

TANGENTIAL REPRESENTATIONS ON A SPHERE

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1. INTRODUCTION

Let G be a finite group. The Smith problem is as follows. Let Σ be a homotopy sphere with smooth G -action such that Σ has just two fixed points, say a and b . Are tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ isomorphic as real G -modules? Two real G -modules U and V are called *Smith equivalent* if there exists a smooth action of G on a sphere Σ such that $S^G = \{a, b\}$, $T_a(\Sigma) \cong U$ and $T_b(\Sigma) \cong V$ as real G -modules. We know infinitely many Oliver groups possessing non-isomorphic Smith equivalent real modules. We consider about the subset $\text{Sm}(G)$ of the real representation ring $\text{RO}(G)$ of G consisting of all differences $U - V$ of Smith equivalent real G -modules. Recently we have several results corresponding to the Smith set. In this note, we study a sufficient condition for the Smith set to be an additive subgroup of the real representation ring $\text{RO}(G)$. This work is a continuous study from [24].

2. SMITH PROBLEM

The Smith problem asks whether the Smith set $\text{Sm}(G)$ is zero or not. There are many results corresponding to the Smith problem.

Atiyah and Bott [1] or Milnor [7] showed that for a homotopy sphere Σ with semi-free smooth compact Lie group with just two fixed points, the tangential representations are isomorphic. Thus, any Smith equivalent real modules over an abelian simple group are isomorphic, that is, $\text{Sm}(C) = 0$ for a prime order cyclic group C . Sanchez [18] generalized the result as follows by computing G -signature and using Franz-Bass's theorem. For a cyclic group P of odd prime power order, Smith equivalent real P -modules are isomorphic. Therefore $\text{Sm}(P) = 0$ for any group P of odd prime power order by combining the Smith theory.

On the other hand, Cappell and Shaneson [2] showed that there exists non-isomorphic, Smith equivalent real module over a cyclic group C_{4n} of order $4n$ for $n \geq 2$, that is, $\text{Sm}(C_{4n}) \neq \{0\}$. Petrie [17] showed that the Smith set of an abelian group of odd order which has at least four non-cyclic subgroups is nontrivial, eg. $\text{Sm}(C_{pqrs} \times C_{pqrs}) \neq 0$,

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where p, q, r, s are distinct odd primes. And in 1980's, Dovermann, Suh, Masuda, etc. studied the Smith equivalent real modules.

Oliver [13] showed that G acts smoothly on a disk without fixed points if and only if there are no subgroups P and H such that P is a p -group, H/P is cyclic, G/H is a q -group for some primes p and q , possibly $p = q$. A group acting on a disk without fixed points is called an Oliver group. Laitinen and Morimoto [5] showed that G is an Oliver group if and only if there exists a one fixed point G -action on sphere. Laitinen and Pawałowski [6] showed that there exists Smith equivalent, non-isomorphic real G -modules for a perfect group G with $r_G \geq 2$ by connecting sum with a sphere with just one fixed point, where r_G is the number of real conjugacy classes of elements of G not of prime power order. After that, Pawałowski and Solomon [14] extended to that $\text{Sm}(G) \neq 0$ if G is a gap Oliver group with $r_G \geq 2$ except $\text{Aut}(A_6)$ and $\text{P}\Sigma\text{L}(2, 27)$. A group G is a gap group if there exists a real G -module V such that

- $\dim V^L = 0$ for any prime power index subgroup L of G and
- for any subgroups P of prime power order and H with $H > P$,

$$\dim V^P \geq 2 \dim V^H.$$

In particular, a perfect group G with $r_G \geq 2$ is a gap Oliver group. A study for gap groups is seen in [12, 19, 20, 22, 23].

Now we need some notations. A real conjugacy class $(x)^\pm$ of an element x of G is the union of the conjugacy class

$$(x) = \{g^{-1}xg \mid g \in G\}$$

of x and one of its inverse x^{-1} . We denote by $\text{NPP}(G)$ the set of elements of G not of prime power order, by $\overline{\text{NPP}}(G)$ the set of elements of the real conjugacy classes of elements of $\text{NPP}(G)$. Then r_G is the cardinality of the set $\overline{\text{NPP}}(G)$. For a prime p , let $\mathcal{N}_p(G)$ be the set of normal subgroups N of G with $[G : N] \leq p$. We denote by $\text{RO}(G)$ the real representation ring, by $\mathcal{P}(G)$ the set of all subgroups of G of prime power, possibly 1, order, by $O^p(G)$ the Dress subgroup of type p for a prime p defined as

$$O^p(G) = \bigcap_{L \leq G; [G:L]=p^a} L,$$

and by $\mathcal{L}(G)$ the set of all prime power, possible 1, index subgroups of G . Then for $L \in \mathcal{L}(G)$, L contains $O^p(G)$ for some prime p . We put

$$\cap p(G) = \cap_{N \in \mathcal{N}_p(G)} N$$

which quotient is an elementary abelian p -group and denote by G^{nil} the smallest normal subgroup of G by which quotient is nilpotent:

$$G^{\text{nil}} = \bigcap_p O^p(G).$$

Note that

$$G \geq \cap p(G) \geq O^p(G) \geq G^{\text{nil}}.$$

For families $\mathcal{F}_1$ and $\mathcal{F}_2$ of subgroups of G and a subset A of $\text{RO}(G)$, we put

$$A_{\mathcal{F}_1} = \bigcap_{P \in \mathcal{F}_1} \ker(\text{Res}_P^G: \text{RO}(G) \rightarrow \text{RO}(P)) \cap A,$$

$$A^{\mathcal{F}_2} = \bigcap_{L \in \mathcal{F}_2} \ker(\text{Fix}^L: \text{RO}(G) \rightarrow \text{RO}(N_G(L)/L)) \cap A,$$

and

$$A_{\mathcal{F}_1}^{\mathcal{F}_2} = A_{\mathcal{F}_1} \cap A^{\mathcal{F}_2} = \bigcap_{P \in \mathcal{F}_1} \ker \text{Res}_P^G \cap \bigcap_{L \in \mathcal{F}_2} \ker \text{Fix}^L \cap A.$$

The automorphism group $\text{Aut}(A_6)$ of the alternating group A_6 is not a gap group, $r_{\text{Aut}(A_6)} = 2$, and $\text{Sm}(G) = 0$ [8]. Morimoto [8] gave a condition

$$\text{Sm}(G) \subset \text{RO}(G)^{\mathcal{N}_2(G)} = \text{RO}(G)^{\cap 2(G)}$$

for Smith equivalent real modules. The rank of $\text{RO}(G)^{\mathcal{N}_2(G)}$ is equal to

$$r_G - r_{G, \cap 2(G)},$$

where $r_{G, \cap 2(G)}$ is the cardinality of the set $\pi(\overline{\text{NPP}}(G))$ for a canonical projection $\pi: G \rightarrow G/\cap 2(G)$ (cf. [14]). This condition implies that there are Oliver solvable groups G such that $r_G \geq 2$ and $\text{Sm}(G) = 0$ [15]. The group $\text{P}\Sigma\text{L}(2, 27)$ is an extension of $\text{PSL}(2, 27)$ by a field automorphism group of order 3 which is a gap non-solvable group, $r_{\text{P}\Sigma\text{L}(2, 27)} = 2$ and $\text{Sm}(\text{P}\Sigma\text{L}(2, 27)) \neq 0$ [9]. Moreover, putting together with [16], for an Oliver non-solvable group G with $r_G \geq 2$, $\text{Sm}(G) = 0$ if and only if G is isomorphic to $\text{Aut}(A_6)$.

3. SUBSETS OF THE SMITH SET

Sanchez's criterion and Petrie's observation says that

$$\text{Sm}(G) \subset \text{RO}(G)_{\mathcal{P}_o(G)}^{(G)},$$

where $\mathcal{P}_o(G)$ is the set of subgroups of G of order 1, 2, 4, or odd prime power. Thus we have

$$\text{Sm}(G) \subset \text{RO}(G)_{\mathcal{P}_o(G)}^{\mathcal{N}_2(G)}.$$

Note that if G has no element of order 8 then $\mathcal{P}_o(G) = \mathcal{P}(G)$. Recall that two real G -modules U and V are Smith equivalent if there exists a smooth action of G on a sphere Σ such that $S^G = \{a, b\}$, $T_a(\Sigma) \cong U$ and $T_b(\Sigma) \cong V$ as real G -modules and put

$$\text{Sm}(G) = \{[U] - [V] \mid U \text{ and } V \text{ are Smith equivalent}\}.$$

Similarly we consider the sets $\text{PSm}^c(G)$ (resp. $\text{LSm}(G)$) of all differences $[U] - [V]$ such that U and V are Smith equivalent and in addition the homotopy sphere Σ satisfies that Σ^P is connected for any prime power order subgroups P of G (resp. for any 2-groups of G).

The set $\text{PSm}^c(G)$ (resp. $\text{LSm}(G)$) is empty if and only if G is of order prime power (resp. 2-power). It holds the inclusions

$$\text{PSm}^c(G) \subset \text{LSm}(G) \subset \text{Sm}(G)$$

and

$$\text{LSm}(G) \subset \text{RO}(G)_{\mathcal{P}(G)}.$$

$$\begin{array}{ccccc}
 \text{PSm}^c(G) & & & & \\
 \cap \downarrow & & & & \\
 \text{LSm}(G) & & \text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} & & \\
 \cap \downarrow & & \cap \downarrow \exists \neq & & \\
 \text{Sm}(G)_{\mathcal{P}(G)} & \xrightarrow{\subset} & \text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{N}_2(G)} = \text{RO}(G)_{\mathcal{P}(G)}^{\{\cap 2(G)\}} & \xrightarrow{\subset} & \text{RO}(G)_{\mathcal{P}(G)}^{(G)} \\
 \cap \downarrow \exists \neq & & \cap \downarrow \exists \neq & & = \downarrow \\
 \text{Sm}(G) & \xrightarrow{\subset} & \text{RO}(G)_{\mathcal{P}_o(G)}^{\mathcal{N}_2(G)} & \xrightarrow{\subset} & \text{RO}(G)_{\mathcal{P}(G)}^{(G)}
 \end{array}$$

TABLE 1. Diagram of inclusions

Theorem 3.1. *Let G be an Oliver group whose nil-quotient G/G^{nil} is not a 2-group. Then*

$$\text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} \subset \text{PSm}^c(G).$$

Moreover, we have

Theorem 3.2. *Let G be an Oliver non-gap group with $[G : O^2(G)] = 2$. Suppose that all elements x of $G \setminus O^2(G)$ of order 2 such that $C_G(x)$ is not a 2-group. Then*

$$\text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} \subseteq \text{PSm}^c(G).$$

We denote by $\text{SG}(m, n)$ the small group of order m and type n which is obtained as `SmallGroup(m, n)` in the software GAP [3]. Morimoto studied (or is studying) the set $\text{Sm}(G)_{\mathcal{P}(G)} \setminus \text{Sm}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)}$. He [9] showed that for $G = \text{P}\Sigma\text{L}(2, 27)$, $\text{SG}(864, 2666)$, $\text{SG}(864, 4666)$, $\text{Sm}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} = 0$ but $\text{Sm}(G)_{\mathcal{P}(G)} = \text{Sm}(G) \cong \mathbb{Z}$. Also he and his colleagues [4] showed that if a Sylow 2-subgroup is normal, then

$$\text{Sm}(G) \subset \text{RO}(G)^{\mathcal{N}_3(G)}$$

and in particular $\text{Sm}(G) = 0$ holds for $G = \text{SG}(1176, 220)$, $\text{SG}(1176, 221)$.

For an Oliver group, we see $\text{PSm}^c(G) \neq 0$ to show $\text{Sm}(G)_{\mathcal{P}(G)} \neq 0$. We have no rich examples so that $\text{Sm}(G)_{\mathcal{P}(G)} \neq \text{Sm}(G)$, whole $\text{Sm}(G) \setminus \text{Sm}(G)_{\mathcal{P}(G)}$ is a finite set. We do

not have an example for an Oliver group G such that $\text{PSm}^c(G) \neq \text{Sm}(G)_{\mathcal{P}(G)}$. It remains the problem whether $\text{PSm}^c(G) = \text{Sm}(G)_{\mathcal{P}(G)}$ for an Oliver group.

4. CRITERION FOR THE SMITH SET TO BE A GROUP

We discuss for Oliver groups G such that $\text{PSm}^c(G)$ is a subgroup of $\text{RO}(G)$. We introduce two condition. One is a part of a sufficient condition to show $\text{Sm}(G)_{\mathcal{P}(G)} \setminus \text{Sm}(G)^{\mathcal{L}(G)} \neq 0$ and the other is a sufficient condition so that $\text{Sm}(G)_{\mathcal{P}(G)}$ is a group.

Let $Q = \bigcap_{p \neq 2} O^p(G)$ be a normal subgroup of G with odd index and let N be a normal subgroup of G with $G^{\text{nil}} \leq N \leq \cap 2(G) \cap Q$. Then

$$Q \geq \cap 2(G) \cap Q \geq N \geq G^{\text{nil}} \geq O^2(Q).$$

Definition 4.1. We say that G satisfies the *quasi- N - $\mathcal{P}$ -condition* if there are real Q -modules U and V such that

- $\dim U^{\cap 2(G) \cap Q} = \dim V^N = 0$ and
- $[\mathbb{R} \oplus U] - [V] \in \text{RO}(Q)_{\mathcal{P}(Q)}$.

In particular, the quasi- G^{nil} - $\mathcal{P}$ -condition is simply called *quasi- Nil - $\mathcal{P}$ -condition*.

Definition 4.2. We say that G satisfies the *weak- Nil - $\mathcal{P}$ -condition* if there are real G -modules U and V such that

- $\dim U^{\cap 2(G)} = \dim V^{G^{\text{nil}}} = 0$ and
- $[\mathbb{R} \oplus U] - [V] \in \text{RO}(G)_{\mathcal{P}(G)}$.

Lemma 4.3. *If G satisfies the quasi- Nil - $\mathcal{P}$ -condition, then G satisfies the weak Nil - $\mathcal{P}$ -condition.*

Proposition 4.4 (cf. [10, Lemma 15]). *Let G be a finite group with $O^2(G) = G$. The following statements are equivalent.*

- (1) G^{nil} has a sub-quotient isomorphic to D_{2pq} for distinct primes p, q .
- (2) G satisfies the quasi- Nil - $\mathcal{P}$ -condition.

Morimoto and Qi [11] obtained a sufficient condition for an Oliver group G to hold that $\text{Sm}(G)_{\mathcal{P}(G)}$ is not equal to $\text{Sm}(G)^{\mathcal{L}(G)}$. This result supplies that $\text{Sm}(G) = \text{Sm}(G)_{\mathcal{P}(G)} \cong \mathbb{Z}$ for $G = \text{SG}(864, 2666)$ or $\text{SG}(864, 4666)$. For $G = \text{SG}(864, 2666)$ or $\text{SG}(864, 4666)$, G/G^{nil} is a cyclic group of order 3 and $\text{RO}(G)_{\mathcal{P}(G)}$ is generated by two element $\mathbb{R}[G/G^{\text{nil}}] + X_1$ and $3(\mathbb{R}[G/G^{\text{nil}}] - \mathbb{R}) + X_2$ for some elements $X_1, X_2 \in \text{RO}(G)^{(G^{\text{nil}})}$ and thus, G satisfies the weak- Nil - $\mathcal{P}$ -condition since G/G^{nil} is a cyclic group of order 3. We see it in the next section. Indeed, G has a sub-quotient isomorphic to D_{12} and G satisfies the quasi- Nil - $\mathcal{P}$ -condition.

Definition 4.5. For a normal subgroup N of G , we say that G satisfies the *N - $\mathcal{P}$ -condition* if there are real G -modules U and V such that $U^N = V^N = 0$ and $[\mathbb{R} \oplus U] - [V] \in \text{RO}(G)_{\mathcal{P}(G)}$. If $N = G^{\text{nil}}$ we say that G satisfies the *Nil - $\mathcal{P}$ -condition*.

Lemma 4.6 or Theorem 4.8 in [9] essentially yields us the following two theorems.

Theorem 4.6. *If a gap Oliver group G satisfies the weak- $\text{Nil-}\mathcal{P}$ -condition with $\text{NPP}(G) \cap G^{\text{nil}} \neq \emptyset$ and has an element of $\text{NPP}(G)$ outside $O^p(G)$ for some prime p , then*

$$\text{PSm}^c(G) \setminus \text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} \neq \emptyset.$$

Note that under the assumption that $\text{NPP}(G) \cap G^{\text{nil}} \neq \emptyset$ the inequality $\text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{N}_2(G)} \neq \text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)}$ if and only if $\text{NPP}(G) \setminus O^p(G)$ is not empty for some prime p . By using the multiplication of $\text{RO}(G)$, we get the following theorem.

Theorem 4.7. *Let G be a gap Oliver group satisfying the $\text{Nil-}\mathcal{P}$ -condition. Then*

$$\text{PSm}^c(G) = \text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{N}_2(G)} = \text{Sm}(G)_{\mathcal{P}(G)}$$

and in particular $\text{Sm}(G)_{\mathcal{P}(G)}$ is an additive group.

If a Sylow 2-subgroup of G is normal, G does not satisfy the $\text{Nil-}\mathcal{P}$ -condition. Although the $\text{Nil-}\mathcal{P}$ -condition is a sufficient one for an Oliver group G such that $\text{Sm}(G)_{\mathcal{P}(G)}$ is an additive group, it is not a necessary condition. For example, $A_5 \times C_4$ does not satisfy the $\text{Nil-}\mathcal{P}$ -condition but the following result holds.

Proposition 4.8. $\text{PSm}^c(A_5 \times C_4) = \text{Sm}(A_5 \times C_4) = \text{RO}(A_5 \times C_4)^{(A_5)}.$

Problem. $\text{PSm}^c(A_5 \times (C_4)^n) = \text{Sm}(A_5 \times (C_4)^n)$ holds. Is it true that $\text{PSm}^c(A_5 \times (C_4)^n) = \text{RO}(A_5 \times (C_4)^n)^{(A_5 \times (C_2)^n)}$ for $n \geq 2$?

5. QUASI- $\text{Nil-}\mathcal{P}$ -CONDITION

In this section we study properties for the weak- $\text{Nil-}\mathcal{P}$ -condition. Remark that there is an Oliver group which satisfies the weak- $\text{Nil-}\mathcal{P}$ -condition but does not satisfy the $\text{Nil-}\mathcal{P}$ -condition (eg. $\text{SG}(864, 2666)$, $\text{SG}(864, 4666)$).

Proposition 5.1. *Let K be a subgroup of G such that $\cap 2(G) \cdot K = G$. If K satisfies the weak- $(G^{\text{nil}} \cap K)$ - $\mathcal{P}$ -condition, then G satisfies the weak- $\text{Nil-}\mathcal{P}$ -condition.*

Theorem 5.2. *Let G be a gap Oliver group. Suppose that $\text{NPP}(G) \cap G^{\text{nil}}$ is not empty and that there is an element $\text{NPP}(G)$ outside of $O^p(G)$ for some prime p . If an odd index subgroup K of G satisfies the weak- $(G^{\text{nil}} \cap K)$ - $\mathcal{P}$ -condition, then*

$$\text{PSm}^c(G) \setminus \text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)} \neq \emptyset.$$

Morimoto and Qi [10, Lemma 21 and Theorem 22] showed that $\text{Sm}(G)_{\mathcal{P}(G)} \neq \text{Sm}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)}$ for an odd integer $n > 1$, an odd prime p , and $G = D_{2n} \wr C_p$, the wreath product of the

dihedral group D_{2n} of order $2n$ by a cyclic group C_p of order p . The group G satisfies the assumption of Proposition 5.1 as follows. The group G has a presentation

$$\begin{aligned} & a_i^n = b_i^2 = (a_i b_i)^2 = 1, \quad (\forall i), \\ \langle a_1, b_1, \dots, a_p, b_p, c \mid & a_i a_j = a_j a_i, a_i b_j = b_j a_i, b_i b_j = b_j b_i, \quad (i \neq j), \quad \rangle, \\ & c^p = 1, c^{-1} a_i c = a_{i+1}, c^{-1} b_i c = b_{i+1}, \quad (\forall i) \end{aligned}$$

where $a_{p+1} = a_1$ and $b_{p+1} = b_1$. The group G^{nil} is a subgroup of G generated by elements $a_1, \dots, a_p$ and $b_i b_j$ ($i < j$), and then $G/G^{\text{nil}} \cong C_{2p}$. Thus G is a gap Oliver group. Put $K = O^p(G)$. Let $f: D_{2n}^p \rightarrow D_{2n}$ be the first projection and let $\hat{U}$ and $\hat{V}$ be $\mathcal{P}(D_{2n})$ -matched real D_{2n} -modules such that $\hat{U}^{D_{2n}} = \mathbb{R}$ and $\hat{V}^{D_{2n}} = 0$. The real K -modules $f^* \hat{U}$ and $f^* \hat{V}$ implies that K satisfies the assumption of Proposition 5.1 since $f(G^{\text{nil}}) = D_{2n}$. (Or directly, two real G -modules $\text{Ind}_K^G f^* \hat{U}$ and $\text{Ind}_K^G f^* \hat{V}$ implies that G satisfies the weak- $\text{Nil-}\mathcal{P}$ -condition.)

Before closing this section, we should say the strongness of the weak- $\text{Nil-}\mathcal{P}$ -condition. Let G be a finite group such that G/G^{nil} is a nilpotent group of odd order and there are an element of G^{nil} not of prime power order and an element of G outside G^{nil} not of prime power order. Then

$$\text{RO}(G)^{\{G^{\text{nil}}\}}_{\mathcal{P}(G)} \neq \text{RO}(G)^{\{G\}}_{\mathcal{P}(G)}.$$

Note that if a Sylow 2-subgroup of G is normal then $\text{Sm}(G) \subset \text{RO}(G)^{\{N_s(G)\}}_{[s]}$ (cf. [4]) and G does not satisfy the weak- $\text{Nil-}\mathcal{P}$ -condition. Otherwise, if G has a sub-quotient isomorphic to D_{2qr} for some distinct primes q and r , there are real G -modules U and V such that the equalities $U^{G^{\text{nil}}} = 0 = V^{G^{\text{nil}}}$ hold and that $\mathbb{R}[G/G^{\text{nil}}] \oplus U$ and V are $\mathcal{P}(G)$ -matched:

$$\mathbb{R} + [(\mathbb{R}[G/G^{\text{nil}}] - \mathbb{R}) \oplus U] - [V] = \mathbb{R}[G/G^{\text{nil}}] + [U] - [V] \in \text{RO}(G)_{\mathcal{P}(G)}.$$

Thus, G satisfies the weak- $\text{Nil-}\mathcal{P}$ -condition and in addition if G is a gap Oliver group then

$$\text{PSm}^c(G)^{\{G^{\text{nil}}\}} \neq \text{PSm}^c(G).$$

6. Nil- $\mathcal{P}$ -CONDITION

In this section we study properties for the Nil- $\mathcal{P}$ -condition.

Proposition 6.1. *If G satisfies the Nil- $\mathcal{P}$ -condition, then G satisfies the weak- $\text{Nil-}\mathcal{P}$ -condition.*

Proposition 6.2. *If a quotient group of G satisfies the Nil- $\mathcal{P}$ -condition, then G also satisfies the Nil- $\mathcal{P}$ -condition.*

Proposition 6.3. *Let N be a normal subgroup of G . If there are a subgroup K of G and an epimorphism $f: K \rightarrow H$ such that $f(K \cap N) = H$, $KN = G$ and H has sub-quotient isomorphic to D_{2pq} , where p and q are distinct primes, then G satisfies the N - $\mathcal{P}$ -condition.*

For a perfect group G , the weak- $\text{Nil-}\mathcal{P}$ -condition and $\text{Nil-}\mathcal{P}$ -condition are equivalent and moreover equivalent to that G has a sub-quotient isomorphic to a dihedral group D_{2pq} for distinct primes p and q .

Proposition 6.4 (cf. [21]). *Simple groups except the following groups satisfy the $\text{Nil-}\mathcal{P}$ -condition.*

- (1) Cyclic group
- (2) Projective special linear groups: $\text{PSL}(2, 4) = \text{PSL}(2, 5) = A_5$, $\text{PSL}(2, 7) = \text{PSL}(3, 2)$, $\text{PSL}(2, 8)$, $\text{PSL}(2, 9) = A_6$, $\text{PSL}(2, 17)$, $\text{PSL}(3, 4)$, $\text{PSL}(3, 8)$
- (3) Suzuki groups $\text{Sz}(8)$, $\text{Sz}(32)$
- (4) Projective unitary groups: $\text{PSU}(3, 3)$, $\text{PSU}(3, 4)$, $\text{PSU}(3, 8)$

Theorem 6.5. *Let $q > 1$ be a prime power. The following groups are gap groups satisfying the $\text{Nil-}\mathcal{P}$ -condition.*

- (1) Symmetric groups S_n , $n \geq 7$
- (2) Projective general linear groups $\text{PGL}(2, q)$, $q \neq 2, 3, 4, 5, 7, 8, 9, 17$
- (3) Projective general linear groups $\text{PGL}(3, q)$, $q \neq 2, 4, 8$
- (4) Projective general linear groups $\text{PGL}(n, q)$, $n \geq 4$
- (5) General linear groups $\text{GL}(2, q)$, $q \neq 2, 3, 4, 5, 7, 8, 9, 17$
- (6) General linear groups $\text{GL}(3, q)$, $q \neq 2, 4, 8$
- (7) General linear groups $\text{GL}(n, q)$, $n \geq 4$
- (8) The automorphism group of sporadic groups

The Smith sets of $\text{PGL}(2, q)$ and $\text{PGL}(3, q)$ have been already obtained in [24]. This can be proved by finding subgroups as in Proposition 6.3. The groups listed up in Theorem 6.5 are non-solvable gap group. Then we have the following theorem.

Theorem 6.6. *Let G be a group which has quotient isomorphic to a group in Theorem 6.5. Then*

$$\text{PSm}^c(G) = \text{Sm}(G)_{\mathcal{P}(G)} = \text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{N}_2(G)}.$$

Corollary 6.7. *Let K be a group in Theorem 6.5 and F any finite group. Then for $G = K \times F$,*

$$\text{PSm}^c(G) = \text{Sm}(G)_{\mathcal{P}(G)} = \text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{N}_2(G)}.$$

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