

Rational points of bounded height on toric varieties

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Let Σ be a complete d -dimensional regular fan in $N_{\mathbf{R}}$ defining a smooth compact d -dimensional toric variety over a number field F , $\Sigma^{(i)}$ the set of all i -dimensional cones in Σ . Let the elements of $\Sigma^{(1)}$ have integral generators $e_1, \dots, e_n$. We define some rational function on $s = (s_1, \dots, s_n) \in \mathbf{C}^n$ associated with the combinatorial structure of the fan Σ

$$f_{\Sigma}(s) = \sum_{\sigma \in \Sigma^{(d)}} f_{\sigma}(s),$$

where $f_{\sigma}(s) = (s_{j_1} \cdots s_{j_d})^{-1}$, if $e_{j_1}, \dots, e_{j_d}$ are generators of the cone σ .

For arhimedian completions of F , we put

$$f_{\Sigma, \mathbf{R}}(s) = 2^d f_{\Sigma}(s), \quad f_{\Sigma, \mathbf{C}}(s) = (2\pi)^d f_{\Sigma}(s).$$

Denote by $P_{\Sigma}(t_1, \dots, t_n)$ the rational function defined as the the Gilbert-Poincare serie of the $\mathbf{Z}_{\geq 0}^n$ -graded Stanley-Reisner ring $R(\Sigma)$ corresponding to Σ .

For any prime $\mathcal{P}$ ideal of F , we denote by $\|\mathcal{P}\|$ the cardinality of the residue field of $\mathcal{P}$, by $\delta_{\mathcal{P}}$ absolute different of the nonarhimedian local field $F_{\mathcal{P}}$, and put

$$f_{\Sigma, \mathcal{P}}(s) = \left(\frac{1}{\sqrt{\delta_{\mathcal{P}}}} \right)^d P_{\Sigma}(\|\mathcal{P}\|^{-s_1}, \dots, \|\mathcal{P}\|^{-s_n}).$$

Denote by $K_{\Sigma}(s)$ the following product

$$f_{\Sigma, \mathbf{R}}^{r_1}(s) f_{\Sigma, \mathbf{C}}^{r_2}(s) \prod_{\mathcal{P}} f_{\Sigma, \mathcal{P}}(s),$$

where r_1 is the number of real embeddings of F , r_2 is the number of complex embeddings of F .

Let r_F the residue of the Dedekind zeta function $\zeta_F(z)$ at $z = 1$;

$$r_F = \frac{2^{r_1} (2\pi)^{r_2} h R}{\sqrt{|D_F|} w}.$$

Theorem. Let $D(s) = s_1 D_1 + \cdots + s_n D_n$ ($s_i > 0$) be an effective divisor on toric variety V_Σ , $H_\Sigma(s, x)$ corresponding height function on F -rational points $x \in T(F) \cong F^*$. Let $T^1(A_F) = (I^1(F))^d$ where $I^1(F)$ is the group of idele with norm 1 of the field F , $d\mu$ the standard Haar measure on $T^1(A_F)$. Then

$$\int_{T^1(A_F)} H_\Sigma(s, x)^{-1} d\mu = (2\pi r_F)^{-d} \int_{M_{\mathbb{R}}} K(s + im) dm.$$

This theorem can be applied to the problem of the asymptotic distribution of rational points of bounded height on toric varieties (cf. [1]).

Example. Let Σ defines $\mathbb{P}^d$. Then

$$f_\Sigma(s) = \frac{s_1 + \cdots + s_{d+1}}{s_1 \cdots s_{d+1}}, \quad P_\Sigma(t_1, \dots, t_{d+1}) = \frac{1 - t_1 \cdots t_{d+1}}{(1 - t_1) \cdots (1 - t_{d+1})},$$

$$K_\Sigma(s) = \left(\frac{2^{r_1} (2\pi)^{r_2}}{\sqrt{|D_F|}} \right)^d \left(\frac{s_1 + \cdots + s_{d+1}}{s_1 \cdots s_{d+1}} \right)^{r_1 + r_2} \frac{\zeta_F(s_1) \cdots \zeta_F(s_{d+1})}{\zeta_F(s_1 + \cdots + s_{d+1})}.$$

Applying the residue formula to the d -dimensional integral, we get

$$\int_{T^1(A_F)} H_\Sigma(s, x)^{-1} = \left(\frac{2^{r_1} (2\pi)^{r_2}}{\sqrt{|D_F|}} \right)^d \left(\frac{s_1 + \cdots + s_{d+1}}{s_1 + \cdots + s_{d+1} - d} \right)^{r_1 + r_2} \frac{\zeta_F(s_1 + \cdots + s_{d+1} - d)}{\zeta_F(s_1 + \cdots + s_{d+1})}.$$

The residue of $\int_{T^1(A_F)} H_\Sigma(s, x)^{-1} d\mu$ at $s = (1, \dots, 1)$ is

$$\left(\frac{2^{r_1} (2\pi)^{r_2}}{\sqrt{|D_F|}} \right)^d (d+1)^{r_1 + r_2 - 1} \left(\frac{2^{r_1} (2\pi)^{r_2} h R}{\sqrt{|D_F|} w} \right) \zeta_F^{-1}(d+1).$$

This number gives the coefficient in the asymptotic formula of Schanuel for the number of rational points in projective spaces [2].

References

- [1] V.V. Batyrev, and Yu. I. Manin, *Sur le nombre des points rationnels de hauteur borné des variétés algébriques*, Math. Ann. **286**, 1990, 27-43.
- [2] S. Schanuel, *Heights in number fields*, Bull. Soc. Math. France, **107**, 1979, 433-449.