Constructions of nearly holomorphic Siegel modular forms of degree two

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Abstract. Pitale, Saha and Schmidt study the representation theoretic aspects of nearly holomorphic modular forms. By their theory, we obtain a classification of $(\mathfrak{g}, K)$-modules which occur in the space of nearly holomorphic modular forms. In this paper, we give two constructions of nearly holomorphic Siegel modular forms of degree 2 which generate reducible indecomposable modules. One construction is given by the Rankin-Cohen bracket of Shimura's Eisenstein series and the other by Klingen Eisenstein series.

1. Introduction

The construction of automorphic forms is a fundamental problem in the theory of automorphic forms. As well-known constructions, we have theta functions, Eisenstein series and Rankin-Cohen brackets e.t.c.. In this paper, we construct nearly holomorphic modular forms by Eisenstein series and Rankin-Cohen brackets.

In [21, section 7], Shimura propose a problem about the analytic behavior of Eisenstein series at critical points. In order to answer the problem, Shimura introduce the notion of nearly holomorphic modular forms. For example, the weight two Eisenstein series E_2 with Fourier expansion

$$E_2(z) = \frac{3}{\pi \operatorname{Im}(z)} - 1 + 24 \sum_{n=1}^{\infty} \left(\sum_{d|n} d \right) \exp(2\pi\sqrt{-1}\, nz), \qquad z \in \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$$

is a nearly holomorphic modular form. Similarly, Shimura constructs a nearly holomorphic degree n Siegel Eisenstein series E^* of weight $(n+3)/2$ in [25, section 16]. The proof of the near holomorphy of E^* is based on the detailed investigation of its Fourier coefficients. In [5, Theorem 5.5], we determine the representations generated by Shimura's Eisenstein series E^* (see Theorem 1.1).

Recently, Pitale, Saha and Schmidt have found a representation theoretic approach to the study of nearly holomorphic Siegel modular forms. They classified the representations generated by nearly holomorphic Siegel modular forms of degree 1 and 2 in [14, Proposition 4.1] and [15, Proposition 4.27]. If a nearly holomorphic modular form f generates an irreducible representation, there exists a holomorphic modular form g and differential operators D_i such that $f = \sum_i D_i g$. If not, we can not obtain f by the same method. Hence, it suffices to determine the multiplicity of the classified modules and to construct nearly holomorphic modular forms which generate indecomposable reducible modules. The goal of this paper is to construct nearly holomorphic Siegel modular forms of degree 2 which generate certain indecomposable reducible modules by Rankin-Cohen brackets and Eisenstein series.

Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{C})$. Then, we have a well-known decomposition:

$$\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-.$$

Here, the Lie subalgebra $\mathfrak{k}$ is the complexification of the Lie algebra of a maximal compact subgroup K_∞ of $\mathrm{Sp}_{2n}(\mathbb{R})$ and the Lie subalgebra $\mathfrak{p}_+$ (resp. the Lie subalgebra $\mathfrak{p}_-$) is the Lie subalgebra of $\mathfrak{g}$ corresponding to the holomorphic tangent space (resp. anti-holomorphic tangent space) of a base point of $\mathfrak{H}_n$.

For a weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for any $1 \leq i \leq n-1$, let $(\rho_\lambda, V_\lambda)$ be the irreducible representation of $\mathfrak{k} \cong \mathrm{Lie}(\mathrm{GL}_n(\mathbb{C}))$ with highest weight λ. We can regard it as an irreducible representation of $\mathfrak{p}_- + \mathfrak{k}$ by letting $\mathfrak{p}_-$ act as trivial. For any Lie algebra $\mathfrak{a}$, we denote by $\mathcal{U}(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$. We define the generalized Verma module $N(\lambda)$ with highest weight λ by

$$N(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_- + \mathfrak{k})} V_\lambda.$$

It is well-known that $N(\lambda)$ has a unique irreducible quotient $L(\lambda)$. We denote by $N(\lambda)^\vee$ the contragredient representation of $N(\lambda)^\vee$ in the sense of [6, section 3.2]. Note that as a $\mathcal{U}(\mathfrak{k})$-module, the $\mathcal{U}(\mathfrak{k})$-module $N(\lambda)^\vee|_{\mathcal{U}(\mathfrak{k})}$ is isomorphic to $N(\lambda)|_{\mathcal{U}(\mathfrak{k})}$. By [15, Proposition 4.27], when $n = 2$, there are no indecomposable

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reducible module other than the module $N(m+1,1)^\vee$ that occur in the space of nearly holomorphic modular forms with integral weight on Sp_4. For general n, we can prove the following:

Theorem 1.1 (Theorem 5.5 [5])**.** *Let E^* be the Shimura's Siegel Eisenstein series of degree n. Then E^* generates $N((n-1)/2,\dots,(n-1)/2)^\vee$.*

When $n=2$, the modular form E^* of weight $5/2$ generates $N(1/2,1/2)^\vee$. This is a key fact to construct the Rankin-Cohen bracket.

Rankin-Cohen bracket is a useful tool to construct modular forms. For example, Aoki and Ibukiyama construct the degree 2 Siegel cusp form χ_{35} of weight 35 by some kind of a Rankin-Cohen bracket in [1, pp. 253]. In [7] and [27], there are many known formulas of Rankin-Cohen brackets for holomorphic modular forms. In this paper, we treat some of them for nearly holomorphic modular forms. Since we can understand the Rankin-Cohen bracket $[\,,\,]$ as a branching rule of tensor products of $\mathcal{U}(\mathfrak{g})$-modules, it may be easy to determine $\mathcal{U}(\mathfrak{g})$-modules generated by $[f,g]$ for nearly holomorphic modular forms f and g. Hence, we expect to construct nearly holomorphic modular forms which generates the indecomposable reducible module by the Rankin-Cohen bracket.

In order to provide the construction of the Rankin-Cohen bracket, we consider the tensor product

$$L(1/2,1/2)\otimes N(1/2,1/2)^\vee.$$

Here $L(1/2,1/2)$ is the unique irreducible quotient of the generalized Verma module $N(1/2,1/2)$. By the branching rules of the tensor product, we get a bracket

$$[\,,\,]_m\colon L(1/2,1/2)\otimes N(1/2,1/2)^\vee \longmapsto N(1+m,1)^\vee$$

for non-negative even integer m. By this fact, we obtain the Rankin-Cohen bracket $[,]_m$ for even integers m. The image of $[,]_m$ is the space of modular forms which generate a quotient module of $N(1+m,1)^\vee$. Let D_- be an element in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ defined in section 2. We regard D_- as a differential operator on the Siegel upper half space of degree 2. We then state the main theorem.

Theorem 1.2 (Theorem 5.2)**.** *Let E^* be the Shimura's Eisenstein series of degree 2. Then, for every non-negative even integer m, the modular form $[D_-E^*,E^*]_m$ is non-zero and generates $N(1+m,1)^\vee$.*

From this theorem, we deduce the positivity of multiplicities of certain indecomposable reducible modules in the space of nearly holomorphic modular forms of degree 2.

Next we consider the Klingen Eisenstein series of degree $n=2$. Let π be an irreducible cuspidal automorphic representation on $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$. Let P be the Klingen parabolic subgroup of Sp_4. For a Hecke character μ of $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times\mathbb{R}_+^\times$, we denote by $I(s,\mu,\pi)$ the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\mu|\cdot|^s\boxtimes\pi)$. Take a standard section $\Phi_s\in I(s,\mu,\pi)$. Let $E(g,s,\Phi_s)$ be the Eisenstein series associated to the standard section Φ. We then consider a $(\mathfrak{g},K_\infty)$-module generated by $E(g,s,\Phi)$. We denote by $L^S(s,\pi,\mu)$ the (partial) standard L function of π twisted by μ, i.e.,

$$L^S(s,\pi,\mu)=\prod_{v\notin S}L_v(s,\pi_v,\mu_v)$$

We then construct a nearly holomorphic modular form with certain properties under some assumptions.

Theorem 1.3 (Theorem 4.7)**.** *Let k be an odd integer with $k\geq 3$, χ a quadratic Hecke character of $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times\mathbb{R}_+^\times$ and π an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$ such that $L^S(s,\pi,\chi)$ has a simple pole at $s=1$. Let $\Phi_s=\otimes_v\Phi_{s,v}$ be a standard section of $I(s,\chi,\pi)$. We denote by S the set of places such that $\infty\in S$ and if $p\notin S$, the local factor $\Phi_{s,p}$ is unramified. Suppose that*

- *$\chi_\infty=\mathrm{sgn}$ and $\pi_\infty\cong D_k^+$,*
- *for $\infty\neq p\in S$, the function $M_{w,s,p}\Phi_{s,p}$ is non-zero at $s=1$, and,*
- *for an archimedean place, the local factor $\Phi_{s,\infty}$ has a K_∞-type $(k,3)$.*
  Here D_k^+ is the holomorphic discrete series representation of $\mathrm{SL}_2(\mathbb{R})$ with the lowest weight k and $M_{w,s,p}$ is the local intertwining operator as in (4.2).

Then, Klingen Eisenstein series $E(g,s,\Phi)$ is holomorphic at $s=1$ as a function in s and generates $N(k,1)^\vee$ as a $(\mathfrak{sp}_4(\mathbb{C}),K_\infty)$-module at $s=1$.

By the theorems, we can construct a nearly holomorphic modular form which generate a module of the form $N(1+m,1)^\vee$ for an even integer m. These are new examples of modular forms which generates the module $N(1+m,1)^\vee$ for any even integer m.

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2. Generalized Verma modules

In this section, we compute some generalized Verma modules. These computations leads to the construction of Rankin-Cohen brackets described in section 5.

2.1. **Definition of generalized Verma modules.** Let M_n be the set of $n \times n$ matrices. For a matrix $m \in \mathrm{M}_{2n}$, we define matrices a_m, b_m, c_m and d_m in M_n by

$$m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}.$$

Let $\mathfrak{g}$ be the complex symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$, i.e.,

$$\mathfrak{g} = \left\{ X \in \mathrm{M}_{2n}(\mathbb{C}) \,\middle|\, {}^tX \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} X = 0 \right\}.$$

Then, we have a decomposition $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$. Here, the Lie algebras $\mathfrak{k}$ and $\mathfrak{p}_\pm$ are described as follows:

$$\begin{aligned} \mathfrak{k} &= \{x \in \mathfrak{g} \mid a_x = d_x, b_x = -c_x\}, \\ \mathfrak{p}_+ &= \{x \in \mathfrak{g} \mid a_x = -\sqrt{-1}\, b_x = -\sqrt{-1}\, c_x = -d_x\}, \\ \mathfrak{p}_- &= \{x \in \mathfrak{g} \mid a_x = \sqrt{-1}\, b_x = \sqrt{-1}\, c_x = -d_x\}. \end{aligned}$$

It is well-known that the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ have the same Cartan subalgebra. We take such a Cartan subalgebra. Then the root system of $\mathfrak{g}$ is

$$\Phi = \{ \pm(e_i + e_j),\ \pm(e_k - e_l), \quad 1 \leq i \leq j \leq n, 1 \leq k < l \leq n \}.$$

We put the set

$$\Phi^+ = \{ -(e_i + e_j),\ e_k - e_l, \quad 1 \leq i \leq j \leq n, 1 \leq k < l \leq n \}$$

to be a positive root system.

Let ρ be half the sum of positive roots. Put

$$\Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z},\ i = 1, \ldots, n-1\}.$$

We say that a weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is $\mathfrak{k}$-dominant if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for any $1 \leq i \leq n-1$. Let Λ^+ be the set of $\mathfrak{k}$-dominant weights. For a $\mathfrak{k}$-dominant weight $\lambda \in \Lambda^+$, we denote by ρ_λ an irreducible $\mathcal{U}(\mathfrak{k})$-module with highest weight λ. Let V_λ be any model of ρ_λ. We regard V_λ as a module for $\mathfrak{p} = \mathfrak{p}_- + \mathfrak{k}$ by letting $\mathfrak{p}_-$ act as trivial. Set

$$N(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V_\lambda.$$

Then $N(\lambda)$ has a natural structure of left $\mathcal{U}(\mathfrak{g})$-module. To simplify the notation, we identify the representation ρ_λ as its representation space V_λ. The modules $N(\lambda)$ are often called the generalized Verma module with highest weight λ with respect to a parabolic subalgebra $\mathfrak{p}$. It is well-known that the module $N(\lambda)$ has a unique irreducible quotient $L(\lambda)$. We denote by $N(\lambda)^\vee$ the contragredient module of $N(\lambda)$ in the sense of [6, section 3.2]. Note that, as a $\mathcal{U}(\mathfrak{k})$-module, $N(\lambda)$ is isomorphic to $N(\lambda)^\vee$. We denote by χ_λ an infinitesimal character of $L(\lambda)$. By the Harish-Chandra isomorphism, $\chi_\lambda = \chi_\mu$ is equivalent to $\lambda = w \cdot \mu$ for some $w \in W$, where W is the Weyl group of $\mathfrak{g}$ and $w \cdot \lambda = w(\lambda + \rho) - \rho$. Note that by χ_0 we mean the infinitesimal character of the trivial representation of $\mathcal{U}(\mathfrak{g})$.

By the Poincare-Birkhoff-Witt Theorem, the module $N(\lambda)$ is isomorphic to

$$\rho_\lambda \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p}_+)$$

as a $\mathbb{C}$-vector space. Since the space $\mathcal{U}(\mathfrak{p}_+)$ is closed under the adjoint action of $\mathfrak{k}$, it is isomorphic to

$$\bigoplus_\omega \rho_\omega$$

where ω runs through the set $\{(\omega_1, \ldots, \omega_n) \in (2\mathbb{Z}_{\geq 0})^n \mid \omega_1 \geq \cdots \geq \omega_n\}$ (cf. [25, Theorem 12.7]). Hence the Verma module $N(\lambda)|_{\mathcal{U}(\mathfrak{k})}$ is isomorphic to

$$\rho_\lambda \otimes_{\mathbb{C}} \left(\bigoplus_\omega \rho_\omega \right) \tag{2.1}$$

as a $\mathcal{U}(\mathfrak{k})$-module. Here ω runs through the same set. We can give the explicit multiplicity formula of $N(\lambda)|_{\mathcal{U}(\mathfrak{k})}$ as a $\mathcal{U}(\mathfrak{k})$-module by the Blattner formula or the Littlewood-Richardson rule.

2.2. **Computations in $n = 2$ case.** Put $n = 2$. Define root vectors of $\mathfrak{g}$ as follows:

$$
\begin{aligned}
Z &= -\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & Z' &= -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \\
N_+ &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{pmatrix}, & N_- &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{pmatrix}, \\
X_+ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_- &= \frac{1}{2}\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
P_{1+} &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ i & 0 & -1 & 0 \end{pmatrix}, & P_{1-} &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -1 \\ -i & 0 & -1 & 0 \end{pmatrix}, \\
P_{0+} &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{pmatrix}, & P_{0-} &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

Here, $i = \sqrt{-1}$. These choices are the same as in [15, Notation (6)].

By computations of the first reduction points in the sense of [3, Theorem 2.4], the generalized Verma module $N(1/2, 1/2)$ is reducible. By [5, Lemma 3.2], we have

$$0 \longrightarrow L(1/2, 1/2) \longrightarrow N(1/2, 1/2)^\vee \longrightarrow L(5/2, 5/2) \longrightarrow 0.$$

For $m \in \mathbb{Z}_{\geq 3}$, by [15, Proposition 2.6], we have the following non-split exact sequence:

$$0 \longrightarrow L(m, 1) \longrightarrow N(m, 1)^\vee \longrightarrow L(m, 3) \longrightarrow 0.$$

Note that we have a decomposition

$$N(\lambda_1, \lambda_2)|_{\mathcal{U}(\mathfrak{k})} \cong N(\lambda_1, \lambda_2)^\vee|_{\mathcal{U}(\mathfrak{k})} \cong \rho_{(\lambda_1, \lambda_2)} \otimes \left(\bigoplus_{(p,q) \in \mathbb{Z}^2, p \geq q \geq 0} \rho_{(2p, 2q)} \right) \tag{2.2}$$

for any $\lambda \in \Lambda^+$ by (2.1).

Since $N(5/2, 5/2)$ is irreducible, by (2.2), we get

$$
\begin{aligned}
L(1/2, 1/2)|_{\mathcal{U}(\mathfrak{k})} &= N(1/2, 1/2)|_{\mathcal{U}(\mathfrak{k})} \Big/ L(5/2, 5/2)|_{\mathcal{U}(\mathfrak{k})} = N(1/2, 1/2)|_{\mathcal{U}(\mathfrak{k})} \Big/ N(5/2, 5/2)|_{\mathcal{U}(\mathfrak{k})} \\
&\cong \left(\bigoplus_{\substack{(p,q) \in \mathbb{Z}^2 \\ p \geq q \geq 0}} \rho_{(2p+1/2, 2q+1/2)} \right) \Big/ \left(\bigoplus_{\substack{(p,q) \in \mathbb{Z}^2 \\ p \geq q \geq 0}} \rho_{(2p+5/2, 2q+5/2)} \right) \\
&\cong \bigoplus_{p \in \mathbb{Z}_{\geq 0}} \rho_{(2p+1/2, 1/2)}.
\end{aligned} \tag{2.3}
$$

Let v be a highest weight vector in $L(1/2, 1/2)$. Note that the vector v is of weight $(1/2, 1/2)$. Define elements D_+ and D_- in $\mathcal{U}(\mathfrak{g})$ by

$$D_+ = P_{1+}^2 - 4X_+ P_{0+}, \qquad D_- = P_{1-}^2 - 4X_- P_{0-}.$$

For any vector u of weight (p, q) such that $N_+ v = 0$, the vector $D_\pm u$ is of weight $(p \pm 2, q \pm 2)$ such that $N_+(D_\pm v) = 0$. Let w be a vector in $N(1/2, 1/2)^\vee$ such that $N_+ w = 0$ and $D_- w = v$. Note that the vector w exists uniquely.

Lemma 2.1. For a positive even integer m and $i, j \in \mathbb{Z}_{\geq 0}$ with $2i + 2j = m$, let

$$a_{i,j} = \frac{m+3}{m - 2i + 3} (-1)^i \binom{m+1}{2i}, \qquad b_{i,j} = \frac{(-1)^{i+1}}{6(m+2)} \binom{m+1}{2i}.$$

We have an isomorphism

$$L(1/2, 1/2) \otimes N(1/2, 1/2)^\vee \cong \left(\bigoplus_{m \in 2\mathbb{Z}_{\geq 0}} N(m+3, 1)^\vee \right) \oplus L(1, 1),$$

as a $\mathcal{U}(\mathfrak{g})$-module. Moreover, a vector

$$\sum_{2i+2j=m} a_{i,j}\, X_+^i v \otimes X_+^j w \; + \sum_{2i+2j=m} b_{i,j}\, D_+(X_+^i v \otimes X_+^j D_- w)$$

generates $N(m+3, 1)^\vee$ for $m \in 2\mathbb{Z}_{\geq 0}$.

Proof. As $\mathcal{U}(\mathfrak{k})$-modules, by (2.3), we have

$$L(1/2,1/2)\otimes N(1/2,1/2)^\vee|_{\mathcal{U}(\mathfrak{k})} \cong \left(\bigoplus_{\ell\in\mathbb{Z}_{\geq 0}} \rho_{(2\ell+1/2,1/2)}\right)\otimes\left(\rho_{(1/2,1/2)}\otimes\bigoplus_{\substack{(p,q)\in\mathbb{Z}^2\\ p\geq q\geq 0}}\rho_{(2p,2q)}\right) \tag{2.4}$$
$$\cong \bigoplus_{\ell\in\mathbb{Z}_{\geq 0}} \rho_{(2\ell+1,1)}\otimes(\mathcal{U}(\mathfrak{p}_+)|_{\mathcal{U}(\mathfrak{k})})$$
$$\cong \bigoplus_{\ell\in\mathbb{Z}_{\geq 0}} N(2\ell+1,1)^\vee|_{\mathcal{U}(\mathfrak{k})}.$$

Here we use the isomorphism

$$\rho_{(2\ell+1,1)}\otimes\bigoplus_{\substack{(p,q)\in\mathbb{Z}^2\\ p\geq q\geq 0}}\rho_{(2p,2q)} \cong \rho_{(2\ell+1,1)}\otimes\mathcal{U}(\mathfrak{p}_+)|_{\mathcal{U}(\mathfrak{k})} \cong N(2\ell+1,1)^\vee|_{\mathcal{U}(\mathfrak{k})}$$

as a $\mathcal{U}(\mathfrak{k})$-module (see (2.2)).

By the straightforward computation, we can show that the vector

$$\sum_{2i+2j=m} a_{i,j}\, X_+^i v\otimes X_+^j w \;+\; \sum_{2i+2j=m} b_{i,j}\, D_+(X_+^i v\otimes X_+^j D_- w)$$

generates $N(m+1,1)^\vee$ for every even integer m in the tensor product module $L(1/2,1/2)\otimes N(1/2,1/2)^\vee$. Here $a_{i,j}$ and $b_{i,j}$ is rational numbers given in the lemma. Hence, for even $m\in\mathbb{Z}_{\geq 0}$, representations $N(m+1,1)^\vee$ must occur in the tensor product module $L(1/2,1/2)\otimes N(1/2,1/2)^\vee$. Note that the product $v\otimes D_- w$ of highest weight vectors generates the irreducible representation $L(1,1)$. This means that we have an inclusion

$$L(1,1)\oplus\bigoplus_{\ell\in\mathbb{Z}_{\geq 0}} N(2\ell+3,1)^\vee \hookrightarrow L(1/2,1/2)\otimes N(1/2,1/2)^\vee$$

as $\mathcal{U}(\mathfrak{g})$-modules. Since the generalized Verma module $N(1,1)$ is irreducible, we have the obvious isomorphisms $N(1,1)\cong N(1,1)^\vee\cong L(1,1)$. The restriction of the above inclusion to $\mathcal{U}(\mathfrak{k})$ is isomorphism by (2.4). Hence the above inclusion is isomorphism as $\mathcal{U}(\mathfrak{g})$-modules. This completes the proof. □

By the explicit formulas of generators of $N(m+1,1)^\vee$ as in the above lemma, we can get Rankin-Cohen brackets in section 5.

3. The space of nearly holomorphic modular forms

In this section, we define the space of nearly holomorphic modular forms of integral or half integral weight. For the convenience, we briefly review the theory of Weil representation.

3.1. **Metaplectic group and Weil representation.** We introduce basic facts of the metaplectic group and the Weil representation. Let $\mathbb{A}$ be the adele ring of the rational number field $\mathbb{Q}$ and $\mathbb{A}_{\mathrm{fin}}$ the finite adele ring. Let Sp_{2n} be the algebraic group defined over $\mathbb{Q}$ defined by

$$\mathrm{Sp}_{2n} = \left\{ g\in\mathrm{GL}_{2n} \;\middle|\; {}^t g\begin{pmatrix}0 & -\mathbf{1}_n\\ \mathbf{1}_n & 0\end{pmatrix} g = \begin{pmatrix}0 & -\mathbf{1}_n\\ \mathbf{1}_n & 0\end{pmatrix}\right\}.$$

For any place v, we then have the Metaplectic group $\mathrm{Mp}_{2n}(\mathbb{Q}_v)$ which is the non-trivial 2-fold cover of $\mathrm{Sp}_{2n}(\mathbb{Q}_v)$. More precisely, there is a non-split exact sequence

$$1\longrightarrow\{\pm 1\}\longrightarrow\mathrm{Mp}_{2n}(\mathbb{Q}_v)\longrightarrow\mathrm{Sp}_{2n}(\mathbb{Q}_v)\longrightarrow 1$$

and the group $\mathrm{Mp}_{2n}(\mathbb{Q}_v)$ equipped with the group multiplication

$$(g_1,\varepsilon_1)(g_2,\varepsilon_2) = (g_1g_2, \varepsilon_1\varepsilon_2 c_v(g_1,g_2))$$

with the Rao's 2-cocycle c_v [16, Theorem 5.3]. It is well-known that there exists a unique splitting $\mathrm{Sp}_{2n}(\mathbb{Z}_p)\hookrightarrow\mathrm{Mp}_{2n}(\mathbb{Q}_p)$ for a finite place $p\neq 2$ (cf. Lemma 3.2). We may regard an open compact group $\mathrm{Sp}_{2n}(\mathbb{Z}_p)$ as an open compact subgroup of $\mathrm{Mp}_{2n}(\mathbb{Q}_p)$ for a finite place $p\neq 2$. We then define the global metaplectic group $\mathrm{Mp}_{2n}(\mathbb{A})$ by the restricted direct product of $\mathrm{Mp}_{2n}(\mathbb{Q}_v)$ over all places of $\mathbb{Q}$ with respect to the open compact group $\prod_{v\neq 2,\infty}\mathrm{Sp}_{2n}(\mathbb{Z}_v)$.

We define a non-trivial additive character $\psi = \otimes_v \psi_v$ of $\mathbb{Q}\backslash\mathbb{A}$ by

$$\psi_p(x) = \exp(-2\pi\sqrt{-1}\,y), \qquad x \in \mathbb{Q}_p, \tag{3.1}$$

$$\psi_\infty(x) = \exp(2\pi\sqrt{-1}\,x), \qquad x \in \mathbb{R}, \tag{3.2}$$

where $y \in \cup_{m=1}^\infty p^{-m}\mathbb{Z}$ such that $x - y \in \mathbb{Z}_p$. For a Schwartz function $f \in \mathcal{S}(\mathbb{Q}_v)$, the Fourier transform $\widehat{f}$ of f is defined by

$$\widehat{f}(x) = \int_F f(y)\psi_v(xy)\,dy,$$

where dy is the self-dual Haar measure on $\mathbb{Q}_v$ with respect to ψ_v. For $a \in \mathbb{Q}_v^\times$, we define the Weil index $\gamma_v(a)$ by

$$\int_{\mathbb{Q}_v} f_v(y)\psi_v\left(\frac{1}{2}ay^2\right)\,dy = \gamma_v(a)|a|_v^{-1/2}\int_{\mathbb{Q}_v} \widehat{f}_v(y)\psi_v\left(-\frac{1}{2}a^{-1}y^2\right)\,dy, \quad f_v \in \mathcal{S}(\mathbb{Q}_v).$$

By definition, we have $\gamma_v(ab^2) = \gamma_v(a)$. We then compute the Weil index γ_v as follows:

Lemma 3.1. For a finite place $p \neq 2$ and $a \in \mathbb{Z}_p^\times$, one has

$$\gamma_p(a) = 1.$$

For an archimedean place ∞, one has

$$\gamma_\infty(\pm 1) = \exp(\pm\pi\sqrt{-1}/4).$$

Proof. For a non-archimedean place $p \neq 2$, let f_p be the characteristic function on $\mathbb{Z}_p$. Since the additive character ψ_p is of order 0, the Fourier transform of f_p is f_p itself. Then we have

$$\int_{\mathbb{Q}_p} f_p(y)\psi_p\left(\frac{1}{2}ay^2\right)\,dy = \int_{\mathbb{Z}_p}\psi_p\left(\frac{1}{2}ay^2\right)\,dy = 1$$

and

$$\int_{\mathbb{Q}_v} \widehat{f_p}(y)\psi_p\left(-\frac{1}{2}a^{-1}y^2\right)\,dy = \int_{\mathbb{Z}_p}\psi_p\left(-\frac{1}{2}a^{-1}y^2\right)\,dy = 1.$$

Hence the first assertion holds.

Next we assume $v = \infty$. Let $f_\infty(x) = \exp(-\pi x^2)$. Then we have $\widehat{f_\infty} = f_\infty$. Hence it is clear that $\gamma_\infty(\pm 1) = \exp(\pm\pi\sqrt{-1}/4)$. $\square$

By this Lemma, the product $\prod_v \gamma_v(a_v)$ is well-defined for $(a_v)_v \in \mathbb{A}^\times$. It is well-known that for $a \in \mathbb{Q}^\times$, we have the product formula $\prod_v \gamma_v(a) = 1$.

Let P be the Siegel parabolic subgroup of Sp_{2n}. For $a \in \mathrm{GL}_n$ and $b \in \mathrm{Sym}_n$, the symmetric matrices of size n, set

$$\mathbf{m}(a) = \begin{pmatrix} a & 0 \\ 0 & {}^ta^{-1}\end{pmatrix}, \qquad \mathbf{n}(b) = \begin{pmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n\end{pmatrix}.$$

Then, we have $P = \{\mathbf{m}(a)\mathbf{n}(b) \mid a \in \mathrm{GL}_n, b \in \mathrm{Sym}_n\}$.

For each place v, we have the local Weil representation ω_{ψ_v} associated to ψ_v. The Weil representation ω_{ψ_v} can be realized on the space of Schwartz-Bruhat functions $\mathcal{S}(\mathbb{Q}_v^n)$. Suppose that the group $\mathrm{Mp}_{2n}(\mathbb{Q}_v)$ acts on $(\omega_\psi, \mathcal{S}(\mathbb{Q}_v^n))$ by the following rule: for $f \in \mathcal{S}(\mathbb{A}^n)$,

$$\begin{aligned}
\omega_{\psi,v}(\mathbf{m}(a),\varepsilon)f(x) &= \varepsilon\frac{\gamma_v(1)}{\gamma_v(\det a)}|\det a|^{1/2}f(xa), && a \in \mathrm{GL}_n(\mathbb{Q}_v), \quad x \in \mathbb{Q}_v^n,\\
\omega_{\psi,v}(\mathbf{n}(b),\varepsilon)f(x) &= \varepsilon\psi_v(xb\cdot {}^tx/2)f(x), && b \in \mathrm{Sym}_n(\mathbb{Q}_v), \quad x \in \mathbb{Q}_v^n,\\
\omega_{\psi,v}\left(\begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0\end{pmatrix},\varepsilon\right)f(x) &= \varepsilon\gamma_v(1)^{-n}\widehat{f}(-x), && x \in \mathbb{Q}_v^n.
\end{aligned}$$

Here the function $\widehat{f}$ is the Fourier transform of f. In general, for $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ with $\det c \neq 0$, we have

$$\begin{aligned}
\omega_{\psi_v}(g,\varepsilon)f(x) = \varepsilon&\frac{\gamma_v(1)^{-n+1}}{\gamma_v(\det c)}|\det c|^{-\frac{1}{2}}\\
&\times\int_{\mathbb{Q}_v^n} f(y)\psi_v\left(\frac{1}{2}(xac^{-1}{}^tx - 2yc^{-1}{}^tx + yc^{-1}d{}^ty)\right)\,dy.
\end{aligned}$$

The following Lemma is well-known:

Lemma 3.2. Let K_2 be an open compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{Q}_2)$ defined by

$$K_2 = \{g \in \mathrm{Sp}_{2n}(\mathbb{Q}_2) \mid g \equiv \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) (\mathrm{mod.}\ 2),\ \det a_g \in 1 + 4\mathbb{Z}_2\},$$

where a_g is the left upper $n \times n$ matrix of g. Then there exists a splitting $K_2 \hookrightarrow \mathrm{Mp}_{2n}(\mathbb{Q}_2)$.

Proof. Let f be the characteristic function on $\mathbb{Z}_2^n$. Set

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \,\middle|\, \det a \in 1 + 4\mathbb{Z}_2 \right\}, \qquad N = \left\{ \begin{pmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n \end{pmatrix} \,\middle|\, b \equiv 0 \pmod{2} \right\}.$$

By the formulas $(\mathbb{Z}_2^\times)^2 = 1 + 4\mathbb{Z}_2$ and $\gamma_v(ab^2) = \gamma_v(a)$, we have

$$\omega_{\psi_2}(m,1)f = f, \qquad \omega_{\psi_2}(n,1)f = f$$

for any $m \in M$ and $n \in N$. We also have

$$\omega_{\psi_2}(1,\zeta)\omega_{\psi_2}(-w_n,1)\omega_{\psi_2}(n,1)\omega_{\psi_2}(w_n,1)f = f$$

for $n \in N$. Here $\zeta = \gamma_2(1)^{2n-1}\gamma_2((-1)^n) \in \{\pm 1\}$ and $w_n = \left(\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right)$. Note that by the product formula of the Weil index γ, we have $\gamma_2(\pm 1) = \exp(\mp\pi\sqrt{-1}/4)$. Set N^w be the subgroup of Mp_{2n} generated by the elements of the form $(1,\zeta)(-w_n,1)(n,1)(w_n,1)$ for $n \in N$.

Let Γ be the subgroup of $\mathrm{Mp}_{2n}(\mathbb{Q}_2)$ generated by $(M,1), (N,1)$ and N^w. We denote by $\mathbf{pr}$ the projection $\mathrm{Mp}_{2n} \longrightarrow \mathrm{Sp}_{2n}$. Then the image $\mathbf{pr}(\Gamma)$ in $\mathrm{Sp}_{2n}(\mathbb{Q}_2)$ contains K_2. Hence we can embed K_2 into $\mathrm{Mp}_{2n}(\mathbb{Q}_2)$ as a subgroup of the group of normalizers of f. This completes the proof. $\square$

3.2. **A factor of automorphy of half integral weight.** We will define a factor of automorphy of half integral weight. Set

$$\mathfrak{H}_n = \left\{ Z \in \mathrm{M}_n(\mathbb{C}) \,\middle|\, {}^t Z = Z, \mathrm{Im}(Z) > 0 \right\}.$$

We call $\mathfrak{H}_n$ the Siegel upper half space of degree n. We also define a canonical factor of automorphy $j(\,\cdot\,,\,\cdot\,)\colon \mathrm{Sp}_{2n}(\mathbb{R}) \times \mathfrak{H}_n \longrightarrow \mathbb{C}$ by

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, Z \right) = \det(cZ + d), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}),\ Z \in \mathfrak{H}_n.$$

Then it is well-known that there exists a factor of automorphy $j^{1/2}(\,\cdot\,,\,\cdot\,)\colon \mathrm{Mp}_{2n}(\mathbb{R}) \times \mathfrak{H}_n \longrightarrow \mathbb{C}$ such that $(j^{1/2}((g,\varepsilon),z))^2 = j(g,z)$. Let K_∞ be a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ defined by

$$K_\infty = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}) \,\middle|\, a = d, b = -c \right\}.$$

Note that K_∞ is the stabilizer of $\sqrt{-1}\,\mathbf{1}_n \in \mathfrak{H}_n$ under the linear fractional transformation of $\mathrm{Sp}_{2n}(\mathbb{R})$ on $\mathfrak{H}_n$. We denote by $\widetilde{K_\infty}$ the inverse image of K_∞ in $\mathrm{Mp}_{2n}(\mathbb{R})$. Set

$$\varphi_\infty((x_1,\ldots,x_n)) = \exp(-\pi(x_1^2 + \cdots + x_n^2)), \qquad (x_1,\ldots,x_n) \in \mathbb{R}^n.$$

Then the function $\omega_{\psi_\infty}(k)\varphi_\infty$ is a constant multiple of φ_∞ for $\widetilde{K_\infty}$. We denote by $\det^{1/2}$ such a constant, i.e., $\omega(g,\varepsilon)\varphi_\infty = \det^{1/2}(g,\varepsilon)\varphi_\infty$ for $(g,\varepsilon) \in \widetilde{K_\infty}$. Then we have

$$\det{}^{1/2}\left(\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \varepsilon \right) = \exp(\sqrt{-1}\,\theta/2)$$

for $\det(\alpha + \sqrt{-1}\,\beta) = \exp(\sqrt{-1}\,\theta)$ with $-\pi < \theta \leq \pi$. We now obtain

$$\left(j^{1/2}((\mathbf{m}(a),1)(\mathbf{n}(b),1)(u,\varepsilon),\mathbf{i}) \right)^{-1} = |\det a|^{1/2}\det{}^{1/2}(u,\varepsilon),$$

for $a \in \mathrm{GL}_n(\mathbb{R})$ with $\det a > 0$, $b \in \mathrm{Sym}_n(\mathbb{R}), (u,\varepsilon) \in \widetilde{K_\infty}$, and $\mathbf{i} = \sqrt{-1}\,\mathbf{1}_n \in \mathfrak{H}_n$.

3.3. **The space of nearly holomorphic modular forms.** Let $K(\mathbb{C})$ be the complexification of K_∞. Then the isomorphism classes of irreducible finite-dimensional representations of $K(\mathbb{C})$ is equal to the set

$$\Lambda_{\mathbb{Z}}^+ = \{\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$$

by the highest weight theory. Let $\lambda = (\lambda_1,\ldots,\lambda_n) \in \lambda^+$. We denote by $(\rho_\lambda, V_\lambda)$ the irreducible representation of $K(\mathbb{C})$ corresponding to λ.

For a finite-dimensional representation (ρ, V) of $K(\mathbb{C})$, we define the factor of automorphy j_ρ of $\mathrm{Sp}_{2n}(\mathbb{R}) \times \mathfrak{H}_n$ by

$$j_\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) = \rho(cz + d).$$

For simplicity, if $\rho = \rho_\lambda$ is irreducible, we write $j_\lambda = j_{\rho_\lambda}$. For $\lambda' = \lambda + (1/2, \dots, 1/2)$ with $\lambda \in \Lambda_{\mathbb{Z}}^+$, we define a factor of automorphy $j_{\lambda'}$ by $j_{\lambda'} = j_\lambda j^{1/2}$. Let Γ be a congruence subgroup of $\{g \in \mathrm{Sp}_{2n}(\mathbb{Z}) \mid g \equiv \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) (\mathrm{mod.}\ 2), \det a_g \equiv 1 (\mathrm{mod.}\ 4)\}$. In general, we can choose a congruence subgroup bigger than $\Gamma(4)$. In this paper, for simplicity, we assume a congruence subgroup is contained in $\Gamma(4)$ in order to use Lemma 3.2. For a factor of automorphy j_ρ and a C^∞-function $f \colon \mathfrak{H}_n \longrightarrow V$, we say that the modular form f is C^∞-modular form of weight ρ with respect to Γ if the automorphy $f(\gamma(z)) = j_\rho(\gamma, z) f(z)$ holds for any $\gamma \in \Gamma$ and $z \in \mathfrak{H}_n$. Here either ρ or $\rho \otimes \det^{1/2}$ is a finite-dimensional representation of K_∞.

Let $\{r_{i,j}(z)\}_{1 \le i,j \le n}$ be scalar valued functions on $\mathfrak{H}_n$ defined by

$$(r_{i,j}(z))_{i,j} = (\mathrm{Im}(z))^{-1}, \qquad z \in \mathfrak{H}_n.$$

For a finite-dimensional complex vector space V and a C^∞ function $f \colon \mathfrak{H}_n \longrightarrow V$, we say that the function f is nearly holomorphic if there exist polynomials $P_k \in \mathbb{C}[r_{i,j}]_{1 \le i,j \le n}$ and V-valued holomorphic functions f_k with $k = 0, \dots m$ such that the function f is expressed by

$$f = \sum_{k=0}^{m} P_k((r_{i,j})_{1 \le i \le j \le n}) f_k.$$

If a C^∞-modular form f is nearly holomorphic and satisfies the cusp condition, we call f a nearly holomorphic modular form. If $n > 1$, we can omit the cusp condition. Indeed, Koechere principle is also established for $n > 1$ in the case of nearly holomorphic modular forms (cf. [5, Proposition 4.1]). We denote by $N_\rho(\Gamma)$ the space of nearly holomorphic modular forms of the factor of automorphy j_ρ with respect to Γ. For $\rho = \det^\ell$ with $2\ell \in \mathbb{Z}$, set $N_\ell(\Gamma) = N_\rho(\Gamma)$. We also define the space of holomorphic modular forms $M_\rho(\Gamma)$ and $M_\ell(\Gamma)$ similarly. The theory of nearly holomorphic modular forms are developed by Shimura (cf. [22, 23, 24, 25]).

For a V-valued modular form $f \in N_\rho(\Gamma)$, set $\varphi_f(g) = j_\rho(g, \mathbf{i})^{-1} f(g(\mathbf{i}))$ for $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ or $\mathrm{Mp}_{2n}(\mathbb{R})$. Take a non-zero vector $v^* \in V^*$, the dual space of V. Then, we get a scalar valued function $\phi(g) = \langle v^*, \varphi_f(g) \rangle$ on the group. A scalar valued function ϕ is $\mathfrak{p}_-$-finite by near holomorphy of f. Conversely, Shimura proved that if a smooth function ϕ on the group is $\mathfrak{p}_-$-finite, the corresponding function f_ϕ on $\mathfrak{H}_n$ is nearly holomorphic. The proof may be found in [23, section 7]. Hence, we can define the space of nearly holomorphic modular forms $\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$ by the space of smooth functions ϕ on $\mathrm{Sp}_{2n}(\mathbb{R})$ which satisfy the following conditions:

- ϕ is left Γ-invariant,
- ϕ is right K_∞-finite,
- ϕ is right $\mathcal{Z}$-finite,
- ϕ is slowly increasing, and
- ϕ is $\mathfrak{p}_-$-finite.

Here $\mathcal{Z}$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Then, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and a maximal compact subgroup K_∞ act on $\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$ by the right translation. For an irreducible representation ρ of K_∞ (resp. $\widetilde{K_\infty}$), we denote by $\mathcal{A}(\Gamma)^\rho_{\mathfrak{p}_-\text{-fin}}$ the ρ-isotypic component i.e.,

$$\mathcal{A}(\Gamma)^\rho_{\mathfrak{p}_-\text{-fin}} = \{\varphi \in \mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}} \mid \langle k \cdot \varphi, k \in K_\infty \text{ (resp. } \widetilde{K_\infty}) \rangle \cong \rho\}.$$

Let ρ^* be the dual representation of ρ. We then have the isomorphism

$$N_\rho \otimes \rho^* \cong \mathcal{A}(\Gamma)^{\rho^*}_{\mathfrak{p}_-\text{-fin}} \colon f \otimes v^* \longmapsto \langle f, v^* \rangle. \tag{3.3}$$

Hence we have a natural correspondence between modular forms on $\mathfrak{H}_n$ and automorphic forms on $\mathrm{Sp}_{2n}(\mathbb{R})$ or $\mathrm{Mp}_{2n}(\mathbb{R})$. Of course, the similar correspondence also holds for modular forms and automorphic forms on the adele groups in a natural way.

4. Eisenstein series

In this section, we review the theory of Shimura's Eisenstein series as in [21] and [25, §16 and §17], and Klingen Eisenstein series of degree 2.

4.1. **Shimura's Eisenstein series.** Let $\mathbb{A}$ be the adele of $\mathbb{Q}$. We only compute the simplest case of Fourier coefficients of Shimura's Eisenstein series of weight $(n+3)/2$. For a non-archimedean place v of $\mathbb{Q}$, set

$$K_v = \begin{cases} \{(g \in \mathrm{Sp}_{2n}(\mathbb{Q}_2) \mid g \equiv \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) (\mathrm{mod.}\ 2), \det a_g \equiv 1 (\mathrm{mod.}\ 4)\} & \text{if } v = p = 2, \\ \mathrm{Sp}_{2n}(\mathbb{Z}_p) & \text{if } v = p \neq 2. \end{cases}$$

Here the matrix a_g is the left upper $n \times n$ matrix of g. We regard K_p as a subgroup of $\mathrm{Mp}_{2n}(\mathbb{Q}_p)$ for a finite place p (cf. Lemma 3.2). We also regard the $\mathbb{Q}$-valued points $\mathrm{Sp}_{2n}(\mathbb{Q})$ is a subgroup of $\mathrm{Mp}_{2n}(\mathbb{Q})$. We then define compact subgroups $\widetilde{K}$ and $\widetilde{K_{\max}}$ by

$$\widetilde{K} = \prod_{p<\infty} K_p \times \widetilde{K_\infty}, \qquad \widetilde{K_{\max}} = \mathbf{pr}^{-1}\left(\prod_{p<\infty} \mathrm{Sp}_{2n}(\mathbb{Z}_p) \times K_\infty\right).$$

Here $\mathbf{pr}$ is the projection $\mathrm{Mp}_{2n}(\mathbb{A}) \longrightarrow \mathrm{Sp}_{2n}(\mathbb{A})$ and $\widetilde{K_{\max}}$ is a maximal compact subgroup of $\mathrm{Mp}_{2n}(\mathbb{A})$.

Let $\chi = \otimes_v \chi_v$ be a genuine character of $\widetilde{\mathrm{GL}_n(\mathbb{A})}$ defined by

$$\chi_v((g, \varepsilon)) = \varepsilon \frac{\gamma_v(1)}{\gamma_v(\det g)}.$$

Here γ_v is the Weil index and the group $\widetilde{\mathrm{GL}_n(\mathbb{A})}$ is the Levi subgroup of the Siegel parabolic subgroup of $\mathrm{Mp}_{2n}(\mathbb{A})$. By Lemma 3.1, the product $\otimes_v \chi_v$ is well-defined. Note that for any $a \in \mathbb{Q}^\times$, we have $\prod_v \gamma_v(a) = 1$. Let $\widetilde{P(\mathbb{A})}$ be the Siegel parabolic subgroup of $\mathrm{Mp}_{2n}(\mathbb{A})$. Let μ be a Hecke character of $\mathbb{A}^\times$. The space $I(\mu, s)$ is the space of functions Φ such that

$$\Phi(pg) = |\det(a_p)|^s \mu(p)\chi(p,\varepsilon)\delta^{1/2}(p)\Phi(g), \qquad (p, \varepsilon) \in \widetilde{P(\mathbb{A})}, g \in \mathrm{Mp}_{2n}(\mathbb{A}).$$

Here, the $n \times n$-matrix a_p is the left upper matrix of p, and the characters χ and μ are characters of $\widetilde{P(\mathbb{A})}$ via the canonical projection $\widetilde{P(\mathbb{A})} \longrightarrow \widetilde{\mathrm{GL}_n(\mathbb{A})}$ and the character δ is the modulus character of the Siegel parabolic subgroup P. This induced representation $I(\mu, s)$ is so-called the degenerate principal series representation. Let $\Phi_s \in I(\mu, s)$ be a section. We say that the section Φ_s is standard if the restriction of Φ_s to $\widetilde{K_{\max}}$ does not depend on s. For a standard section Φ_s, we denote by $E(g, s, \Phi)$ the Eisenstein series defined by

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{Sp}_{2n}(\mathbb{Q})} \Phi_s(\gamma g), \qquad g \in \mathrm{Mp}_{2n}(\mathbb{A}).$$

Then the sum is formally well-defined and converges absolutely and uniformly on any compact subset in g. We also define the function E^* on $\mathrm{Mp}_{2n}(\mathbb{R})$ by

$$E^*(g, s, \Phi) = E(g\zeta, s, \Phi), \qquad g \in \mathrm{Mp}_{2n}(\mathbb{R}),$$

where

$$\zeta_p = (w_n, \gamma_p(1)^{2n-1}\gamma_p((-1)^n)), \qquad \zeta_\infty = 1, \qquad w_n = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}. \tag{4.1}$$

In [25, Proposition 16.9], Shimura computed Fourier coefficients of E^* for a special standard section Φ_s.

For a trivial character $\mathbf{1}$, we take a standard section $\Phi = \otimes_v \Phi_v$ in $I(\mathbf{1}, s)$ as follows. Suppose $n \equiv 2 \pmod 4$. For a non-archimedean place $p = 2$, let

$$\Phi_2(pk) = \begin{cases} \varepsilon |\det a_p|^{s+(n+1)/2} \chi_2(d_p d_k)^{-1}, & \text{if } pk \in \widetilde{P(\mathbb{Q}_2)}K_2 \\ 0, & \text{if otherwise} \end{cases}$$

where $p = ((\begin{smallmatrix} a_p & b_p \\ 0 & d_p \end{smallmatrix}), \varepsilon) \in \widetilde{P(\mathbb{Q}_2)}$ and $k = ((\begin{smallmatrix} a_k & b_k \\ c_k & d_k \end{smallmatrix}), 1) \in K_2$. Note that the set $\widetilde{P(\mathbb{Q}_2)}K_2$ is equal to the set $\{((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), \varepsilon) \mid d \in \mathrm{GL}_n(\mathbb{Q}_2), d^{-1}c \in \mathrm{M}_n(2\mathbb{Z}_2)\}$. For a non-archimedean place $v \neq 2$, let

$$\Phi_v|_{K_v} = \text{the characteristic function on } K_v.$$

For an archimedean place ∞, let

$$\Phi_\infty(k) = \left(j^{1/2}(k, \mathbf{i})\right)^{-(n+3)}, \qquad \text{for } k \in \widetilde{K_\infty}.$$

Then, since the rank n is congruent to 2 modulo 4, the standard section $\otimes_v \Phi_{s,v} \in I(\mathbf{1}, s)$ is well-defined.

For an automorphic form φ on $\mathrm{Mp}_{2n}(\mathbb{A})$ and a semipositive matrix h in $\mathrm{Sym}_n(\mathbb{Q})$, we denote by $\mathrm{Wh}_h(\varphi)$ the h-th Whittaker-Fourier coefficient of φ, which is defined by

$$\mathrm{Wh}_h(\varphi)(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\psi(-\mathrm{tr}(hn))\, dn, \qquad g \in \mathrm{Mp}_{2n}(\mathbb{A}).$$

Here, $N \cong \mathrm{Sym}_n$ is the unipotent radical of the Siegel parabolic subgroup P and ψ is the non-trivial additive character defined in (3.1) and (3.2).

Next we define the Siegel series and the confluent hypergeometric function. For a non-archimedean place p and $h \in \mathrm{Sym}_n(\mathbb{Z}_p)^*$, the dual space of $\mathrm{Sym}_n(\mathbb{Z}_p)$ with respect to the trace form, let

$$\alpha_p(h,s) = \sum_{\sigma \in \mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)} \omega_p(\sigma)\psi_p(-\mathrm{tr}(h\sigma))\nu(\sigma)^{-s}.$$

Here $\nu(\sigma) = [\sigma\mathbb{Z}_p^n + \mathbb{Z}_p^n \colon \mathbb{Z}_p^n]$ and

$$\omega_p(\sigma) = \nu(\sigma)^{1/2} \int_{\mathbb{Z}_p^n} \psi_p(x\sigma{}^t x/2)\,dx.$$

We normalize the measure dx by $\int_{\mathbb{Z}_p^n} dx = 1$. These functions α_p is so-called the Siegel series. For $h \in \mathrm{Sym}_n(\mathbb{R})$, we define the confluent hypergeometric function $\xi(g,h;\alpha,\beta)$ by

$$\int_{\mathrm{Sym}_n(\mathbb{R})} \det(x+\sqrt{-1}\,g)^{-\alpha}\det(x-\sqrt{-1}\,g)^{-\beta}\exp(-2\pi\sqrt{-1}\,\mathrm{tr}(hx))\,dx,$$

where $0 < g \in \mathrm{Sym}_n(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$. Here for $s \in \mathbb{C}$ and $z \in \mathfrak{H}_n$ we choose the branches of $\det(z)^s$ and $\overline{\det(z)}$ so that their values at $z = \sqrt{-1}\,\mathbf{1}_n$ are $(\sqrt{-1})^{ns}$ and $\overline{(\sqrt{-1})}^{-ns}$, respectively. Here $(\sqrt{-1})^s$ is defined by

$$(\sqrt{-1})^s = \exp(\pi\sqrt{-1}\,s/2).$$

Then, we can state the Shimura's result [25].

Proposition 4.1 (Proposition 16.9 in [25])**.** If the value $\mathrm{Wh}_h(E^*)(\mathbf{n}(b)\mathbf{m}(a))$ is non-zero for some $a \in \mathrm{GL}_n(\mathbb{R})$ with $\det a > 0$ and $b \in N(\mathbb{R})$, we have $h \in (1/2)\mathrm{Sym}_n(\mathbb{Z})$. With the same notation, one has

$$\begin{aligned}\mathrm{Wh}_h(E^*(\,\cdot\,,s,\Phi))(\mathbf{n}(b)\mathbf{m}(a)) &= \psi(\mathrm{tr}(hb))\exp(\pi\sqrt{-1}\,n/4)\cdot 2^{-n(n+1)/2}\det(y)^{(2s+n+1)/4} \\ &\quad \times \left(\prod_{2\neq p<\infty} \alpha_p\left(\frac{2s+n+1}{2}, h\right)\right) \times \xi\left(y,h;\frac{(s+n+2)}{2},\frac{(s-1)}{2}\right).\end{aligned}$$

Proof. For $1 \le j \le n$, let

$$w_j = \begin{pmatrix} 0 & 0 & -1_j & 0 \\ 0 & 1_{n-j} & 0 & 0 \\ 1_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-j} \end{pmatrix}.$$

By the Bruhat decomposition, we have

$$\mathrm{Sp}_{2n}(\mathbb{Q}) = \coprod_{j=1}^{n} P(\mathbb{Q})w_jP(\mathbb{Q})$$

where $P = MN$ is the Siegel parabolic subgroup of Sp_{2n}. Fix $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Q})$. Suppose $\Phi(\gamma n g_\infty \zeta)$ is non-zero for some $n \in N(\mathbb{A})$ and $g_\infty \in \mathrm{Sp}_{2n}(\mathbb{R})$. Here ζ is defined in (4.1). Since the support of Φ_2 is contained in the set $\mathbf{pr}^{-1}(P(\mathbb{Q}_2)w_nP(\mathbb{Q}_2))$, we have $\gamma \in P(\mathbb{Q})w_nP(\mathbb{Q}) = P(\mathbb{Q})w_nN(\mathbb{Q})$. This shows that the h-th Whittaker-Fourier function $\mathrm{Wh}_h(E(\,\cdot\,,s,\Phi))$ is equal to

$$\begin{aligned}&\int_{N(\mathbb{A})} \Phi_s\left((w_n,1)ng_\infty\zeta\right)\overline{\psi}(\mathrm{tr}(hn))\,dn \\ &= \prod_{p<\infty}\int_{N(\mathbb{Q}_p)} \Phi_{s,p}\left((-w_n,1)n(w_n,\gamma_p(1)^{2n})\right)\overline{\psi_p}(\mathrm{tr}(hn))\,dn \times \int_{N(\mathbb{R})}\Phi_{s,\infty}\left((w_n,1)ng_\infty\right)\overline{\psi_\infty}(\mathrm{tr}(hn))\,dn.\end{aligned}$$

For a non-archimedean place $p \neq 2$, the above integral is equal to the Siegel series $\alpha_p(h, s+(n+1)/2)$. For $v = 2$, we have

$$\begin{aligned}\int_{N(\mathbb{Q}_2)} \Phi_{s,2}\left((-w_n,1)n(w_n,\varepsilon)\right)\overline{\psi_2}(\mathrm{tr}(hn))\,dn &= \int_{N(2\mathbb{Z}_2)} \Phi_{s,2}\left((-w_n,1)n(w_n,\varepsilon)\right)\overline{\psi_2}(\mathrm{tr}(hn))\,dn \\ &= \int_{N(2\mathbb{Z}_2)} \overline{\psi_2}(\mathrm{tr}(hn))dn,\end{aligned}$$

where $\varepsilon = \gamma_2(1)^{2n}$. If the above integral is non-zero, we have $h \in (2\cdot\mathrm{Sym}_n(\mathbb{Z}_2))^*$ and hence h is contained in $(1/2)\mathrm{Sym}_n(\mathbb{Z})$. Suppose $h \in (1/2)\mathrm{Sym}_n(\mathbb{Z})$. Then we have

$$\int_{N(2\mathbb{Z}_2)} \psi_2(\mathrm{tr}(hn))dn = \int_{N(2\mathbb{Z}_2)} dn = 2^{-n(n+1)/2}\int_{N(\mathbb{Z}_2)} dn = 2^{-n(n+1)/2}.$$

Finally we compute the case of $v = \infty$. Since $\Phi_{s,\infty}$ is of weight $(n+3)/2$, we have $\Phi_{s,\infty}(g(k,\varepsilon)) = \Phi_{s,\infty}(g)(\det^{1/2}(k,\varepsilon))^{n+3}$ and $\det^{1/2}(w_n,1) = \gamma(1)^{-n}$. Hence for $n(b)m(a) \in P(\mathbb{R})$ with $y = a^t a$ and $\det a > 0$, we have

$$\begin{aligned}
&\int_{N(\mathbb{R})} \Phi_{s,\infty}\left((w_n,1)(n,1)(n(b)m(a),1)\right)\overline{\psi_\infty}(\mathrm{tr}(hn))\,dn \\
&= \int_{N(\mathbb{R})} \Phi_{s,\infty}\left((w_n,1)(n,1)(n(b)m(a),1)(-w_n,1)(w_n,1)(1,\gamma_\infty(1)^{2n}\right)\overline{\psi_\infty}(\mathrm{tr}(hn))\,dn \\
&= \gamma_\infty(1)^{-n(n+3)}\gamma_\infty(1)^{2n}\det(y)^{(2s+n+1)/2}\xi\left(y,h;\frac{(s+n+2)}{2},\frac{(s-1)}{2}\right).
\end{aligned}$$

Here we use the formula $(w_n,1)(-w_n,1) = (1,\gamma_\infty(1)^{2n})$. Since n is congruent to 2 modulo 4 and $\gamma_\infty(1)^8 = 1$, we have $\gamma_\infty(1)^{-n(n+3)} = \gamma_\infty(1)^{-n}$. This completes the proof. □

Remark 4.2. Seemingly, Proposition 4.1 and [25, Proposition 16.9] look different. However, Proposition 4.1 is same as [25, Proposition 16.9] by replacing $2s$ in [25] to $s + (n+1)/2$ in this paper. In our notation, we consider modules character $\delta^{1/2}$ and a character $|a_p|^s$. In [25], the modules character are not considered and, moreover, we must replace $|a_p|^s$ to $|a_p|^{2s}$. Hence Proposition 4.1 states the same assertion as in [25, Proposition 16.9].

We consider the case of $n = 2$. We define the Shimura's Eisenstein series E^* on the Siegel upper half space $\mathfrak{H}_n$ by

$$E^*(g_\infty(\mathbf{i})) = j^{(n+3)/2}(g_\infty,\mathbf{i})E^*(g_\infty,1,\Phi).$$

Then the above explicit formula implies the following Corollary:

Corollary 4.3. Let $n = 2$. We wright the Fourier expansion of Shimura's Eisenstein series by

$$E^*(x+\sqrt{-1}\,y) = \sum_{h\geq 0} c(h,y)\exp(2\pi\sqrt{-1}\,\mathrm{tr}(h(x+\sqrt{-1}\,y))), \qquad x+\sqrt{-1}\,y \in \mathfrak{H}_n.$$

Then the following statements hold:

(1) If $c(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), y)$ is non-zero, we have $a, d \in (1/2)\mathbb{Z}$ and $b, c \in \mathbb{Z}$.

(2) If h is positive definite, the Fourier coefficient $c(h,y)$ is not depending on y, i.e., $c(h,y)$ is a constant.

(3) If $h = 0$, we have
$$c(0,y) = \frac{4}{\pi^2\det y}.$$

(4) If h is singular but non-zero, the Fourier coefficient $c(h,y)$ is a nearly holomorphic function on $\mathfrak{H}_n$. Moreover if h is of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & 0\end{smallmatrix}\right)$ and $c(h,y)$ is non-zero, we have $a = m^2/2$ for some integer m. We also have
$$c\left(\begin{pmatrix} 1/2 & 0 \\ 0 & 0\end{pmatrix}, y\right) = \frac{8(1+4\pi y_1)}{\pi^2\det y}$$
and
$$c\left(\begin{pmatrix} 2 & 0 \\ 0 & 0\end{pmatrix}, y\right) = \frac{8(1+16\pi y_1)}{\pi^2\det y},$$
where $y = \left(\begin{smallmatrix} y_1 & * \\ * & *\end{smallmatrix}\right)$.

Proof. The statement (1) follows immediately from the previous proposition. If the matrix h is singular, by the explicit formulas of the Siegel series [25, §16] the confluent hypergeometric functions [20], the statement (2), (3) and the first part of (4) are obvious. By the explicit formula of Siegel series [25, Proposition 16.10], we obtain the final assertion. For details, see the proof of [25, Proposition 17.6]. □

4.2. **Klingen Eisenstein series of degree 2.** In this section, we construct nearly holomorphic modular forms by Klingen Eisenstein series. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$. Let $G = \mathrm{Sp}_4$ and let K be a canonical maximal compact subgroup of $G(\mathbb{A})$, i.e.,

$$K = \prod_v K_v, \qquad K_v = \begin{cases} \mathrm{Sp}_4(\mathbb{Z}_v) & \text{if } v \text{ is non-archimedean} \\ \{\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{R})\} & \text{if } v \text{ is archimedean.}\end{cases}$$

Let P be the Klingen parabolic subgroup of G with a Levi decomposition $(\mathrm{GL}_1 \times \mathrm{SL}_2)N = MN = P$. For a Hecke character χ on $\mathbb{Q}^\times\mathbb{R}_+^\times\backslash\mathbb{A}^\times$ and an irreducible cuspidal automorphic representation π of $\mathrm{SL}_2(\mathbb{A})$, we denote by $I(s,\chi,\pi)$ the normalized induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} |\cdot|_{\mathbb{A}}^s\chi \boxtimes \pi$.

We review an automorphic realization of $I(s,\chi,\pi)$. Take $\phi \in I(s,\chi,\pi)$. By definition, we have a vector valued function $\phi(g) \in |\cdot|_{\mathbb{A}}^s\chi \boxtimes \pi$ for all $g \in G(\mathbb{A})$. We evaluate $\phi(g)$ at 1, the unit element of $M(\mathbb{A})$. Then, we get a scalar valued function F_ϕ defined by

$$F_\phi(g) = \phi(g)(1), \qquad g \in G(\mathbb{A}).$$

For elements $m \in M(\mathbb{A})$, $n \in N(\mathbb{A})$ and $k \in K$, we have

$$F_\phi(nmk) = \phi(nmk)(1) = \phi(k)(m).$$

For $k \in K$, we define a function $F_{\phi,k}$ on $M(\mathbb{A})$ by

$$F_{\phi,k}(m) = F_\phi(mk), \qquad m \in M(\mathbb{A}).$$

Then for every $k \in K$, a function $F_{\phi,k}$ belongs to the representation space of $|\cdot|_{\mathbb{A}}^s\chi \boxtimes \pi$. With the notation as in [12, section 2.1], the function F_ϕ belongs to $|\cdot|_{\mathbb{A}}^s\chi \boxtimes \pi$-isotypic component of the space of automorphic forms $A(N(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A}))_{|\cdot|_{\mathbb{A}}^s\chi\boxtimes\pi}$ on $N(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})$. By the above discussions, we can regard $I(s,\chi,\pi)$ as the set of scalar valued functions on $G(\mathbb{A})$ with certain properties. Hence we then can discuss the infinite sum of functions and its convergence in induced representations.

Take a standard section $\Phi_s \in I(s,\chi,\pi)$. By the automorphic realization, we may regard the function Φ_s as a scalar valued function on $G(\mathbb{A})$. We then consider the Eisenstein series $E(g,s,\Phi)$ defined by

$$E(g,s,\Phi) = \sum_{P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi_s(\gamma g), \qquad g \in G(\mathbb{A}).$$

We say that the Eisenstein series $E(g,s,\Phi)$ is Klingen Eisenstein series.

Let

$$w = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G(\mathbb{Q}).$$

Then, we have the intertwining operator $M_{s,w} : I(s,\chi,\pi) \longrightarrow I(-s,\chi,\pi)$. If the real part $\mathrm{Re}(s)$ is sufficiently large, the intertwining operator M_w is expressed by

$$(M_{s,w}\varphi)(g) = \int_{N(\mathbb{A})} \varphi(wng)\,dn, \qquad g \in G(\mathbb{A}). \tag{4.2}$$

It is well-known that $M_{s,w}$ is meromorphically continued to whole s-plane. For details, see [18, section 4]. For each place v, let $M_{s,w,v}$ be the local intertwining operator of $I_v(s,\chi_v,\pi_v) = \mathrm{Ind}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} |\cdot|^s\chi_v \boxtimes \pi_v$. Note that the global intertwining operator $M_{s,w}$ decomposes as the product of the local intertwining operators $\otimes_v M_{s,w,v}$.

Suppose that Φ_s is decomposable, i.e., $\Phi_s = \otimes_v \Phi_{s,v}$. Let S be the finite set of places such that $\infty \in S$ and if $p \notin S$, then $\Phi_{s,p}$ is unramified. By [18, Theorem 6.2.2], the constant term of $E(g,s,\Phi)$ along the Klingen parabolic subgroup P is equal to

$$\Phi_s + \frac{L^S(s, |\cdot|_{\mathbb{A}}^s\chi \boxtimes \pi, \mathrm{Ad})}{L^S(s+1, |\cdot|_{\mathbb{A}}^s\chi \boxtimes \pi, \mathrm{Ad})} \left(\bigotimes_{v \notin S} \Phi_{-s,v} \right) \otimes \left(\bigotimes_{v \in S} M_{s,w,v}\Phi_{s,v} \right). \tag{4.3}$$

It is easy to show that $L^S(s, |\cdot|_{\mathbb{A}}^s\chi \boxtimes \pi, \mathrm{Ad})$ is equal to the standard L-function $L^S(s,\pi,\chi)$ of π twisted by χ by the definition of the (partial) Adjoint L-function (cf. [18, Definition 2.3.5]). By [9, section 7], we have

- If $\chi^2 \neq 1$, then $L^S(s,\pi,\chi)$ is entire.
- If $\chi^2 = 1$, then $L^S(s,\pi,\chi)$ has a possible pole at $s = 1$. Moreover, if $L^S(s,\pi,\chi)$ has a pole at $s = 1$, then the pole is simple.

In order to construct Klingen Eisenstein series which generates indecomposable reducible modules, we consider an induced representation $I_\infty(s,\chi_\infty,\pi_\infty)$ at the archimedean place.

For a pair $(p,q) \in \mathbb{Z}^2$ with $p \geq q$, let $X(p,q)$ be the discrete series representation of $G(\mathbb{R})$ with minimal K_∞-type (p,q). Note that if $q \geq 3$, then $X(p,q)$ is the holomorphic discrete series representation of lowest K_∞-type (p,q). If $p \geq 0$ and $q < 0$, then $X(p,q)$ is a large discrete series representation. We remark that this "lowest weight representation" is corresponding to the "usual" choice of a positive root system. If we choose a positive root system as in section 2, any holomorphic discrete series representations will be a highest weight representation. As a $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module, the module $X(p,q)$ is isomorphic to the representation $L(p,q)$ if $q \geq 3$. Here, the module $L(p,q)$ is equal to the irreducible quotient of the generalized Verma module $N(p,q)$. The following Proposition is due to G. Muić.

Proposition 4.4 (Theorem 10.1 in [13])**.** Let k be an odd integer with $k \geq 3$. Then, the representation $I_\infty(1, \mathrm{sgn}, D_k^+)$ satisfies the non-split exact sequence

$$0 \longrightarrow X(k,3) \oplus X(k,-1) \longrightarrow I_\infty(1, \mathrm{sgn}, D_k^+) \longrightarrow X \longrightarrow 0.$$

Here, the irreducible infinite-dimensional representation D_k^+ is the lowest weight representation of $\mathrm{SL}_2(\mathbb{R})$ with lowest weight k and X is an irreducible representation of $G(\mathbb{R})$. Moreover, the representation X is isomorphic to $L(k,1)$, the unique irreducible quotient of the generalized Verma module $N(k,1)$, as a $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module.

The above module X is also considered in [17, Lemma 2.1]. We then get the following Corollary:

Corollary 4.5. A module $N(k,1)^\vee$ is contained in $I_\infty(-1, \mathrm{sgn}, D_k^+)$.

Proof. By [13, Lemma 6.1], we can find a vector v in $I_\infty(-1, \mathrm{sgn}, D_k^+)$ of weight $(k,3)$ such that $N_+ v = 0$. We denote by M the $\mathcal{U}(\mathfrak{g})$-module generated by v. By Proposition 4.4, we have a non-split exact sequence

$$0 \longrightarrow L(k,1) \longrightarrow M \longrightarrow L(k,3) \longrightarrow 0.$$

By the definition of the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ (cf. [6, section 9]), the module M is an object of $\mathcal{O}^{\mathfrak{p}}$. It is easy to show that the dimension of the space of extensions $\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(k,3), L(k,1))$ is 1 (cf. [6, section 3.12], [5, Lemma 3.5]). Since the exact sequence

$$0 \longrightarrow L(k,1) \longrightarrow N(k,1)^\vee \longrightarrow L(k,3) \longrightarrow 0$$

is non-split, the module M is isomorphic to $N(k,1)^\vee$. □

Suppose an archimedean component $\Phi_{s,\infty}$ of Φ_s has a K_∞-type $(k,3)$. Note that the K_∞-type (k,k) and $(k,3)$ is multiplicity one in $I_\infty(s, \mathrm{sgn}, D_k^+)$ by [13, Lemma 6.1]. Then, if $M_w \Phi_s$ is a non-zero finite function at $s = 1$, the constant term of Klingen Eisenstein series $E(g, s, \Phi)$ along P generates $N(k,1)^\vee$ as a $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module by Corollary 4.5. For $\infty \neq p \in S$, we may assume $M_{s,w,v}\Phi_{s,v}$ is non-zero finite function at $s = 1$. Hence, it remains to consider an archimedean component $M_{s,w,\infty}\Phi_{s,\infty}$. We call this factor the Gamma factor. In general, it is hard to compute the Gamma factor directly. We replace an archimedean component $\Phi_{s,\infty}$ by $\Phi'_{s,\infty}$ which has a K_∞-type (k,k). Then, there exists differential operators $\mathcal{D} \in \mathcal{U}(\mathfrak{sp}_4(\mathbb{C}))$ such that $\mathcal{D}\Phi_{s,\infty} = c(s)\Phi'_{s,\infty}$ with some meromorphic function $c(s)$. By Proposition 4.4 and Corollary 4.5, we may assume that $c(1)$ takes a non-zero finite value. Hence, it is sufficient to consider the Gamma factor with respect to $\Phi'_{s,\infty}$. Indeed we have $\mathcal{D}M_{w,\infty}\Phi_{1,\infty} = M_{w,\infty}\mathcal{D}\Phi_{1,\infty} = c(1)M_{w,\infty}\Phi'_{1,\infty}$. By Corollary 4.5 and the choice of $\mathcal{D}$, the function $\mathcal{D}M_{w,\infty}\Phi_{1,\infty}$ is zero if and only if $M_{w,\infty}\Phi_{1,\infty}$ is zero. We then can show that

$$M_{w,\infty}\Phi'_{s,\infty}(1) = \frac{c_0(s)\Gamma(s+k+3)\Gamma(s-k+1)\Gamma(s+1)}{\Gamma\left(\frac{s+k+2}{2}\right)^2 \Gamma\left(\frac{s-k+2}{2}\right)^2 \Gamma\left(\frac{s+k+1}{2}\right)\Gamma\left(\frac{s-k+3}{2}\right)},$$

where $c_0(s)$ is some function on $\mathbb{C}$ with $c_0(1) \neq 0$. By the above discussions, the archimedean component $M_{w,s,\infty}\Phi_{s,\infty}$ has a simple zero at $s = 1$. Note that $k \geq 3$ is odd. Summarizing the argument so far, the second term of (4.3) is non-zero.

Lemma 4.6. The constant term (4.3) generates $N(k,1)^\vee$ as a $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module at $s = 1$.

Proof. By Proposition 4.4 and Corollary 4.5, we may regard the constant term (4.3) at $s = 1$ as the sum $f + g \in L(k,3) \oplus N(k,1)^\vee$ where $f \in L(k,3)$ and $g \in N(k,1)^\vee$ are $\mathcal{U}(\mathfrak{k})$-dominant vectors of weight $(k,3)$. Take an element $P_{0,-} \in \mathcal{U}(\mathfrak{g})$ such that $P_{0,-}f$ and $P_{0,-}g$ are $\mathcal{U}(\mathfrak{k})$-dominant of weight $(k,1)$ (cf. [15, Table 1]). Then by (2.2) and the irreducibility of $N(k,3)$, $P_{0,-}f$ is zero and $P_{0,-}g$ is non-zero. We denote by τ and σ the $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module generated by $f + g$ and $P_{0,-}g$, respectively. Since $L(k,1)$ is the unique proper submodule in $N(k,1)^\vee$, σ is isomorphic to $L(k,1)$. In the quotient space τ/σ, $f + g$ generates highest weight module of weight $(k,3)$. Hence we obtain the exact sequence

$$0 \longrightarrow L(k,1) \longrightarrow \tau \longrightarrow L(k,3) \longrightarrow 0.$$

This shows that τ is isomorphic to $N(k,1)^\vee$ or $L(k,1) \oplus L(k,3)$ by $\dim \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(k,3), L(k,1)) = 1$. If τ is isomorphic to $L(k,1) \oplus L(k,3)$, this contradicts to the choice of g. This completes the proof. □

We then state the main theorem in this section.

Theorem 4.7. Let k be an odd integer with $k \geq 3$, χ a quadratic Hecke character and π an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ such that $L(s, \pi, \chi)$ has a simple pole at $s = 1$. Let $\Phi_s = \otimes_v \Phi_{s,v}$ be a standard section of $I(s, \chi, \pi)$. We denote by S the set of places such that $\infty \in S$ and if $p \notin S$, the local factor $\Phi_{s,p}$ is unramified. Suppose that

- $\chi_\infty = \mathrm{sgn}$ and $\pi_\infty = D_k^+$,
- for $\infty \neq p \in S$, the function $M_{w,s,p}\Phi_{s,p}$ are non-zero at $s = 1$, and,
- for an archimedean place, the local factor $\Phi_{s,\infty}$ has a K_∞-type $(k, 3)$.

Then, Klingen Eisenstein series $E(g, s, \Phi)$ is holomorphic at $s = 1$ as a function in s and generates $N(k, 1)^\vee$ as a $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module at $s = 1$.

Proof. Let Q be the Siegel parabolic subgroup of G. By [12, Proposition 2.I.7], the constant term of $E(g, s, \Phi)$ along Q is zero. We already showed that the constant term of $E(g, s, \Phi)$ along P is holomorphic at $s = 1$. Hence, by the result of Langlands [11], the Eisenstein series $E(g, s, \Phi)$ is holomorphic at $s = 1$, since the location of poles of Eisenstein series is equal to the location of poles of its constant terms.

Let Π be the $G(\mathbb{A}_{\mathrm{fin}}) \times (\mathfrak{g}, K_\infty)$-module generated by $E(g, s, \Phi)|_{s=1}$. Then the map

$$\Pi \longrightarrow I(1, \chi, \pi) \oplus I(-1, \chi, \pi) \colon \varphi \longmapsto \varphi_P$$

is injective, since the constant term $E(g, s, \Phi)_Q$ is zero. Hence the $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module generated by the Eisenstein series $E(g, s, \Phi)|_{s=1}$ is isomorphic to the $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module generated by its constant term along P. Such a constant term generates $N(1 + m, 1)^\vee$ as a $(\mathfrak{sp}_4(\mathbb{C}), K_\infty)$-module by Lemma 4.6. This completes the proof. □

By the theorem, we get nearly holomorphic Klingen Eisenstein series which generate indecomposable reducible modules.

To conclude this section, we discuss the analytic properties of $L(s, \pi, \chi)$ at $s = 1$. If $\chi^2 \neq 1$, $L(s, \pi, \chi)$ is entire by [9, Corollary 7.2.3]. Suppose $\chi^2 = 1$. Take an automorphic cuspidal representation τ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that $\tau|_{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})} \supset \pi$. By the definition of $L^S(s, \pi, \chi)$, we have

$$L(s, \pi, \chi) = \frac{L(s, (\tau \otimes \chi) \times \widetilde{\tau})}{L(s, \chi)},$$

where $\widetilde{\tau}$ is the contragredient representation of τ. Hence if $L(s, \pi, \chi)$ has a pole at $s = 1$, $L(s, (\tau \otimes \chi) \times \widetilde{\tau})$ has a pole at $s = 1$. Poles of $L(s, (\tau \otimes \chi) \times \widetilde{\tau})$ are simple and hence we may assume χ is non-trivial. We have $\tau \simeq \tau \otimes \chi$ and $\chi \neq 1$ by [4, Theorem 2.2], i.e., σ is monomial (see [8, p. 137]). By [10, Proposition 6.5], χ determines a quadratic extension E of $\mathbb{Q}$ and there exists a Hecke character μ of E such that σ is isomorphic to the automorphic representation attached to μ (see [4, p. 491]).

We translate the above discussion to the classical language. Let f be an elliptic cusp form of weight k and a character ω with respect to $\Gamma_1(N)$, i.e., f lies in $M_k(N, \omega)$ (see [2, p. 119]). Suppose f is a normalized eigenform with the Fourier expansion $f(z) = \sum_{n=1}^\infty c_n \exp(2\pi\sqrt{-1}\, nz)$. For a primitive Dirichlet character χ, we denote by $L(s, f, \chi)$ the standard L-function of f twisted by χ. This is equal to $D(s, f, \chi)$ as in [19]. Put

$$R(s) = \pi^{-3s/2}\Gamma(s/2)\Gamma((s+1)/2)\Gamma((s-k+2)/2)L(s, f, \chi).$$

Then $R(s)$ is holomorphic everywhere except possibly at $s = k$ and $k - 1$ by Shimura [19, Theorem 1]. We may regard ω and χ as Hecke characters of $\mathbb{Q}^\times\mathbb{R}_+^\times\backslash\mathbb{A}_\mathbb{Q}^\times$. Let π be the automorphic representation of $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$ generated by f. By [4, p. 496], we have

$$R(s + k - 1) = L(s, \pi, \omega\chi),$$

where $L(s, \pi, \omega\chi)$ is the completed L-function of π twisted by $\omega\chi$. For a place $v \in S$, the local L-function $L_v(s, \pi_v, \omega_v\chi_v)$ is holomorphic and non-zero at $s = 1$ (cf. [8, p. 137]). Note that, for the archimedean place ∞, the local representation π_∞ is temperd, by $\pi_\infty \cong D_k^+$. Hence, $L^S(s, \pi, \omega\chi)$ has a pole at $s = 1$ if and only if $L(s, f, \chi)$ has a pole at $s = k$. Put $g(z) = \sum_{n=1}^\infty \overline{\chi}(n)\overline{c_n} \exp(2\pi\sqrt{-1}\, nz)$. Then g is a modular form with respect to certain congruence subgroup. By [19, Theorem 2], $L(s, f, \chi)$ has a pole at $s = k$ if and only if the following conditions are satisfied:

- The Hecke character $\omega\chi$ is a non-trivial quadratic character.
- The Petersson inner product $\langle f, g\rangle$ is non-zero.

An example of f that $R(s)$ has a pole at $s = k$ may be found in [19, p. 97] .

5. Rankin-Cohen brackets

In this section, we construct some modular forms via Rankin-Cohen brackets.

5.1. **Definition.** Let X_+ and $D_\pm$ be the differential operators on $\mathfrak{H}_2$ defined in [15, section 3.3]. Let Γ be a congruence subgroup of $\mathrm{Sp}_4(\mathbb{Q})$. We denote by a subspace $N^*_{5/2}(\Gamma) \subset N_{5/2}(\Gamma)$ the space of modular forms which generates $N(1/2,1/2)^\vee$ or $L(5/2,5/2)$. Note that the set of modular forms which generates $N(1/2,1/2)$ is not a linear space. Indeed, for a holomorphic modular form $f \in M_{5/2}$ and a nearly holomorphic modular form g which generates $N(1/2)^\vee$, a modular form $f+g$ generates $N(1/2)^\vee$. Since f generates $L(5/2,5/2)$, a modular form $(f+g)-g$ does not generate $N(1/2,1/2)^\vee$. By Lemma 2.1 and (3.3), we can construct Rankin-Cohen bracket

$$[\,,\,]_m \colon M_{1/2}(\Gamma) \otimes N^*_{5/2}(\Gamma) \longrightarrow N_{\rho_{(m+1,3)}}(\Gamma)$$

by

$$f \otimes g \longmapsto \sum_{2i+2j=m, i,j \geq 0} a_{i,j}\, X_+^i f \otimes X_+^j g + \sum_{2i+2j=m, i,j\geq 0} b_{i,j}\, D_+(X_+^i f \otimes X_+^j D_- g)$$

for any positive even integer m. Here, the coefficients $a_{i,j}$ and $b_{i,j}$ are as in Lemma 2.1. Images of $[\,,\,]_m$ generate $N(m+1,1)^\vee$ or $L(m+1,3)$. Note that if an image $[f,g]_m$ generates $L(m+1,3)$, the modular form $[f,g]_m$ is holomorphic. Conversely, it is easy to show that if an image $[f,g]_m$ is holomorphic, the modular form $[f,g]_m$ generates $L(m+1,3)$. In order to construct a modular form which generates $N(m+1,1)^\vee$, we need to find a nearly holomorphic (but non-holomorphic) modular form $f \in N^*_{5/2}$. One of such forms is Shimura's Eisenstein series explained in section 4. In the next subsection, we give an example.

5.2. **Example.** Let E^* be Shimura's Eisenstein series of weight $5/2$ of degree 2 defined in section 4. We then get the following:

Proposition 5.1 (Theorem 5.5 in [5])**.** Let M be a $(\mathfrak{g}, K_\infty)$-module generated by the Shimura's Eisenstein series E^*. The module M is isomorphic to the contragredient module $N(1/2,1/2)^\vee$ as in section 2.

The Fourier expansion of $E^*(Z) = E^*(X + \sqrt{-1}\,Y)$ is given by

$$\frac{4}{\pi^2 \det Y)} + 8\left(\frac{1-4\pi y}{\pi^2 \det Y}\right)\exp(\pi i \tau) + 8\left(\frac{1-16\pi y}{\pi^2 \det Y}\right)\exp(4\pi i\tau) + \cdots,$$

where

$$Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}, \qquad Y = \begin{pmatrix} y & v \\ v & y' \end{pmatrix}.$$

By definition of D_-, we have a holomorphic modular form $D_- E^*$ of weight $1/2$ with the Fourier expansion

$$-24\pi^{-2} - 48\pi^{-2}\exp(\pi\sqrt{-1}\,\tau) - 48\pi^{-2}\exp(4\pi\sqrt{-1}\,\tau) + \cdots.$$

Then, the Fourier expansion of $\langle [D_-E^*, E^*]_m, v^*\rangle$ is

$$\pi^{-3}\frac{2^8\cdot 3}{m+2}(m-2+4(m+1)(-1)^{m/2+1})(-2\pi)^{m/2}\left(\frac{y}{\det Y}\right)\exp(\pi\sqrt{-1}\,\tau) + \cdots,$$

where v^* is a lowest weight vector of the dual representation of $\rho_{(m,1)}$. This is non-zero and non-holomorphic for all even positive integer m. Summarize these, we have the theorem as follows:

Theorem 5.2. With the above notation, the modular form $[D_-E^*, E^*]_m$ generates $N(m+1,1)^\vee$ for all even integer $m \geq 0$.

Remark 5.3. The author does not know the kernel and the image of $[\,,\,]_m$. By Theorem 5.2, we only show that the image of $[\,,\,]_m$ contains non-holomorphic modular forms. Hence, we have the following questions:

(1) Is $[\,,\,]_m$ injective or not? If it is not injective, how can we characterize the kernel?
(2) Is $[\,,\,]_m$ surjective or not? If it is not surjective, how can we characterize the image?

We can regard Rankin-Cohen brackets as a "product" of modular forms. Therefore, the images $[\,,\,]_m$ may have non-trivial cuspidal components. For example, in SL_2-case, the difference $E_6^2 - E_{12}$ is a constant multiple of the Ramanujan's delta function. Then, we have the following questions:

(3) What are cuspidal components of images $[f,g]_m$?
(4) Can we construct cuspidal forms via images $[f,g]_m$?

The author does not have any answers to these questions. We hope to consider these problem in some future.

5.3. **Remarks on Rankin-Cohen brackets.** To conclude the paper, we will give some remarks on Rankin-Cohen brackets. First we discuss a $(\mathfrak{g}, K_\infty)$-module generated by holomorphic Siegel modular forms. We will follow the notation as in section 2 and 3.

Let f be a holomorphic Siegel modular form of a factor of automorphy j_ρ. Suppose that the representation ρ of $\mathrm{GL}_n(\mathbb{C})$ is irreducible with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathbb{Z}}^+$. Recall the result of Weissauer [26, Satz 3]. Let

$$f(z) = \sum_{h \geq 0} c_h \exp(2\pi\sqrt{-1}\,\mathrm{tr}(hz))$$

be the Fourier expansion of f. We define the corank(f) by

$$\mathrm{Corank}(f) = \max_{h, c_h \neq 0}(n - \mathrm{rank}(h)).$$

Theorem 5.4 (Weissauer [26])**.** With the above notation, if a modular form f is a cusp form or satisfies $\mathrm{Corank}(f) \leq 2(n - \lambda_n)$, the modular form f is square-integrable.

If the modular form f is square integrable, the $(\mathfrak{g}, K_\infty)$-module M_f is irreducible. Since the weight of f is λ, we have a surjective map $N(\lambda) \longrightarrow M_f$ and hence we have the isomorphism

$$M_f \cong L(\lambda).$$

In particular, if $\mathrm{Corank}(f) \leq 2(n - \lambda_n)$ holds, the module M_f is irreducible. Hence, we may assume that the inequality $\mathrm{Corank}(f) > 2(n - \lambda_n)$ holds.

Recall the definition of first reduction point in the sense of [3]. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a highest weight such that $\mu_n = n$ holds. For a module $N(\mu + r(-1, \ldots, -1))$ with $r \in \mathbb{R}$, we say that a real number r_0 is the first reduction point if the module $N(\mu + r_0(-1, \ldots, -1))$ is reducible and a module $N(\mu + r(-1, \ldots, -1)$ is irreducible for $r < r_0$.

Theorem 5.5 (Theorem 2.10 in [3])**.** Let $\mu = (\mu_1, \ldots, \mu_n)$ be a $\mathfrak{k}$-dominant weight with $\mu_n = n$. Set $p(\lambda) = \#\{i \mid \lambda_i = n\}$ and $q(\lambda) = \#\{i \mid \lambda_i = \lambda_n + 1\}$. Then, the first reduction point r_0 equals to $(p(\lambda) + q(\lambda) + 1)/2$.

By this fact, we have the following:

Proposition 5.6. Let f be a holomorphic Siegel modular form of degree n of weight ρ_λ. Then, the module M_f is isomorphic to the irreducible highest weight representation $L(\lambda)$.

Proof. We may assume that $\mathrm{Corank}(f) > 2(n - \lambda_n)$ holds. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a $\mathfrak{k}$-dominant weight such that there exists r such that $\lambda = \mu + r(-1, \ldots, -1)$ and $\mu_n = n$. Note that such a μ exists uniquely and we have $p(\lambda) = p(\mu)$. By [26, Satz 1], we have

$$\mathrm{Corank}(f) \leq p(\lambda).$$

We then have

$$2(n - \lambda_n) < \mathrm{Corank}(f) \leq p(\mu).$$

Hence we have

$$n - \lambda_n < (p(\mu) + q(\lambda) + 1)/2.$$

The right hand side is just equal to the first reduction point. The left hand side satisfies

$$\mu + (n - \lambda_n)(-1, \ldots, -1) = \lambda.$$

Therefore the module $N(\lambda)$ is irreducible. By the universality of Verma modules, the module M_f is a quotient of $N(\lambda)$. This means that the module M_f is isomorphic to the irreducible quotient $L(\lambda)$. This completes the proof. □

By [3], we conclude that the module M_f is unitarizable for a holomorphic modular form f. Hence any tensor product $M_f \otimes M_g$ is unitarizable for holomorphic modular forms f and g. This means that if we would like to construct nearly holomorphic modular forms which generate indecomposable reducible module, we should consider a tensor product module $M_f \otimes M_g$ such that the module M_f or M_g is indecomposable and reducible.

For an object M of parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$, let λ be the highest weight of M. Our choice of positive roots are found in section 2. Then, if the weight λ satisfies $\lambda_n \geq n$ and M is indecomposable, the module M is isomorphic to $L(\lambda) \cong N(\lambda)$. Indeed, we have $\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(\lambda), N(\lambda)) = 0$. For objects M_1 and M_2 in $\mathcal{O}^{\mathfrak{p}}$, let $\lambda_1 = (\lambda_{1,1}, \ldots \lambda_{1,n})$ and $\lambda_2 = (\lambda_{2,1}, \ldots \lambda_{2,n})$ be highest weights of M_1 and M_2, respectively. Then, the weight $\lambda_1 + \lambda_2$ is the highest weight of $M_1 \otimes M_2$. Hence, if we puled like to get indecomposable reducible module by branching of a tensor product module $M_1 \otimes M_2$, we should assume

$\lambda_{1,n} + \lambda_{2,n} < n$. Under this condition, we have $\lambda_{1,n} < n/2$ or $\lambda_{2,n} < n/2$. If a modular form f is of weight λ with $\lambda_n < n/2$, the modular form f is a singular form. To sum it up, in order to obtain desired modular forms, we should consider singular forms and nearly holomorphic modular forms which generate an indecomposable reducible module. One of such the nearly holomorphic modular form is Shimura's Eisenstein series explained in section 4.

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