

A CLASS OF FUNCTIONS DEFINED BY USING
HADAMARD PRODUCT. II

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ABSTRACT. The object of the present paper is to obtain closure theorems, integral operators and several interesting results for the modified Hadamard products of functions belonging to the class $P_{\alpha}[\beta, \gamma]$ consisting of analytic functions with negative coefficients and defined by using Hadamard product $f * S_{\alpha}(z)$ of $f(z)$ and $S_{\alpha}(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$. Also we obtain distortion theorem for certain fractional integral operator of functions in the class $P_{\alpha}[\beta, \gamma]$.

KEY WORDS- Analytic, starlike, modified Hadamard product.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the unit disc $U = \{z: |z| < 1\}$. And let S denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disc U . A function $f(z)$ in S is said to be starlike of order α if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U) \quad (1.2)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α . Further, a function $f(z)$ in S is said to be convex of order α if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U) \quad (1.3)$$

for some $\alpha (0 \leq \alpha < 1)$. And We donote by $K(\alpha)$ the class of all convex functions of order α . It is well-known that

$$f(z) \in K(\alpha) \text{ if and only if } zf'(z) \in S^*(\alpha). \quad (1.4)$$

These classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [8], and later were studied by Schild [9], MacGregor [3] and Pinchuk [7].

Now, the function

$$S_{\alpha}(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (1.5)$$

is the well-known extremal function for the class $S^*(\alpha)$ (see [8,1]). Setting

$$C(\alpha, n) = \frac{\prod_{k=2}^n (k-2\alpha)}{(n-1)!} \quad (n \geq 2). \quad (1.6)$$

$S_{\alpha}(z)$ can be written in the form

$$S_{\alpha}(z) = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n. \quad (1.7)$$

Then we note that $C(\alpha, n)$ is decreasing in α and satisfies

$$\lim_{n \rightarrow \infty} C(\alpha, n) = \begin{cases} \infty & (\alpha < \frac{1}{2}) \\ 1 & (\alpha = \frac{1}{2}) \\ 0 & (\alpha > \frac{1}{2}). \end{cases} \quad (1.8)$$

Let $f * g(z)$ be the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (1.9)$$

then

$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (1.10)$$

We say that the function $f(z)$ defined by (1.1) belongs to the class $P_{\alpha}(\beta, \gamma)$ if $f(z)$ satisfies the following condition

$$\left| \frac{(f * S_{\alpha}(z))' - 1}{(f * S_{\alpha}(z))' + (1-2\beta)} \right| < \gamma \quad (z \in U) \quad (1.11)$$

for $\beta(0 \leq \beta < 1)$ and $\gamma(0 < \gamma \leq 1)$.

Let T denote the subclass of A consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \quad (1.12)$$

And we denote by $P_{\alpha}[\beta, \gamma]$ the class obtained by taking intersection of $P_{\alpha}(\beta, \gamma)$ with T , that is,

$$P_{\alpha}[\beta, \gamma] = P_{\alpha}(\beta, \gamma) \cap T. \quad (1.13)$$

The class $P_{\alpha}[\beta, \gamma]$ was studied by Owa and Ahuja [6]. The class $P_{\alpha}[\beta, \gamma]$ is the generalization of the class $P^*(\beta, \gamma)$ which was introduced by Gupta and Jain [2]. In particular, $P_{1/2}[\beta, \gamma] = P^*(\beta, \gamma)$ when $\alpha = \frac{1}{2}$. Further we note that many classes defined by using the Hadamard product $f * S_{\alpha}(z)$ of $f(z)$ and $S_{\alpha}(z)$ were introduced by Sheil-Small, Silverman and

Silvia [11], Silverman and Silvia ([12], [13]), and Ahuja and Silverman [1].

In order to prove our results for functions belonging to the class $P_\alpha[\beta, \gamma]$, we shall require the following lemma given by Owa and Ahuja [6].

LEMMA 1. Let the function $f(z)$ be defined by (1.12). Then $f(z)$ is in the class $P_\alpha[\beta, \gamma]$ if and only if

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n) a_n \leq 2\gamma(1 - \beta). \quad (1.14)$$

The result is sharp.

2. Closure Theorems

Let the functions $f_i(z)$ be defined, for $i = 1, 2, \dots, m$, by

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \quad (a_{n,i} \geq 0) \quad (2.1)$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $P_\alpha[\beta, \gamma]$.

THEOREM 1. Let the functions $f_i(z)$ ($i = 1, 2, \dots, m$) defined by (2.1) be in the class $P_\alpha[\beta, \gamma]$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (2.2)$$

also belongs to the class $P_\alpha[\beta, \gamma]$, where

$$b_n = \frac{1}{m} \sum_{i=1}^m a_{n,i}. \quad (2.3)$$

PROOF. Since $f_i(z) \in P_\alpha[\beta, \gamma]$, it follows from Lemma 1 that

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)a_{n,i} \leq 2\gamma(1-\beta) \quad (i=1, 2, \dots, m). \quad (2.4)$$

Therefore

$$\begin{aligned} \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)b_n &= \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n) \left[\frac{1}{m} \sum_{i=1}^m a_{n,i} \right] \\ &= \frac{1}{m} \sum_{i=1}^m \left\{ \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)a_{n,i} \right\} \leq 2\gamma(1-\beta). \end{aligned} \quad (2.5)$$

Hence, by Lemma 1, $h(z) \in P_\alpha[\beta, \gamma]$. Thus we have the theorem.

THEOREM 2. Let the functions $f_i(z)$ defined by (2.1) be in the class $P_\alpha[\beta, \gamma]$ for each $i = 1, 2, \dots, m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^m d_i f_i(z) \quad (d_i \geq 0) \quad (2.6)$$

is also in the same class $P_\alpha[\beta, \gamma]$, where

$$\sum_{i=1}^m d_i = 1. \quad (2.7)$$

PROOF. According to the definition of $h(z)$, we can write that

$$h(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^m d_i a_{n,i} \right) z^n. \quad (2.8)$$

By means of Lemma 1, we have

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)a_{n,i} \leq 2\gamma(1-\beta) \quad (2.9)$$

for every $i = 1, 2, \dots, m$. Hence we can observe that

$$\begin{aligned} & \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n) \left(\sum_{i=1}^m d_i a_{n,i} \right) \\ &= \sum_{i=1}^m d_i \left\{ \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)a_{n,i} \right\} \end{aligned}$$

$$\leq \left[\sum_{i=1}^m d_i \right] 2\gamma(1-\beta) = 2\gamma(1-\beta) \quad (2.10)$$

which implies that $h(z) \in P_{\alpha}[\beta, \gamma]$. Thus we have the theorem.

THEOREM 3. The class $P_{\alpha}[\beta, \gamma]$ is closed under convex linear combination.

PROOF. Let the functions $f_i(z)$ ($i = 1, 2$) defined by (2.1) be in the class $P_{\alpha}[\beta, \gamma]$. Then it is sufficient to prove that the function

$$h(z) = \lambda f_1(z) + (1-\lambda)f_2(z) \quad (0 \leq \lambda \leq 1) \quad (2.11)$$

is in the class $P_{\alpha}[\beta, \gamma]$. Since, for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{n=2}^{\infty} \left\{ \lambda a_{n,1} + (1-\lambda)a_{n,2} \right\} z^n, \quad (2.12)$$

with the aid of Lemma 1, we have

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n) \left\{ \lambda a_{n,1} + (1-\lambda)a_{n,2} \right\} \leq 2\gamma(1-\beta) \quad (2.13)$$

which implies $h(z) \in P_{\alpha}[\beta, \gamma]$.

3. Integral Operators

THEOREM 4. Let the function $f(z)$ defined by (1.12) be in the class $P_\alpha[\beta, \gamma]$, and let d be a real number such that $d > -1$. Then the function $F(z)$ defined by

$$F(z) = \frac{d+1}{z^d} \int_0^z t^{d-1} f(t) dt \quad (3.1)$$

also belongs to the class $P_\alpha[\beta, \gamma]$.

PROOF. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad (3.2)$$

where

$$b_n = \left(\frac{d+1}{d+n} \right) a_n. \quad (3.3)$$

Therefore,

$$\begin{aligned} \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)b_n &= \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n) \left(\frac{d+1}{d+n} \right) a_n \\ &\leq \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)a_n \leq 2\gamma(1-\beta), \end{aligned} \quad (3.4)$$

since $f(z) \in P_\alpha[\beta, \gamma]$. Hence, by Lemma 1, $F(z) \in P_\alpha[\beta, \gamma]$.

THEOREM 5. Let the function $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$ ($a_n \geq 0$) be in the class $P_{\alpha}[\beta, \gamma]$, and let d be a real number such that $d > -1$. Then the function $f(z)$ defined by (3.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_n \left[\frac{[(1+\gamma)C(\alpha, n)(d+1)]^{\frac{1}{n-1}}}{2\gamma(1-\beta)(d+n)} \right] \quad (n \geq 2). \quad (3.5)$$

The result is sharp.

PROOF. From (3.1), we have

$$\begin{aligned} f(z) &= \frac{z^{1-d} (z^d F'(z))'}{(d+1)} \quad (d > -1) \\ &= z - \sum_{n=2}^{\infty} \left(\frac{d+n}{d+1} \right) a_n z^n. \end{aligned} \quad (3.6)$$

In order to obtain the required result it suffices to show that $|f'(z) - 1| < 1$ in $|z| < R^*$. Now

$$|f'(z) - 1| = \left| -\sum_{n=2}^{\infty} n \left(\frac{d+n}{d+1} \right) a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n \left(\frac{d+n}{d+1} \right) a_n |z|^{n-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{n=2}^{\infty} n \left[\frac{d+n}{d+1} \right] a_n |z|^{n-1} < 1. \quad (3.7)$$

But Lemma 1 confirms that

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} a_n \leq 1. \quad (3.8)$$

Hence (3.7) will be satisfied if

$$\frac{n(d+n)}{(d+1)} |z|^{n-1} \leq \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} \quad (n \geq 2)$$

or if

$$|z| \leq \left[\frac{(1+\gamma)C(\alpha, n)(d+1)}{2\gamma(1-\beta)(d+n)} \right]^{\frac{1}{n-1}} \quad (n \geq 2). \quad (3.9)$$

The required result follows now from (3.9). The result is sharp for the function

$$f(z) = z - \frac{2\gamma(1-\beta)(d+n)}{n(1+\gamma)C(\alpha, n)(d+1)} z^n \quad (n \geq 2). \quad (3.10)$$

4. Modified Hadamard Products

Let the functions $f_i(z)$ ($i = 1, 2$) be defined by (2.1).

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \quad (4.1)$$

THEOREM 6. Let the functions $f_i(z)$ ($i = 1, 2$) defined by (2.1) be in the class $P_{\alpha}[\beta, \gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$. Then $f_1 * f_2(z) \in P_{\alpha}[\delta(\alpha, \beta, \gamma), \gamma]$, where

$$\delta(\alpha, \beta, \gamma) = 1 - \frac{\gamma(1-\beta)^2}{2(1+\gamma)(1-\alpha)}. \quad (4.2)$$

The result is sharp.

PROOF. Employing the technique used earlier by Schild and Silverman [10], we need to find the largest $\delta = \delta(\alpha, \beta, \gamma)$ such that

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\delta)} a_{n,1} a_{n,2} \leq 1. \quad (4.3)$$

Since

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} a_{n,1} \leq 1 \quad (4.4)$$

and

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} a_{n,2} \leq 1, \quad (4.5)$$

By the Cauchy-Schwarz inequality we have

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} \sqrt{a_{n,1} a_{n,2}} \leq 1. \quad (4.6)$$

Thus it is sufficient to show that

$$\frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\delta)} a_{n,1} a_{n,2} \leq \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} \sqrt{a_{n,1} a_{n,2}} \quad (n \geq 2), \quad (4.7)$$

that is, that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(1-\delta)}{(1-\beta)}. \quad (4.8)$$

Note that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)} \quad (n \geq 2). \quad (4.9)$$

Consequently, we need only to prove that

$$\frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)} \leq \frac{(1-\delta)}{(1-\beta)} \quad (n \geq 2). \quad (4.10)$$

or, equivalently, that

$$\delta \leq 1 - \frac{2\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha, n)} \quad (n \geq 2). \quad (4.11)$$

Since

$$A(n) = 1 - \frac{2\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha, n)} \quad (4.12)$$

is an increasing function of n ($n \geq 2$), for $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, letting $n = 2$ in (4.12), we obtain

$$\delta \leq A(2) = 1 - \frac{\gamma(1-\beta)^2}{2(1+\gamma)(1-\alpha)}, \quad (4.13)$$

which completes the proof of Theorem 6.

Finally, by taking the functions given by

$$f_i(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \quad (i = 1, 2) \quad (4.14)$$

we can see that the result is sharp.

THEOREM 7. Let the function $f_1(z)$ defined by (2.1) be in the class $P_\alpha[\beta, \gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, and the function $f_2(z)$ defined by (2.1) be in the class $P_\alpha[\tau, \gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \tau < 1$, and $0 < \gamma \leq 1$, then $f_1 * f_2$

$f_2(z) \in P_\alpha[\zeta(\alpha, \beta, \tau, \gamma), \gamma]$, where

$$\zeta(\alpha, \beta, \tau, \gamma) = 1 - \frac{\gamma(1-\beta)(1-\tau)}{2(1+\gamma)(1-\alpha)}. \quad (4.15)$$

The result is sharp.

PROOF. Proceeding as in the proof of Theorem 6, we get

$$\zeta \leq B(n) = 1 - \frac{2\gamma(1-\beta)(1-\tau)}{n(1+\gamma)C(\alpha, n)} \quad (n \geq 2). \quad (4.16)$$

Since the function $B(n)$ is an increasing function of n ($n \geq 2$), for $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, $0 \leq \tau < 1$, and $0 < \gamma \leq 1$, letting $n = 2$ in (4.16) we obtain

$$\zeta \leq B(2) = 1 - \frac{\gamma(1-\beta)(1-\tau)}{2(1+\gamma)(1-\alpha)}, \quad (4.17)$$

which evidently proves Theorem 7.

Finally the result is best possible for the functions

$$f_1(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \quad (4.18)$$

and

$$f_2(z) = z - \frac{\gamma(1-\tau)}{2(1+\gamma)(1-\alpha)} z^2. \quad (4.19)$$

COROLLARY 1. Let the functions $f_i(z)$ ($i = 1, 2, 3$) defined by (2.1) be in the class $P_\alpha[\beta, \gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, then $f_1 * f_2 * f_3(z) \in P_\alpha[\eta(\alpha, \beta, \gamma), \gamma]$, where

$$\eta(\alpha, \beta, \gamma) = 1 - \frac{\gamma^2(1-\beta)^3}{4(1+\gamma)^2(1-\alpha)^2}. \quad (4.20)$$

The result is best possible for the functions

$$f_i(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \quad (i = 1, 2, 3). \quad (4.21)$$

PROOF. From Theorem 6, we have $f_1 * f_2(z) \in P_\alpha[\delta(\alpha, \beta, \gamma), \gamma]$, where δ is given by (4.2). We use now Theorem 7, we get $f_1 * f_2 * f_3(z) \in P_\alpha[\eta(\alpha, \beta, \gamma), \gamma]$, where

$$\eta(\alpha, \beta, \gamma) = 1 - \frac{\gamma(1-\beta)(1-\delta)}{2(1+\gamma)(1-\alpha)} = 1 - \frac{\gamma^2(1-\beta)^3}{4(1+\gamma)^2(1-\alpha)^2}.$$

This completes the proof of Corollary 1.

THEOREM 8. Let the functions $f_i(z)$ ($i = 1, 2$) defined by (2.1) be in the class $P_\alpha[\beta, \gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$. Then the function

$$h(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \quad (4.21)$$

belongs to the class $P_\alpha[\varphi(\alpha, \beta, \gamma), \gamma]$, where

$$\varphi(\alpha, \beta, \gamma) = 1 - \frac{\gamma(1-\beta)^2}{(1+\gamma)(1-\alpha)}. \quad (4.22)$$

The result is sharp for the functions $f_i(z)$ ($i = 1, 2$) defined by (4.14).

PROOF. By virtue of Lemma 1, we obtain

$$\sum_{n=2}^{\infty} \left[\frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} \right]^2 a_{n,1}^2 \leq \left[\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} a_{n,1} \right]^2 \leq 1 \quad (4.23)$$

and

$$\sum_{n=2}^{\infty} \left[\frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} \right]^2 a_{n,2}^2 \leq \left[\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} a_{n,2} \right]^2 \leq 1. \quad (4.24)$$

It follows from (4.23) and (4.24) that

$$\sum_{n=2}^{\infty} \frac{1}{2} \left[\frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} \right]^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \quad (4.25)$$

Therefore, we need to find the largest $\varphi = \varphi(\alpha, \beta, \gamma)$ such that

$$\frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\varphi)} \leq \frac{1}{2} \left[\frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} \right]^2 \quad (n \geq 2) \quad (4.26)$$

that is,

$$\varphi \leq 1 - \frac{4\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha, n)} \quad (n \geq 2). \quad (4.27)$$

Since

$$D(n) = 1 - \frac{4\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha, n)} \quad (4.28)$$

is an increasing function of n ($n \geq 2$), for $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, we readily have

$$\varphi \leq D(2) = 1 - \frac{\gamma(1-\beta)^2}{(1+\gamma)(1-\alpha)}, \quad (4.29)$$

and Theorem 8 follows at once.

5. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [14].

DEFINITION 1. For real numbers $\eta > 0$, ρ and δ , the fractional integral operator $I_{0,z}^{\eta,\rho,\delta}$ is defined by

$$I_{0,z}^{\eta,\rho,\delta} f(z) = \frac{z^{-\eta-\rho}}{\Gamma(\eta)} \int_0^z (z-t)^{\eta-1} F(\eta+\rho, -\delta; \eta; 1-\frac{t}{z}) f(t) dt \quad (5.1)$$

where $f(z)$ is an analytic function in a simply-connected

region of the z -plane containing the origin with the order

$$f(z) = O(|z|^\epsilon), \quad z \rightarrow 0,$$

where

$$\epsilon > \max(0, \rho - \delta) - 1,$$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad (5.2)$$

where $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu+1)\dots(\nu+n-1) & (n \in \mathbb{N} = \{1, 2, \dots\}), \end{cases} \quad (5.3)$$

and the multiplicity of $(z-t)^{\eta-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

REMARK. For $\rho = -\eta$, we note that

$$I_{0,z}^{\eta, -\eta, \delta} f(z) = D_z^{-\eta} f(z),$$

where $D_z^{-\eta} f(z)$ is the fractional integral of order η which was introduced by Owa ([4], [5]).

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [14].

LEMMA 2. If $\eta > 0$ and $n > \rho - \delta - 1$, then

$$I_{0,z}^{\eta,\rho,\delta} z^n = \frac{\Gamma(n+1) \Gamma(n-\rho+\delta+1)}{\Gamma(n-\rho+1) \Gamma(n+\eta+\delta+1)} z^{n-\rho}. \quad (5.4)$$

With the aid of the Lemma 2, we prove

THEOREM 9. Let $\eta > 0$, $\rho < 2$, $\eta+\delta > -2$, $\rho-\delta < 2$, $\rho(\eta+\delta) \leq 3\eta$. If the function $f(z)$ defined by (1.12) is in the class $P_\alpha[\beta, \gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$. Then

$$\begin{aligned} |I_{0,z}^{\eta,\rho,\delta} f(z)| &\geq \frac{\Gamma(2-\rho+\delta) |z|^{1-\rho}}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)} \left\{ 1 \right. \\ &\quad \left. - \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z| \right\} \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} |I_{0,z}^{\eta,\rho,\delta} f(z)| &\leq \frac{\Gamma(2-\rho+\delta) |z|^{1-\rho}}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)} \left\{ 1 \right. \\ &\quad \left. + \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z| \right\} \end{aligned} \quad (5.6)$$

for $z \in U_0$, where

$$U_0 = \begin{cases} U & (\rho \leq 1) \\ U - \{0\} & (\rho > 1). \end{cases}$$

The equalities in (5.5) and (5.6) are attained for the

function $f(z)$ given by

$$f(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2. \quad (5.7)$$

PROOF. By using Lemma 2, we have

$$\begin{aligned} I_{0,z}^{\eta,\rho,\delta} f(z) &= \frac{\Gamma(2-\rho+\delta)}{\Gamma(2-\rho) \Gamma(2+\eta+\delta)} z^{1-\rho} \\ &\quad - \sum_{n=2}^{\infty} \frac{\Gamma(n+1) \Gamma(n-\rho+\delta+1)}{\Gamma(n-\rho+1) \Gamma(n+\eta+\delta+1)} a_n z^{n-\rho}. \end{aligned} \quad (5.8)$$

Letting

$$\begin{aligned} H(z) &= \frac{\Gamma(2-\rho) \Gamma(2+\eta+\delta)}{\Gamma(2-\rho+\delta)} z^{\rho} I_{0,z}^{\eta,\rho,\delta} f(z) \\ &= z - \sum_{n=2}^{\infty} h(n) a_n z^n, \end{aligned} \quad (5.9)$$

where

$$h(n) = \frac{(2-\rho+\delta)_{n-1} (1)_n}{(2-\rho)_{n-1} (2+\eta+\delta)_{n-1}} \quad (n \geq 2), \quad (5.10)$$

we can see that $h(n)$ is non-increasing for integers $n \geq 2$,

and we have

$$0 < h(n) \leq h(2) = \frac{2(2-\rho+\delta)}{(2-\rho)(2+\eta+\delta)}. \quad (5.11)$$

We note that $C(\alpha, n+1) \geq C(\alpha, n)$ for $0 \leq \alpha \leq \frac{1}{2}$ and $n \geq 2$.

Hence by using Lemma 1, we obtain that

$$2(1+\gamma)C(\alpha, 2) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)a_n \leq 2\gamma(1-\alpha) \quad (5.12)$$

which implies that

$$\sum_{n=2}^{\infty} a_n \leq \frac{\gamma(1-\alpha)}{2(1+\gamma)(1-\alpha)}. \quad (5.13)$$

Therefore, by using (5.11) and (5.13), we have

$$\begin{aligned} |H(z)| &\geq |z| - h(2) |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{\gamma(1-\alpha)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z|^2 \end{aligned} \quad (5.14)$$

and

$$|H(z)| \leq |z| + h(2) |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\leq |z| + \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z|^2. \quad (5.15)$$

This completes the proof of Theorem 9.

REMARK 1. Taking $\rho = -\eta = -\lambda$ in Theorem 9, we get the result obtained by Owa and Ahuja [6, Theorem 10].

REMARK 2. Taking $\alpha = 1/2$ in the above results we obtain corresponding results for the class $P^*[\alpha, \gamma]$ defined by Gupta and Jain [2].

References

- [1] O.P. Ahuja and H. Silverman, Convolutions of prestarlike functions, Internat. J. Math. Math. Sci. 6 (1983), 59-68.
- [2] V.P. Gupta and P.K. Jain, Certain classes of univalent functions with negative coefficients. II, Bull. Austral. Math. Soc. 15 (1976), 467-473.
- [3] T.H. MacGregor, The radius of convexity for starlike functions of order $\frac{1}{2}$, Proc. Amer. Math. Soc. 14 (1963), 71-76.
- [4] S. Owa, On the distortion theorems. I, Kyungpook Math. J. 18 (1978), 53-59.
- [5] S. Owa, Some applications of the fractional calculus,

Research Notes in Math. 138, Pitman, Boston, London and Melbourne, 1985, 164-175.

- [6] S. Owa and O.P. Ahuja, A class of functions defined by using Hadamard product, Hokkaido Math. J. 15 (1986), 217-232.
- [7] B. Pinchuk, On starlike and convex functions of order α , Duke Math. J. 35 (1968), 721-734.
- [8] M.S. Robertson, On the theory of univalent functions, Ann. of Math. 37 (1936), 374-408.
- [9] A. Schild, On starlike functions of order α , Amer. J. Math. 87 (1965), 65-70.
- [10] A. Schild and H. Silverman, Convolutions of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska Sect. A, 29 (1975), 99-107.
- [11] T. Sheil-Small, H. Silverman and E.M. Silvia, Convolution multipliers and starlike functions, J. Analyse Math. 41 (1982), 181-192.
- [12] H. Silverman and E. M. Silvia, Prestarlike functions with negative coefficients, Internat. J. Math. Math. Sci. 2 (1979), 427-439.
- [13] H. Silverman and E. M. Silvia, Subclasses of

prestarlike functions, Math. Japon. 29 (1984), 929-935.

- [14]. H. M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, J. Math. Anal. Appl. 131 (1988), 412-420.

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