

Paris-Harrington Theory and reflection Principles

Reijiro Kurata (倉田 令二朗)

Department of Applied Science, Faculty of Engineering
Kyushu University, Fukuoka 812, Japan

Introduction

In their paper [PH], Paris and Harrington showed that in Peano Arithmetic PA Harrington principle (H) is equivalent to the uniform reflection principle RFN_{Σ_1} . Since uniform reflection principles RFN_{Σ_p} ($p = 1, 2, 3, \dots$) make a hierarchy over PA ($[Sm], *4, 1^a$), it is natural to ask for a hierarchy of extensions of (H) which corresponds to the hierarchy of reflection principles.

In order to prove the unprovability of (H) in PA, Paris and Harrington considered a theory T, showing that (H) implies $\text{Con}(T)$ and $\text{Con}(T)$ implies $\text{Con}(\text{PA})$ in PA. If one see the proof precisely, then we can find that (H) is equivalent to $\text{Mod}(T)_\omega$ and $\text{Mod}(T)_\omega$ implies $\text{Con}(T)$, where $\text{Mod}(T)_\omega$ means that every finite subset of T has a model on ω . We consider theories T_n ($n \in \omega$) and T_∞ , which are extensions of $T = T_0$. However all the sentences $\text{Mod}(T_n)_\omega$ ($n \in \omega$) and $\text{Mod}(T_\infty)$ become equivalent to (H). In addition, all of $\text{Con}(T_n)$ ($n \in \omega$) are equivalent to $\text{Con}(\text{PA})$. By considering T_n ($n \in \omega$) we cannot produce any hierarchy corresponding to RFN_{Σ_p} ($p = 1, 2, 3, \dots$).

Next we extend Harrington principle directly and define sentences (H_p) ($p = 1, 2, 3, \dots$) where (H_1) is (H). Then this hierarchy is the one just seeked for, since (H_p) is exactly equivalent to RFN_{Σ_p} for every $p = 1, 2, 3, \dots$. So the problem is solved in one sense. However, since the principles (H_p) are rather complicated in the view point of arithmetical and combinatorial formula, so it is desirable to find more simple hierarchies.

§1. Equiconsistency of PA and Theory T_n and equivalency of $\text{Mod}(T_n)_\omega$ to (H)

1.1 Definitions and notations

(1) PA ; Peano Arithmetic with μ -symbol.

(2) (H) ; Harrington Principle i.e.

$$\forall e \forall r \forall k \exists M (M \xrightarrow{*} (k)_r^e)$$

(3) Theory T_n ($n \in \omega$), T_∞

T_0 is the theory of T in [PH].

T_n ($n \geq 1$) is as follows;

Language of T_n ; $0, 1, +, \cdot, <$ and constants c_i ($i \in \omega$).

Axioms of T_n ;

(i) Defining equations for $0, 1, +, \cdot, <$ and the mathematical induction axioms for Σ_n -formulas.

(ii) $c_i^2 < c_{i+1}$.

(iii) For any $i < k, k'$ and Σ_n -formula $\phi(y, z)$ (where k, k' and z have the same length),

$$\forall y < c_i [\phi(y, c(k)) \leftrightarrow \phi(y, c(k'))].$$

T_∞ is obtained from T_n by changing the Σ_n -formulas into unrestricted ones.

Remark) $T_0 \subset T_n \subset T_{n+1} \subset T_\infty$.

1.2 Equiconsistency and conservativity.

Proposition 1. $PA \vdash \text{Con}(T_0) \rightarrow \text{Con}(PA)$ (2.2 in [FH]).

Proposition 2. $PA \vdash \text{Con}(PA) \rightarrow \text{Con}(T_\infty)$.

So T_n ($n \in \omega$), T_∞ , and PA are provably equiconsistent in PA.

Proposition 3. T_∞ is a conservative extension of PA.

Lemma. Let $\phi_j(y, z) \equiv \phi_j(y_1, \dots, y_m, z_1, z_2, \dots, z_n)$ ($j = 1, \dots, l$)

be a finite set of formulas in PA. For any number $k > n$, there is a sequence of terms $\tilde{c}_0, \dots, \tilde{c}_{k-1}$ in PA which satisfies

(i) $PA \vdash \tilde{c}_i^2 < \tilde{c}_{i+1}$ for $0 \leq i < k-1$ and

(ii) $PA \vdash \forall y < c_i [\phi_j(y, \tilde{c}(k)) \leftrightarrow \phi_j(y, \tilde{c}(k'))]$ $j = 1, \dots, l$ for $i < k$,

$k' < k$.

From this Lemma, we can derive Proposition 2 and 3 immediately. In [PH], both Proposition 2 for T_0 and the Lemma are not mentioned explicitly. But Dr. Uesu pointed out to me that the proofs of 2.10 and 2.11 of [PH] can be regarded the proof of the above Lemma. For, (H) is not provable in PA, but for each number e , the formula $\forall k \forall r \exists M (M \xrightarrow{*} (k)_r^e)$ is provable in PA. In this proof the fact that " ϕ_j is limited" is never used, so the Lemma holds for any formulas ϕ_j .

1.3 Models of finite subsets of T_n on ω .

Let $\text{Mod}(T_n)_\omega$ be the formula expressing "Every finite set of axioms of T_n has a model on ω ".

Proposition 4. For all $n \in \omega$, $PA \vdash (H) \leftrightarrow \text{Mod}(T_n)_\omega$

Proof) The Proposition 2.11 in [PH] leads to one direction,

$PA \vdash (H) \rightarrow \text{Mod}(T_n)_\omega$, by constructing a model. The axioms in (i) of T_n are also satisfied in this model, because the truth-definition for Σ_n -formulas can be constructed in PA itself. On the other hand, Paris and Harrington proved that $PA \vdash \text{Mod}(T_0)_\omega \rightarrow \text{RFN}_{\Sigma_1}$ and $PA \vdash \text{RFN}_{\Sigma_1} \leftrightarrow (H)$, so $PA \vdash \text{Mod}(T_n)_\omega \rightarrow (H)$.

$\text{Con}(T_n)$ and $\text{Mod}(T_n)_\omega$ give no hierarchy corresponding to RFN_{Σ_n} .

§2. Extensions of Harrington principle and Reflection principles

2.1 Harrington Principles and Reflection principles

In this section we suppose that PA and T have all the symbols of primitive recursive functions and their defining equations as axioms.

A Π_p -sentence ϕ of PA can be written as

$$\phi := \forall x_0 Qx_1 \dots Qx_{p-1} A(x_0, x_1, \dots, x_{p-1}) \quad (1)$$

where Q'_s are $\exists$ or $\forall$ alternately, and A is a quantifier free formula. Then define

$$\phi^*(z_0, z_1, \dots, z_{p-1}) := \forall x_0 < z_0 Qx_1 < z_1 \dots Qx_{p-1} < z_{p-1} A(x_0, x_1, \dots, x_{p-1}).$$

Definition 1. $M \xrightarrow[\ast]{\phi} (k)_r^e$

For $k, e, r, M \in \omega$ and a Π_p -sentence ϕ , $M \xrightarrow[\ast]{\phi} (k)_r^e$ is the following formula:

For every partition $P : [M]^e \rightarrow r$ there is a subset $Y \subseteq M$ ($Y = \{y_0, y_1, \dots, y_{q-1}\}$, $y_0 < y_1 < \dots < y_{q-1}$) such that

(i) Y is homogeneous for P ,

(ii) $\text{card}(Y) \geq k$,

(iii) Y is relatively large, i.e. $\text{card}(Y) \geq \min(Y)$, and

(iv) $\phi^*(y_0, y_1, \dots, y_{p-1})$ holds.

Note that $M \xrightarrow[\ast]{\phi} (k)_r^e$ is a primitive recursive formula.

Definition 2. (H_p)

For $p = 1, 2, \dots$, (H_p) means the following sentence:

For all k, e, r and for all true Π_p -sentence ϕ , there exists an $M \in \omega$ such that $M \xrightarrow[\ast]{\phi} (k)_r^e$.

If $p = 1$, since the condition (iv) holds trivially, (H_1) coincides with the Harrington principle (H) . Clearly (H_{p+1}) implies (H_p) .

Proposition 1.

(H_p) is a Π_{p+1} -sentence of PA.

Proof).

For every Π_p -sentence ϕ of the form (1) there is a number f such that ϕ is equivalent to the formula

$$\tau(f) := \forall x_0 Qx_1 \dots Qx_{p-1} T'_{p-1}(f, x_0, \dots, x_{p-1})$$

where T'_{p-1} is Kleene's T_{p-1} if Qx_{p-1} is $\exists x_{p-1}$ and is $\neg T_{p-1}$ if Qx_{p-1} is $\forall x_{p-1}$.

So (H_p) is expressed as

$$\begin{aligned} & \forall k \forall e \forall r \forall f [\forall x_0 Qx_1 \dots Qx_{p-1} T'_{p-1}(f, x_0, \dots, x_{p-1}) \\ & \rightarrow \exists M (M \xrightarrow[*]{\tau(f)} (k)_r^e)], \end{aligned}$$

which is a Π_{p+1} -sentence.

Definition 3.

For every true Π_p -sentence ϕ of the form (1) we define a finite sequence of arithmetical functions

$$f_1(x_0), f_3(x_0, x_2), f_5(x_0, x_2, x_4) \dots$$

by the following way:

$$f_1(x_0) = \mu x_1 Qx_2 \dots Qx_{p-1} A(x_0, x_1, \dots, x_{p-1})$$

$$f_3(x_0, x_2) = \mu x_3 Qx_4 \dots Qx_{p-1} A(x_0, f_1(x_0), x_2, \dots, x_{p-1})$$

$$f_5(x_0, x_2, x_4) = \mu x_5 Qx_6 \dots Qx_{p-1} A(x_0, f_1(x_0), x_2, f_3(x_0, x_2), x_4, x_5, \dots, x_{p-1})$$

...

We call $(f_1, f_3, \dots, f_s)$ the function sequence of ϕ .

Proposition 2.

Let ϕ be a true Π_p -sentence and $(f_1, f_3, f_5, \dots)$ be its function sequence.

Put the functions $f_1^*, f_3^*, f_5^* \dots$ as following:

$$f_1^*(y_0) = \max\{f_1(x_0), x_0 < y_0\}$$

$$f_3^*(y_0, y_2) = \max\{f_3(x_0, x_2); x_0 < y_0, x_2 < y_2\}$$

$$f_5^*(y_0, y_2, y_4) = \max\{f_5(x_0, x_2, x_4); x_0 < y_0, x_2 < y_2, x_4 < y_4\}$$

.....

Then the condition (iv) of Definition 1 is equivalent to:

$$(iv)' \quad f_1^*(y_0) < y_1, f_3^*(y_0, y_2) < y_3, f_5^*(y_0, y_2, y_4) < y_5, \dots$$

(The proof is obvious.)

Proposition 3.

(H_p) is equivalent to the sentence obtained from the definition of (H_p) by replacing the condition (iv) of $M \xrightarrow[\ast]{\phi} (k)_r^e$ with the following condition:

$$(iv)'' \quad \text{For all } i_0, i_1, \dots, i_{p-1} \in \omega$$

$$i_0 < i_1 < \dots < i_{p-1} < q-1 \rightarrow \phi^*(x_{i_0}, x_{i_1}, \dots, x_{i_{p-1}}).$$

Proof).

Use 2.9 in [PH].

2.2 Truth of (H_p)

Proposition 4.

- (i) (H_p) is true.
- (ii) For each e and each true Π_p -sentence ϕ

$$PA \vdash \forall k \forall r \exists M (M \xrightarrow[\ast]{\phi} (k)_r^e).$$

- (iii) $PA \vdash \text{RFN}_{\Sigma_p} \rightarrow (H_p).$

Proof) (Cf. 2.1 and 3.1 in [PH])

(i) Suppose H_p were false, construct the tree of counter examples $\langle P, M \rangle$, take an infinite path by König lemma, and put a homogeneous infinite set by infinite Ramsey theorem. Then we can find its finite subset that satisfies the conditions (i)-(iv) for Y in Definition 1.

(ii), (iii) Formalize the above proof.

2.4 Relation to reflection principles

Proposition 5. (2.4 in [PH])

For every model A of T there is a model $\mathcal{J}$ of PA such that for all prenex formula $\theta(y)$ in PA and for all $i < k$ and $a < c_i$.

$$\mathcal{J} \models \theta(a) \quad \text{iff} \quad A \models \theta^*(a, c(k)).$$

Proposition 6.

In $PA + (H_p)$ it is proved that for all true Π_p -sentence ϕ and finite subset S of T , $S + \{\phi^*(c_0, \dots, c_{p-1})\}$ has a model on ω .

Proof)

Similar to 2.11 on [PH].

Proposition 7.

$$PA + (H_p) \vdash \text{RFN}_{\Sigma_p}.$$

Proof)

In PA, suppose (H_p) and let ϕ be a true Π_p -sentence. By Proposition 6 and Compactness theorem $T + \{\phi^*(c_0, \dots, c_{p-1})\}$ has a model. Then by Proposition 5 $PA + \{\phi\}$ is consistent.

Formalizing the above discussion, we can obtain

$$PA + (H_p) \vdash \text{Tr}_p(\ulcorner \phi \urcorner) \rightarrow \neg \text{Pr}_{pA}(\ulcorner \neg \phi \urcorner). \quad (2)$$

where Tr_p is the partial truth-definition of order p (Cf. [Sm]). Let $\phi(a)$ be a Π_p -formula whose only free variable is a . Since for the sentences $\phi(\bar{n})$ for all numeral $\bar{n}$ (2) holds, we have

$$PA + (H_p) \vdash \text{Tr}_p(\ulcorner \phi(\dot{a}) \urcorner) \rightarrow \neg \text{Pr}_{pA}(\ulcorner \neg \phi(\dot{a}) \urcorner).$$

$$\text{And } PA \vdash \text{Tr}_p(\ulcorner \phi(\dot{a}) \urcorner) \leftrightarrow \phi(a) \quad (5.21 \text{ in } [Sc]),$$

so for all Σ_p -formula $\psi(a)$,

$$PA + (H_p) \vdash \text{Pr}_{pA}(\ulcorner \psi(\dot{a}) \urcorner) \rightarrow \psi(a).$$

Combining this proposition and Proposition 4 (iii), we have the following theorem.

Theorem.

$$PA \vdash (H_p) \leftrightarrow \text{RFN}_{\Sigma_p} \quad (p = 1, 2, 3, \dots).$$

References

- [PH] J. Paris and L. Harrington, A mathematical incompleteness in Peano Arithmetic (Handbook of Mathematical Logic, D.8.), 1977.
- [Sm] C. Smorynski, The incompleteness theorem (ibid. D.1.).
- [Sc] H. Schwichtenberg, Proof theory: Some applications of cut elimination (ibid. D.2.).