

On the Dade -Tasaka correspondence between blocks of finite groups

渡辺アツミ (Atumi Watanabe)

熊本大学大学院自然科学研究科

Graduate School of Science and Technology, Department of Mathematics, Faculty of
Science, Kumamoto University

1 Introduction

In this report we state a generalization of Tasaka's isotypy between blocks of finite groups obtained by the Dade character correspondence. Let p be a prime and $(\mathcal{K}, \mathcal{O}, k)$ be a p -modular system such that $\mathcal{K}$ is a splitting field for all finite groups which we consider in this talk. Let S denote $\mathcal{O}$ or k . For a finite abelian group F , we denote by $\hat{F}$ the character group of F and by $\hat{F}_q$ the subgroup of $\hat{F}$ of order q for $q \in \pi(F)$, where $\pi(F)$ is the set of all primes dividing $|F|$. Let G be a finite group and N be a normal subgroup of G . We denote by $\text{Irr}(G)$ the set of ordinary irreducible characters of G and $\text{Irr}^G(N)$ be the set of G -invariant irreducible characters of N . For $\phi \in \text{Irr}(N)$, we denote by $\text{Irr}(G|\phi)$ the set of irreducible characters χ of G such that ϕ is a constituent of the restriction χ_N of χ to N .

Hypothesis 1 *G is a finite group which is a normal subgroup of a finite group E such that the factor group $F = E/G$ is a cyclic group of order r . λ is a generator of $\hat{F}$. $E_0 = \{x \in E \mid \bar{x} \text{ is a generator of } F\}$ where $\bar{x} = xG$. E' is a subgroup of E such that $E'G = E$, $G' = G \cap E'$ and $E'_0 = E' \cap E_0$. Moreover $(E'_0)^\tau \cap E'_0$ is the empty set, for all $\tau \in E - E'$.*

Under the above hypothesis, in [2], E.C. Dade constructed a bijection between $\text{Irr}^E(G)$ and $\text{Irr}^{E'}(G')$ which is a generalization of the cyclic case of the Glauberman correspondence ([3] or, [6], Chap.13).

Theorem 1 ([2], Theorem 6.8, Theorem 6.9) *Assume Hypothesis 1 and $|F| \neq 1$. For each prime $q \in \pi(F)$, we choose some non-trivial character $\lambda_q \in \hat{F}_q$. There is a bijection*

$$\rho(E, G, E', G') : \text{Irr}^E(G) \rightarrow \text{Irr}^{E'}(G') \quad (\phi \mapsto \phi' = \phi_{(G')})$$

which satisfies the following conditions. If r is odd, then there are a unique integer $\epsilon_\phi = \pm 1$ and a unique bijection $\psi \mapsto \psi_{(E')}$ of $\text{Irr}(E|\phi)$ onto $\text{Irr}(E'|\phi')$ such that

$$(1.1) \quad \left(\prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \psi \right)_{E'} = \epsilon_\phi \prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \psi_{(E')},$$

for any $\psi \in \text{Irr}(E|\phi)$. If r is even, and we choose $\epsilon_\phi = \pm 1$ arbitrarily, then there is a unique bijection $\psi \mapsto \psi_{(E')}$ of $\text{Irr}(E|\phi)$ onto $\text{Irr}(E'|\phi')$ such that (1.1) holds for all $\psi \in \text{Irr}(E|\phi)$. In both cases we have

$$(\lambda\psi)_{(E')} = \lambda\psi_{(E')}$$

for any $\lambda \in \hat{F}$ and $\psi \in \text{Irr}(E|\phi)$. Furthermore, the resulting bijection is independent of the choice of the non-trivial character $\lambda_q \in \hat{F}_q$, for any $q \in \pi(F)$.

Assume Hypothesis 1. We call $\rho(E, G, E', G')$ the Dade correspondence, where $\rho(E, G, E', G')$ denotes the identity map of $\text{Irr}^E(G)$ when $|F| = 1$. Following the notations in [7], for $\phi' \in \text{Irr}^{E'}(G)$, we set $\phi'_{(G)} = \rho(E, G, E', G')^{-1}(\phi')$, and for $\psi' \in \text{Irr}(E'|\phi')$, we set $\psi'_{(E)} = \psi$ if $\psi' = \psi_{(E')}$. From (1.1) ψ' is a constituent of $(\lambda\psi'_{(E)})_{E'}$ for some $\lambda \in \hat{F}$, hence $\phi_{(G')}$ is a constituent of $\phi_{G'}$. In particular if ϕ is the trivial character of G , then $\phi_{(G')}$ is the trivial character of G' .

The Generalized Glauberman case: Let G and A be finite groups such that A is cyclic, A acts on G via automorphism and that $(|C_G(A)|, |A|) = 1$. We set $E = G \rtimes A$, $G' = C_G(A)$ and $E' = G' \times A \leq E$. By [2], Lemma 7.5, E, G, E' and G' satisfy Hypothesis 1. Moreover by [2], Proposition 7.8, if $(|A|, |G|) = 1$, then $\rho(E, G, E', G')$ coincides with the Glauberman correspondence.

Theorem 2 (Horimoto[4]) *Assume the generalized Glauberman case. Suppose that $p \nmid |A|$ and that a Sylow p -subgroup of G is contained in G' . Then there is an isotypy between $b(G)$ and $b(G')$ induced by the Dade correspondence where $b(G)$ is the principal block of G .*

Isotypy is a concept introduced in [1].

Hypothesis 2 *Assume Hypothesis 1. $(p, r) = 1$. b is an E -invariant block of G covered by r distinct blocks of E .*

Hypothesis 3 *Assume Hypothesis 1. $(p, r) = 1$. b' is an E' -invariant block of G' covered by r distinct blocks of E' .*

Theorem 3 (Tasaka [7], Theorem 5.5) *Assume Hypotheses 2 and 3, and r is a prime power. Moreover assume some $\phi \in \text{Irr}(b)$, $\phi_{(G')} \in \text{Irr}(b')$. If r is odd, or $r = 2$, or b is the principal block of G , then there is an isotypy between b and b' induced by the Dade correspondence.*

In this report we state that the arguments in [7] can be extended to the general case (see Theorem 8 below).

2 Dade correspondence and blocks

Let G be a finite group. We denote by $G_0(\mathcal{K}G)$ the Grothendieck group of the group algebra $\mathcal{K}G$. If L is a $\mathcal{K}G$ -module, then let $[L]$ denote the element in $G_0(\mathcal{K}G)$ determined by the isomorphism class of L . For $\phi \in \text{Irr}(G)$, we denote by $\check{\phi}$. For a block b of G , we denote by $\text{Irr}(b)$ the set of irreducible characters belonging to b , and by $\mathcal{R}_{\mathcal{K}}(G, b)$ the additive group of generalized characters belonging to b . For other notations, see [5] and [8].

Note that under the Hypothesis 2, any irreducible character in $\text{Irr}(b)$ is E -invariant.

Theorem 4 (see [7], Proposition 3.5)

- (i) Assume Hypothesis 2. Then $\{\phi_{(G')} | \phi \in \text{Irr}(b)\}$ is contained in a block $b_{(G')}$ of G' .
- (ii) Assume Hypothesis 3. Then $\{\phi'_{(G)} | \phi' \in \text{Irr}(b')\}$ is contained in a block $b'_{(G)}$ of G .

Assume Hypothesis 2. We denote by $\hat{b}_0$ a block of E covering b . For each $\phi \in \text{Irr}(b)$, we denote $\hat{\phi}$ by a unique extension of ϕ which belongs to $\hat{b}_0$. For any $i \in \mathbb{Z}$, we denote by $\hat{b}_i$ be the block of E which contains $\lambda^i \hat{\phi}$ where $\phi \in \text{Irr}(b)$.

Proposition 1 (see [7], Proposition 3.5, (3)) Assume Hypotheses 2 and 3, and assume $b' = b_{(G')}$ using the notation in Theorem 4. Then there exists a block $(\hat{b}_0)_{(E')}$ of E' such that $\text{Irr}((\hat{b}_0)_{(E')}) = \{(\hat{\phi})_{(E')} | \phi \in \text{Irr}(b)\}$. If r is odd, then $(\hat{b}_0)_{(E')}$ is uniquely determined, and if r is even, we have exactly two choices for $(\hat{b}_0)_{(E')}$.

With the notation in the above proposition, we denote by $(\hat{b}_i)_{(E')}$ the block of E' containing $\lambda^i(\hat{\phi})_{(E')}$ ($\phi \in \text{Irr}(b)$). Moreover, when r is even, we fix one of two $(\hat{b}_0)_{(E')}$.

3 Local structure

Lemma 1 ([7], Lemma 3.3) Assume $p \nmid r$. For a block b of G , b satisfies Hypothesis 2 if and only if there exists $s \in E_0$ such that $\widehat{C(s)}b$ is invertible in $Z(\mathcal{O}Eb)$.

Assume Hypothesis 2. By the above lemma and [7], Lemma 2.4, there exists an element $s \in E'_0$ such that $\widehat{C(s)}b \in Z(\mathcal{O}Eb)^\times$. Hence there exists a defect group D of b centralized by s , and hence contained in G' . Let $P \leq D$. Then by [7], Lemma 3.9, $C_E(P)$, $C_G(P)$, $C_{E'}(P)$ and $C_{G'}(P)$ satisfy Hypothesis 1. Here we note $F \cong C_E(P)/C_G(P)$. Let $e \in \text{Bl}(C_G(P), b)$. Then we see that $\text{Br}_P^{\mathcal{O}E}(\widehat{C(s)}b)e^* \in (Z(kC_E(P)e^*))^\times$. This implies that e is covered by r blocks of $C_E(P')$. Similarly assume Hypothesis 3. Let D' be a defect group of b' and $e' \in \text{Bl}(C_{G'}(P'), b')$ for a subgroup P' of D' . Then e' is covered by r blocks of $C_{E'}(P')$.

Theorem 5 (see [7], Proposition 3.11) Using the same notations as in Theorem 4 we have the following.

- (i) Assume Hypothesis 2. Let D be a defect group of b obtained in the above and let $P \leq D$. Let $e \in \text{Bl}(C_G(P), b)$. Then $e_{(C_{G'}(P))} \in \text{Bl}(C_{G'}(P), b_{(G')})$. In particular, $b_{(G')}$ have a defect group containing D .
- (ii) Assume Hypothesis 3. Let D' be a defect group of b' and let $P' \leq D'$. Let $e' \in \text{Bl}(C_{G'}(P'), b')$. Then $e'_{(C_G(P'))} \in \text{Bl}(C_G(P'), b'_{(G)})$. In particular, $b'_{(G)}$ have a defect group containing D' .

Assume Hypotheses 2 and 3, and $b' = b_{(G')}$. The Dade correspondence $\rho(E, G, E', G')$ gives a bijection between $\text{Irr}(b)$ and $\text{Irr}(b')$ by Theorem 4. By Theorem 5, b and b' have a common defect group D . Let (D, b_D) be a maximal b -Brauer pair. For $P \leq D$, let (P, b_P) be a b -Brauer pair contained in (D, b_D) . We set

$$(b_P)' = (b_P)_{(C_{G'}(P))}.$$

By the above theorem $(b_P)'$ is associated with b' and $(D, (b_D)')$ is a maximal b' -Brauer pair. The following holds.

Theorem 6 (see [7], Theorem 5.2) Assume Hypotheses 2 and 3, and assume $b' = b_{(G')}$. Then the Brauer categories $\mathbf{B}_G(b)$ and $\mathbf{B}_{G'}(b')$ are equivalent.

4 Perfect isometry and isotopy

Assume Hypotheses 2 and 3, and $b' = b_{(G')}$ using the notations in Theorem 4. With the notations in the previous section, we put

$$b_i = \sum_{l=0}^{r-1} (\hat{b}_l)_{(E')} \hat{b}_{l+i} \quad (\forall i \in \mathbf{Z}).$$

Then $(b_i)^2 = b_i$ and $b_i \in (\mathcal{O}Gb b')^{E'}$ for each i . For each prime $q \in \pi(F)$, let $\lambda_q \in \hat{F}_q$ be a non-trivial character as in Theorem 1. Set $l = |\pi(F)|$. Moreover we set for t ($1 \leq t \leq l$) distinct primes $q_1, q_2, \dots, q_t \in \pi(F)$

$$\lambda_{q_1} \cdots \lambda_{q_t} = \lambda^{m_{\{q_1, \dots, q_t\}}} \quad (m_{\{q_1, \dots, q_t\}} \in \mathbf{Z})$$

where λ is a generator of $\hat{F}$. Then we have

$$\prod_{q \in \pi(F)} (1 - \lambda_q) = 1 + \sum_{t=1}^l (-1)^t \sum_{\{q_1, \dots, q_t\} \subseteq \pi(F)} \lambda^{m_{\{q_1, \dots, q_t\}}}$$

where $\{q_1, \dots, q_t\}$ runs over the set of t -element subsets of $\pi(F)$.

Proposition 2 (see [7], Proposition 4.4) *With the above notations we have*

$$\begin{aligned} [b_0 \mathcal{K}G] + \sum_{t=1}^l (-1)^t \sum_{\{q_1, \dots, q_t\} \subseteq \pi(F)} [b_{m_{\{q_1, \dots, q_t\}}} \mathcal{K}G] \\ = \sum_{\phi \in \text{Irr}(b)} \epsilon_\phi [L_{\phi_{(G')}} \otimes_{\mathcal{K}} L_{\hat{\phi}}] \end{aligned}$$

in $G_0(\mathcal{K}(G' \times G))$.

From the above proposition and [1], Proposition 1.2, we have the following.

Theorem 7 (see [7], Theorem 4.5) *Assume Hypotheses 2 and 3, and that $b' = b_{(G')}$. Set $\mu = \sum_{\phi \in \text{Irr}(b)} \epsilon_\phi \phi_{(G')} \phi$. Then μ induces a perfect isometry $R_\mu : \mathcal{R}_{\mathcal{K}}(G, b) \rightarrow \mathcal{R}_{\mathcal{K}}(G', b')$ which satisfies $R_\mu(\phi) = \epsilon_\phi \phi_{(G')}$.*

Let D be a common defect group of b and b' . For $P \leq D$, R^P be the perfect isometry between $\mathcal{R}_{\mathcal{K}}(C_G(P), b_P)$ and $\mathcal{R}_{\mathcal{K}}(C_{G'}(P), (b_P)_{(C_{G'}(P))})$ obtained by the Dade correspondence.

Theorem 8 (see [7], Theorem 5.5) *Assume Hypotheses 2 and 3, and assume $b' = b_{(G')}$. Then b and b' are isotypic with the local system $(R^P)_{\{P(\text{cyclic}) \leq D\}}$.*

Example Suppose $p = 5$. Let $G = Sz(2^{2n+1})$, the Suzuki group, $A = \langle \sigma \rangle$ where σ is the Frobenius automorphism of G with respect to $\text{GF}(2^{2n+1})/\text{GF}(2)$. Set $G' = Sz(2) = C_G(A)$, $E = G \rtimes A$, $E' = G' \rtimes A$. Suppose that $5 \nmid 2n+1$. Then $(2n+1, |G'|) = 1$. Moreover a Sylow 5-subgroup of G has order 5. By the above theorem, the Dade correspondence gives an isotopy between $b(G)$ and $b(G')$. Moreover, if $5 \mid (2^{2n+1} + 2^{n+1} + 1)$, then $b(G)$ and $b(G')$ are splendidly Morita equivalent.

References

- [1] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, *Astérisque*, **181-182**(1990), 61-92.
- [2] E.C. Dade, A new approach to Glauberman's correspondence, *J. Algebra*, **270**(2003), 583-628.
- [3] G. Glauberman, Correspondences of characters for relatively prime operator groups, *Canad. J. Math.*, **20**(1968), 1465-1488.
- [4] H. Horimoto, The Glauberman-Dade correspondence and perfect isometries for principal blocks, preprint.
- [5] H. Nagao and Y. Tsushima, "Representations of Finite Groups", Academic Press, Boston, 1989.
- [6] I.M. Isaacs, "Character Theory of Finite Groups" Academic Press, New York, 1976.
- [7] F. Tasaka, On the isotopy induced by the Glauberman-Dade correspondence between blocks of finite groups, *J. Algebra*, **319**(2008), 2451-2470.
- [8] J. Thévenaz, "G-algebras and Modular Representation Theory", Oxford University Press, New York, 1995.