be the class of all functions $p(z)$ which are analytic in the unit disc $\mathbb{E} = \{z : |z| < 1\}$ and equal to 1 at $z = 0$. We say $p(z) \in \mathcal{N}$ a Carathéodory function if it satisfies the condition $\operatorname{Re} p(z) > 0$ in $\mathbb{E}$.

If $F(z)$ and $G(z)$ are analytic in $\mathbb{E}$, then $F(z)$ is subordinate to $G(z)$, written by $F(z) \prec G(z)$, if $G(z)$ is univalent in $\mathbb{E}$, $F(0) = G(0)$ and $F(z) \subset G(z)$.

In [1, Theorem 5], Miller and Mocanu proved the following theorem.

Theorem A. *Let $p(z) \in \mathcal{N}$ and suppose that*

$$p(z) + zp'(z) \prec \left[\frac{1+z}{1-z} \right]^\alpha \implies p(z) \prec \left[\frac{1+z}{1-z} \right]^\beta$$

where $\alpha = \alpha(\beta) = \beta + (2/\pi)\tan^{-1}\beta$, $0 < \beta < \beta_0 = 1.21872 \dots$ and β_0 is the root of the equation

$$\beta\pi = \frac{3}{2}\pi - \tan^{-1}\beta.$$

On the other hand, Nunokawa [2] proved the following lemma.

Lemma 1. *Let $p(z) \in \mathcal{N}$, $p(z) \neq 0$ in $\mathbb{E}$ and suppose that there exists a point $z_0 \in \mathbb{E}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{in } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

where $0 < \alpha$. Then we have

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$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2}\alpha$$

where

$$p(z_0)^{1/\alpha} = \pm ia \quad \text{and} \quad a > 0.$$

Applying Lemma 1, we can easily obtain the following result.

Theorem B. Let $p(z) \in \mathcal{N}$, $p(z) \neq \mathbb{E}$ and suppose that

$$|\arg(p(z) + zp'(z))| < \frac{\pi}{2} \left(\beta + \frac{2}{\pi} \tan^{-1} \beta \right) \quad \text{in } \mathbb{E}$$

where $0 < \beta$. Then we have

$$|\arg p(z)| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{E}.$$

Remark 1. For the case $0 < \beta < \beta_0$, Theorem B is obtained from Theorem A but Theorem B holds to be true for all the case $0 < \beta$ if we consider the function $p(z)$ on the infinitely many sheeted Riemann surfaces which are cut along the negative half of real axis.

Applying Lemma 1, Nunokawa [2] obtained Theorem C.

Theorem C. Let $p(z) \in \mathcal{N}$, $p(z) \neq 0$ in $\mathbb{E}$ and suppose that

$$\left| \arg \left(p(z) + \frac{zp'(z)}{p(z)} \right) \right| < \frac{\pi}{2} \alpha(\beta) \quad \text{in } \mathbb{E}$$

where $0 < \beta \leq 1$,

$$\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2}(1-\beta)}{p(\beta) + \beta q(\beta) \cos \frac{\pi}{2}(1-\beta)},$$

$$p(\beta) = (1+\beta)^{(1+\beta)/2} \quad \text{and} \quad q(\beta) = (1-\beta)^{(\beta-1)/2}.$$

Then we have

$$|\arg p(z)| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{E}.$$

Remark 2. Theorem C holds to be true for all the case $0 < \beta$ if we also consider it like as Remark 1.

ON THE ARGUMENT INEQUALITY OF ANALYTIC FUNCTIONS

In the distortion theorem of analytic function theory, if we suppose some assumptions for $|f'(z)|$, then we can easily get some results for $|f(z)|$ by applying integral inequality

$$|f(z) - f(0)| \leq \int_0^z |f'(t)| |dt|.$$

On the other hand, we can not find out any results for the rotation theorem of analytic functions between $|\arg p'(z)|$ and $|\arg p(z)|$.

2. MAIN RESULT.

Theorem. Let $p(z) \in \mathcal{N}$, $p(z) \neq 0$ in $\mathbb{E}$ and suppose that

$$|\arg p'(z)| < \frac{\pi}{4}(\alpha - 1) \quad \text{in } \mathbb{E},$$

where $1 < \alpha < 2$. Then we have

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{in } \mathbb{E}.$$

Proof. Let us suppose that if there exists a point $z_0 \in \mathbb{E}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha.$$

Applying Lemma 1, let us suppose $\arg p(z_0) = \pi\alpha/2$, then we have

$$\begin{aligned} p'(z_0) &= \frac{p(z_0)}{z_0} i\alpha k \\ &= \left(\frac{p(z_0) - 1}{z_0} \right) \left(\frac{i\alpha k p(z_0)}{p(z_0) - 1} \right) \\ &= \left(\frac{1}{z_0} \int_0^{z_0} p'(t) dt \right) \left(\frac{i\alpha k p(z_0)}{p(z_0) - 1} \right) \\ &= \left(\frac{1}{r} \int_0^r p'(\rho e^{i\theta}) d\rho \right) \left(\frac{i\alpha k p(z_0)}{p(z_0) - 1} \right) \end{aligned}$$

where $z_0 = re^{i\theta}$, $t = \rho e^{i\theta}$ and $0 \leq \rho \leq r$. Therefore we have

$$\arg p'(z_0) = \frac{\pi}{2} + \frac{\pi}{2}\alpha + \arg \left(\frac{1}{r} \int_0^r p'(\rho e^{i\theta}) d\rho \right) + \arg \left(\frac{\overline{p(z_0)} - 1}{|p(z_0) - 1|^2} \right).$$

Applying the property of integral mean of the integral (See Pommerenke [3, Lemma 1]), we have

$$\begin{aligned} \arg p'(z_0) &\geq \frac{\pi}{2} + \frac{\pi}{2}\alpha - \frac{\pi}{4}(\alpha - 1) - \pi \\ &= \frac{\pi}{4}(\alpha - 1). \end{aligned}$$

This contradicts the assumption.

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For the case $\arg p(z_0) = -\pi\alpha/2$, we also have the following

$$\arg p'(z_0) = \arg \frac{p(z_0)}{z_0}(-i\alpha k) = \arg \left(\frac{1}{z_0} \int_0^{z_0} p'(t) dt \right) + \arg \left(\frac{-i\alpha k p(z_0)}{p(z_0) - 1} \right)$$

where $1 \leq k$. Therefore, we have

$$\begin{aligned} \arg p'(z_0) &= -\frac{\pi}{2} - \frac{\pi}{2}\alpha + \arg \left(\frac{1}{r} \int_0^r p'(\rho e^{i\theta}) d\rho \right) + \arg \left(\frac{\overline{p(z_0)} - 1}{|p(z_0) - 1|^2} \right) \\ &\geq -\frac{\pi}{2} - \frac{\pi}{2}\alpha + \frac{\pi}{4}(\alpha - 1) + \pi \\ &= -\frac{\pi}{4}(\alpha - 1). \end{aligned}$$

This contradicts the assumption and so this completes the proof. $\square$

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