

Fourier ultra-hyperfunctions as the boundary values of smooth solutions of heat equations

上智大学理工学研究科 諏訪 将範
Sophia University, Masanori Suwa

1 Introduction

1987, Matsuzawa characterized the space of hyperfunctions with compact support K (denote by $\mathcal{A}'(K)$) as the boundary value of $\mathcal{C}^\infty$ -solutions of the heat equations:

Theorem 1.1 (Matsuzawa [4]). *Let $u \in \mathcal{A}'(K)$ and $U(x, t) := \langle u_y, E(x \cdot y, t) \rangle$, $E(x, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{x^2}{4t}}$ (heat kernel). Then $U(x, t) \in \mathcal{C}^\infty(\mathbb{R}_+^{n+1})$, $\mathbb{R}_+^{n+1} := \{(x, t); x \in \mathbb{R}^n, t > 0\}$ and $U(\cdot, t) \in \mathcal{A}(\mathbb{C}^n)$, $\mathcal{A}(\mathbb{C}^n)$ is the space of entire functions, for each $t > 0$. Furthermore $U(x, t)$ satisfies the heat equation:*

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x, t) = 0 \quad \text{in } \mathbb{R}_+^{n+1}, \quad (1)$$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ (Laplacian).

For every $\varepsilon > 0$, we have

$$|U(x, t)| \leq C_\varepsilon e^{\frac{\varepsilon}{t}} \quad \text{in } \mathbb{R}_+^{n+1}. \quad (2)$$

We have for any $\delta > 0$,

$$U(\cdot, t) \rightarrow 0 \quad (3)$$

uniformly in $\{x \in \mathbb{R}^n; \text{dis}(x, K) \geq \delta\}$ as $t \rightarrow 0_+$.

We have $U(\cdot, t) \rightarrow u$ in $\mathcal{A}'(K)$ as $t \rightarrow 0_+$, i.e.

$$u(\varphi) = \lim_{t \rightarrow 0_+} \int U(x, t) \chi(x) \varphi(x) dx, \quad \varphi \in \mathcal{A}(\mathbb{C}^n), \quad (4)$$

for any $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ in a neighborhood of K .

Conversely, every C^∞ -function $U(x, t)$ defined in $\mathbb{R}_+^{n+1}$ satisfying the conditions (1), (2) and (3) can be expressed in the form $U(x, t) = \langle u_y, E(x - y, t) \rangle$ with unique element $u \in \mathcal{A}'(K)$.

Furthermore 1989, Matsuzawa characterized the space of distributions with compact support K (denote by $\mathcal{E}'_K$) and the space of ultradistributions with compact support K (denote by $\mathcal{E}_K^{\{s\}'}$, $\mathcal{E}_K^{(s)'}$, $s > 1$) by the same way (for details of definitions of ultradistributions, we refer the reader to [3]):

Theorem 1.2 (Matsuzawa [5]). *Let $u \in \mathcal{E}_K^{\{s\}'}$ ($\mathcal{E}_K^{(s)'}$) with $s > 1$ and $U(x, t) = \langle u_y, E(x - y, t) \rangle$. Then $U(x, t) \in C^\infty(\mathbb{R}_+^{n+1})$. Furthermore $U(x, t)$ satisfies the following conditions:*

$$\left(\frac{\partial}{\partial t} - \Delta \right) U(x, t) = 0 \quad \text{in } \mathbb{R}_+^{n+1}. \quad (5)$$

For every $\varepsilon > 0$, $\delta > 0$, there exists a positive constant $C_{\varepsilon, \delta}$ such that

$$|U(x, t)| \leq C_{\varepsilon, \delta} e^{\left(\frac{\varepsilon}{t}\right)^{\frac{1}{2s-1}} - \frac{\text{dis}(x, K_\delta)^2}{8t}} \quad \text{in } \mathbb{R}_+^{n+1}, \quad (6)$$

where $K_\delta := \{x \in \mathbb{R}^n; \text{dis}(x, K) \leq \delta\}$.

We have $U(x, t) \rightarrow u$ as $t \rightarrow 0_+$ in $\mathcal{E}^{\{s\}'}(\mathbb{R}^n)$ (resp. $\mathcal{E}^{(s)' }(\mathbb{R}^n)$).

Conversely, every C^∞ -function $U(x, t)$ defined in $\mathbb{R}_+^{n+1}$ satisfying conditions (5) and (6) can be expressed in the form $U(x, t) = \langle u_y, E(x - y, t) \rangle$ with unique element $u \in \mathcal{E}_K^{\{s\}'}$ (resp. $\mathcal{E}_K^{(s)'}$).

Theorem 1.3 (Matsuzawa [5]). *Let $u \in \mathcal{E}'_K$ and $U(x, t) = \langle u_y, E(x - y, t) \rangle$. Then $U(x, t) \in C^\infty(\mathbb{R}_+^{n+1})$. Furthermore $U(x, t)$ satisfies the following conditions:*

$$\left(\frac{\partial}{\partial t} - \Delta \right) U(x, t) = 0 \quad \text{in } \mathbb{R}_+^{n+1}. \quad (7)$$

There exists a nonnegative integer $N = N(u)$ such that

$$|U(x, t)| \leq C_\delta t^{-N} e^{-\frac{\text{dis}(x, K_\delta)^2}{8t}} \quad \text{in } \mathbb{R}_+^{n+1}, \quad (8)$$

We have $U(x, t) \rightarrow u$ as $t \rightarrow 0_+$ in $\mathcal{E}'_K$.

Conversely, every C^∞ -function $U(x, t)$ defined in $\mathbb{R}_+^{n+1}$ satisfying conditions (7) and (8) can be expressed in the form $U(x, t) = \langle u_y, E(x - y, t) \rangle$ with unique element $u \in \mathcal{E}'_K$.

1993, K.W.Kim, S.-Y.Chung and D.Kim characterized the space of Fourier hyperfunctions with compact support K in $\mathbb{D}^n$ (denote by $\mathcal{F}'(K)$) by the same way ([2]):

Theorem 1.4 (K.W.Kim, S.-Y.Chung and D.Kim, [2]). *Let $u \in \mathcal{F}'(K)$ and $U(x, t) = \langle u_y, E(x - y, t) \rangle$. Then $U(x, t) \in C^\infty(\mathbb{R}_+^{n+1})$ and satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta \right) U(x, t) = 0 \quad \text{in } \mathbb{R}_+^{n+1}. \quad (9)$$

For any $\varepsilon > 0$, there is a constant $C \geq 0$ such that

$$|U(x, t)| \leq C e^{\frac{\varepsilon}{t} + \varepsilon t + \varepsilon |x| - \frac{\text{dis}(x, K \cap \mathbb{R}^n)^2}{8t}} \quad \text{in } \mathbb{R}_+^{n+1}. \quad (10)$$

We have $U(x, t) \rightarrow u$ as $t \rightarrow 0_+$ in $\mathcal{F}'(K)$.

Conversely, every C^∞ -function $U(x, t)$ defined in $\mathbb{R}_+^{n+1}$ satisfying conditions (9) and (10) can be expressed in the form $U(x, t) = \langle u_y, E(x - y, t) \rangle$ with unique element $u \in \mathcal{F}'(K)$.

For $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions, the following result is known:

Theorem 1.5 (Matsuzawa [6]). *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $U(x, t) = \langle u_y, E(x - y, t) \rangle$. Then $U(x, t) \in C^\infty(\mathbb{R}_+^{n+1})$ and satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta \right) U(x, t) = 0 \quad \text{in } \mathbb{R}_+^{n+1}. \quad (11)$$

There exists constants $C \geq 0$, $\nu \geq 0$ and $k \geq 0$ such that

$$|U(x, t)| \leq C t^{-\nu} (1 + |x|)^k \quad \text{in } \mathbb{R}_+^{n+1}. \quad (12)$$

We have $U(x, t) \rightarrow u$ as $t \rightarrow 0_+$ in $\mathcal{S}'(\mathbb{R}^n)$.

Conversely, every C^∞ -function $U(x, t)$ defined in $\mathbb{R}_+^{n+1}$ satisfying conditions (11) and (12) can be expressed in the form $U(x, t) = \langle u_y, E(x - y, t) \rangle$ with unique element $u \in \mathcal{S}'(\mathbb{R}^n)$.

Besides many authors research generalized function by the same way. For example, 1999, M.Budinčević, Z.L.-Crvenković and D.Perošić characterized the spaces of Beurling and Roumieu type tempered ultradistributions (for details, we refer the reader to [1]).

2 Main theorem

Now, we can obtain the same result for Fourier ultra-hyperfunctions. First, we give some notations:

Notations

$$\begin{aligned}\mathbb{C}^n &= \mathbb{R}^n + i\mathbb{R}^n. \\ z &= x + iy, \quad \zeta = \xi + i\eta. \\ z &= (z_1, z_2, \dots, z_n), \quad z_j = x_j + iy_j, \quad j = 1, 2, \dots, n. \\ \zeta &= (\zeta_1, \zeta_2, \dots, \zeta_n), \quad \zeta_j = \xi_j + i\eta_j, \quad j = 1, 2, \dots, n. \\ \langle \zeta, z \rangle &= \sum_{j=1}^n \zeta_j z_j. \quad \text{In particular, } z^2 = \langle z, z \rangle. \\ E(z, t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{z^2}{4t}}, \quad z \in \mathbb{C}^n, \quad t > 0.\end{aligned}$$

Let K be a convex compact set in $\mathbb{R}^n$. Then we define supporting function $h_K(x)$ by

$$h_K(x) = \sup_{\xi \in K} \langle \xi, x \rangle.$$

We denote “complex Laplacian” by Δ :

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}.$$

Let L be a closed set in $\mathbb{C}^n$ and $\overset{\circ}{L}$ be interior of L . We denote by $\mathcal{H}(\overset{\circ}{L})$ the spaces of holomorphic functions on $\overset{\circ}{L}$ and by $\mathcal{C}(L)$ the spaces of continuous functions on L .

Definition 2.1. Let K and K' be convex compact sets in $\mathbb{R}^n$. Then we define $Q_b(\mathbb{R}^n + \imath K, K')$ as follows:

$$Q_b(\mathbb{R}^n + \imath K, K') := \{f \in \mathcal{H}(\mathring{\mathbb{R}^n + \imath K}) \cap \mathcal{C}(\mathbb{R}^n + \imath K) : \sup_{z \in \mathbb{R}^n + \imath K} |f(z)e^{h_{K'}(x)}| < +\infty\}.$$

Definition 2.2. We define the space Q_0 as follows:

$$Q_0 := \varprojlim_{K, K' \subset \subset \mathbb{R}^n} Q_b(\mathbb{R}^n + \imath K, K'),$$

where $\varprojlim$ means projective limit.

Definition 2.3. We denote by Q'_0 the dual space of Q_0 . The element of Q'_0 is called Fourier ultra-hyperfunctions.

For details of Fourier ultra-hyperfunctions, we refer the reader to [7].

The following theorem is a main result:

Theorem 2.4. Let $T \in Q'_0$ and $U(z, t) = \langle T_\zeta, E(z - \zeta, t) \rangle$. Then $U(z, t)$ is an entire function of z and C^∞ -function of t , $t > 0$ satisfying the following conditions:

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(z, t) = 0, \quad (13)$$

$$U(z, t) \longrightarrow T, \quad (t \rightarrow 0_+), \quad \text{in } Q'_0 \quad (14)$$

$\exists R \geq 0, \exists b \geq 0, \exists C \geq 0$, s.t.

$$|U(z, t)| \leq C e^{\frac{1}{4t} \sum_{j=1}^n (b + |y_j|)^2 + R \sum_{j=1}^n |x_j| + nR^2 t}. \quad (15)$$

Conversely, for a function $U(z, t)$, $t > 0$, entire function of z , C^∞ -function of t , satisfying (13) and (15), there exists unique $T \in Q'_0$ such that $\langle T_\zeta, E(z - \zeta, t) \rangle = U(z, t)$.

For details we refer the reader to [8].

At present:

Recently we obtained the same results for tempered distributions with support in a proper convex cone. This paper will be soon appeared.

Reference

- [1] M.Budinčević, Z.Lozanov-Crvenković and D.Perošić : *Representation theorems for tempered ultradistributions*, Publications de L'Institut mathématique, Nouvelle série, tome 65, (1999), 142-160.
- [2] K.W.Kim, S.-Y.Chung and D.Kim : *Fourier hyperfunctions as the boundary values of smooth solutions of heat equations*, Publ. RIMS, Kyoto Univ. 29 (1993), 289-300.
- [3] H.Komatsu : *Introduction to the theory of distributions (in Japanese)*, Iwanami Shoten, (1978).
- [4] T.Matsuzawa : *A calculus approach to the hyperfunctions I*, Nagoya Math. J. Vol.108, (1987), 53-66.
- [5] T.Matsuzawa : *A calculus approach to the hyperfunctions II*, Trans. Amer. Math. Soc. 313 (1989), 619-654.
- [6] T.Matsuzawa : *An introduction to the theory of partial differential equations*, JSPS-DOST Lecture Notes in Mathematics, Vol.4, (1997).
- [7] P.Sargos and M.Morimoto : *Transformation des fonctionnelles analytiques à porteurs non compacts*, Tokyo J.Math. Vol.4, (1981), 457-492.
- [8] M.Suwa : *Fourier ultra-hyperfunctions as the boundary values of smooth solutions of heat equations*, Tokyo J.Math., (to appear).