

A remark on Alperin's conjecture

Atumi WATANABE

熊本大学 教養部
渡辺アツミ

Let p be a prime number and F be an algebraically closed field of characteristic p . Let G be a finite group and B be a $(p-)$ block of G with abelian defect group D and Brauer correspondent B_0 , a block of $N_G(D)$. Alperin's weight conjecture states that the number of isomorphism classes of irreducible $(FG)B$ -modules is equal to that of isomorphism classes of irreducible $(FN_G(D))B_0$ -modules (see [1]). So if Alperin's conjecture is true, then the number of isomorphism classes of indecomposable $(FG)B$ -modules with source module $F_{\{1_G\}}$ is equal to that of isomorphism classes of indecomposable $(FN_G(D))B_0$ -modules with source module $F_{\{1_G\}}$. We generalize this to an arbitrary source module. Let Q be a p -subgroup of G and L be a finitely generated indecomposable FQ -module with vertex Q . We denote by $\text{Ind}(B|L)$ the set of isomorphism classes of indecomposable FG -modules which belong to B and have L as a source module. $\text{Ind}(B|L)$ is a finite set. We denote by $|\text{Ind}(B|L)|$ the cardinality. In this paper we prove the following .

PROPOSITION. *Under the above notation, if Alperin's conjecture is true, then we have*

$$|\text{Ind}(B|L)| = \sum'_{x \in T(L) \backslash G/N_G(D)} |\text{Ind}(B_0|L^x)|,$$

where $T(L) = \{t \in N_G(Q) \mid L \otimes_{FQ} t \subset L^G\}$ is isomorphic to L , $L^x = L \otimes_{FQ} x$, and where x ranges over a complete set of representatives for $(T(L), N_G(D))$ -double cosets $T(L)xN_G(D)$ of G such that $D \supset Q^x$.

§ 1. Preliminaries.

Let M be a finitely generated right FG -module and put $E = \text{End}_{FG} M$. There is a one to one correspondence between the decompositions of M into direct sums of indecomposable FG -submodules and the decompositions of E into direct sums of principal indecomposable right modules. Moreover, if $\text{id}_M = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n$ is a decomposition of the identity map of M into the sum of primitive idempotents of E , then $\varepsilon_i(M)$ and $\varepsilon_j(M)$ are isomorphic if and only if $\varepsilon_i E$ and $\varepsilon_j E$ are isomorphic as E -modules. Hence the number of isomorphism classes of indecomposable components of M is equal to that of isomorphism classes of irreducible E -modules. Let $Z(FG)$ be the center of FG and for $z \in Z(FG)$, $z_R \in E : m \longrightarrow mz, m \in M$. Then the map φ defined by $\varphi(z) = z_R$ is a homomorphism from $Z(FG)$ into $Z(E)$. Let B be a block and assume that $\varphi(B) \neq 0$. It is clear that, by the above, the number of isomorphism classes of indecomposable components of M belonging to B is equal to that of isomorphism classes of irreducible $E\varphi(B)$ -modules.

Let Q be a normal abelian subgroup of G and L be a finitely generated G -invariant indecomposable FQ -module. We put $E_G = \text{End}_{FG} L^G$, where $L^G = L \otimes_{FQ} FG$. For $x \in G$ let $E_x = \{f \in E_G \mid f(L \otimes_{FQ} 1) = L \otimes_{FQ} x\}$.

$\subset L \otimes_{FQ} x \}$. We have

$$E_G = \sum_{x \in G/Q} E_x.$$

By the assumption there exists a unit ϕ_x in E_x , and it holds that $E_x = E_1 \phi_x = \phi_x E_1$ and $E_x E_y = E_{xy}$. Moreover if $c \in C_G(Q)$, then we can choose (and do so) ϕ_c such that $\phi_c(l \otimes y) = l \otimes cy$, $l \in L$ and $y \in G$. The F -linear map $\phi : FC_G(Q) \rightarrow E_G$ defined by $\phi(c) = \phi_c$, $c \in C_G(Q)$ is an algebra homomorphism. Moreover, for $z \in FC_G(Q) \cap Z(FG)$, $\phi(z) = z_R$, $z_R : l \otimes x \mapsto l \otimes xz = l \otimes zx$, $l \in L$ and $x \in G$. Let $J(E_1)$ be the radical of E_1 and let $\bar{E}_G = E_G / (J(E_1)E_G)$. $E_1/J(E_1) \cong F$ and $J(E_1)E_G \subset J(E_G)$. For each $\bar{x} \in \bar{G} = G/Q$, let $\phi_{\bar{x}}$ be one of $\overline{\phi_y}$, $y \in Qx$. Here $\overline{\phi_y} = \phi_y + J(E_1)E_G$. We have

$$(1) \quad \phi_{\bar{x}} \phi_{\bar{y}} = \alpha(\bar{x}, \bar{y}) \phi_{\overline{xy}}, \quad \bar{x}, \bar{y} \in \bar{G},$$

where $\alpha(\bar{x}, \bar{y}) \in F^\times = F - \{0\}$. It is clear that α is an F -cocycle of $\bar{G}$ and $\bar{E}_G$ is isomorphic to the twisted group algebra $F(\bar{G}, \alpha)$ of $\bar{G}$ with cocycle α . We identify $\bar{E}_G$ with $F(\bar{G}, \alpha)$. Moreover ϕ induces an algebra isomorphism from $\overline{FC_G(Q)}$ into $\bar{E}_G$, because $\overline{\phi_{cz}} = \overline{\phi_c}$ for $c \in C_G(Q)$, $z \in Q$. We embed $\overline{FC_G(Q)}$ into $\bar{E}_G$. Let H be a subgroup of G containing Q . We can embed $E_H (= \text{End}_{FH} L^H)$ into E_G , and $\bar{E}_H$ into $\bar{E}_G$. Moreover we have $\bar{E}_H = F(H, \alpha_H)$ and $\bar{E}_{C_G(Q)} = \overline{FC_G(Q)}$.

By [3, Theorem 1.1], there exists a finite group $\tilde{G}$ with the following properties: (i) $\tilde{G}$ has a p' -subgroup A which is contained in the center of $\tilde{G}$. (ii) There is an isomorphism $\sigma : \bar{G} \rightarrow \tilde{G} / A$. (iii) There is an F -valued linear character λ of A such that the special cocycle of $\bar{G}$ associated with λ is cohomologous to α .

For each $\bar{x} \in \bar{G}$, let $r(\bar{x})$ be an element of $\tilde{G}$ such that $\sigma(\bar{x}) = Ar(\bar{x})$. Then $\tilde{G} = \bigcup_{\bar{x} \in \bar{G}} Ar(\bar{x})$ and there exist $a(\bar{x}, \bar{y}) \in A$ such that

$r(\bar{x})r(\bar{y}) = a(\bar{x}, \bar{y})r(\bar{x}\bar{y})$, for $\bar{x}, \bar{y} \in \bar{G}$. By the above (iii), there exists a map $\delta : \bar{G} \longrightarrow F^\times$ such that

$$(2) \quad \alpha(\bar{x}, \bar{y}) = \delta(\bar{x})\delta(\bar{y})\delta(\bar{x}\bar{y})^{-1} \lambda(a(\bar{x}, \bar{y})), \quad \bar{x}, \bar{y} \in \bar{G}.$$

Let e_λ be the primitive idempotent of FA corresponding to λ . $(FA)e_\lambda = Fe_\lambda$ and $(F\tilde{G})e_\lambda = \sum_{\bar{x} \in \bar{G}} F(r(\bar{x})e_\lambda)$. Now let θ be an F -linear map such that

$$\begin{aligned} \theta : F(\bar{G}, \alpha) = \bar{E}_G &\longrightarrow (F\tilde{G})e_\lambda \\ \bar{x} &\longmapsto \delta(\bar{x})r(\bar{x})e_\lambda, \quad \bar{x} \in \bar{G}. \end{aligned}$$

From (1) and (2), θ is an algebra isomorphism. For each subgroup H of G containing Q , let $\tilde{H}$ be a subgroup of $\tilde{G}$ such that $\sigma(\tilde{H}) = \tilde{H} / A$. θ induces the isomorphism

$$F(\tilde{H}, \alpha_{\tilde{H}}) = \bar{E}_H \longrightarrow (F\tilde{H})e_\lambda.$$

§ 2. Proof of Proposition.

Step 1. We may assume that Q is normal in G .

Proof. Let $\mathcal{G} = \{[P, \beta] \mid P \text{ is a } p\text{-subgroup of } G \text{ and } \beta \text{ is a block of } N_G(P) \text{ with } \beta^G = B\}$ and let b be a root of B in $C_G(D)$. G acts on $\mathcal{G}$ by conjugation. Since D is abelian, any element of $\mathcal{G}$ is G -conjugate to some $[P, b^{N_G(P)}]$, $P \subset D$. Suppose that $[P, b^{N_G(P)}]$ and $[R, b^{N_G(R)}]$ are G -conjugate : $[P, b^{N_G(P)}] = [R, b^{N_G(R)}]^x$, $x \in G$. b^x is a root of $b^{N_G(P)}$ in $C_G(D^x)$, and hence there exists $y \in N_G(P)$ such that $D = D^{xy}$ and $b = b^{xy}$. So $xy \in T(b)$ and $P = R^{xy}$. Let $\mathcal{P}$ be a complete set of representatives for $T(b)$ -conjugacy classes of subgroups of D , where $T(b) = \{t \in N_G(D) \mid b^t = b\}$. $\{[P, b^{N_G(P)}] \mid P \in \mathcal{P}\}$ forms a complete

set of representatives for G -conjugacy classes in $\mathcal{P}$.

By [2, Theorem 2], we have $|\text{Ind}(B|L)| = \sum_{\beta} |\text{Ind}(\beta|L)|$, where β ranges over the set of $\text{Bl}(N_G(Q), B)$ of blocks of $N_G(Q)$ associated with B . On the other hand, for $\beta \in \text{Bl}(N_G(Q), B)$ there exists $P \in \mathcal{P}$ and $u \in G$ such that $[P, b^{N_G(P)}] = [Q, \beta]^u$. Since $|\text{Ind}(\beta|L)| = |\text{Ind}(\beta^u|L^u)| = |\text{Ind}(b^{N_G(P)}|L^u)|$, we can see

$$(3) \quad |\text{Ind}(B|L)| = \sum_{P \in \mathcal{P}_Q} |\text{Ind}(b^{N_G(P)}|L^{u_P})|,$$

where $\mathcal{P}_Q = \{P \in \mathcal{P} \mid P \text{ is } G\text{-conjugate to } Q\}$ and u_P is an element of G with $P = Q^{u_P}$. We note for P , $|\text{Ind}(b^{N_G(P)}|L^{u_P})|$ does not depend on the choice of u_P . For each Q^v , $v \in G$, we set $\mathcal{P}_{Q^v} = \{P \in \mathcal{P} \mid P \text{ is } N_G(D)\text{-conjugate to } Q^v\}$ and for $P \in \mathcal{P}_{Q^v}$, denote by $u_{v,P}$ an element of $N_G(D)$ such that $P = Q^{vu_{v,P}}$. Now, if $Q^z \subset N_G(D)$, $z \in G$, then by applying (3) for $N_G(D)$, B_0 and L^z , we have

$$(4) \quad |\text{Ind}(B_0|L^z)| = \sum_{P \in \mathcal{P}_{Q^z}} |\text{Ind}(b^{N_G(P) \cap N_G(D)}|L^{zu_{z,P}})|.$$

From (3) we have

$$(5) \quad |\text{Ind}(B|L)| = \sum_{v \in N_G(Q) \backslash G/N_G(D)} \sum_{P \in \mathcal{P}_{Q^v}} |\text{Ind}(b^{N_G(P)}|L^{vu_{v,P}})|,$$

where v ranges over a complete set of representatives for the

$(N_G(Q), N_G(D))$ -double cosets of G . If $\mathcal{P}_{Q^v}$ is not empty, then $D \supset Q^v$. Here we assume that Proposition holds for $N_G(Q)$. Then we have for $P \in \mathcal{P}_{Q^v}$,

$$\begin{aligned}
 (6) \quad & | \text{Ind}(b^{N_G(P)} | L^{vu_{v,P}}) | = | \text{Ind}((b^{(vu_{v,P})^{-1}})^{N_G(Q)} | L) | \\
 = & \sum'_{y \in T(L) \setminus N_G(Q) / (N_G(D^{v^{-1}}) \cap N_G(Q))} | \text{Ind}((b^{(vu_{v,P})^{-1}})^{N_G(Q) \cap N_G(D^{v^{-1}})}) | L^y) | \\
 = & \sum'_{y \in T(L) \setminus N_G(Q) / (N_G(D^{v^{-1}}) \cap N_G(Q))} | \text{Ind}((b^{N_G(P) \cap N_G(D)}) | L^{yvu_{v,P}}) |,
 \end{aligned}$$

where y ranges over a complete set of representatives for $(T(L), N_G(D^{v^{-1}}) \cap N_G(Q))$ -double cosets $T(L)y(N_G(D^{v^{-1}}) \cap N_G(Q))$ of $N_G(Q)$. By substituting (6) in (5) and using (4), we can show

$$\begin{aligned}
 & | \text{Ind}(B | L) | \\
 = & \sum_v \sum_{P \in \mathcal{P}_{Q^v}} \sum'_{y \in T(L) \setminus N_G(Q) / (N_G(D^{v^{-1}}) \cap N_G(Q))} | \text{Ind}((b^{N_G(P) \cap N_G(D)}) | L^{yvu_{v,P}}) | \\
 = & \sum_{v \in N_G(Q) \setminus G / N_G(D)} \sum'_{y \in T(L) \setminus N_G(Q) / (N_G(D^{v^{-1}}) \cap N_G(Q))} | \text{Ind}(B_0 | L^{yv}) | \\
 = & \sum'_{x \in T(L) \setminus G / N_G(D)} | \text{Ind}(B_0 | L^x) |,
 \end{aligned}$$

where $D \supset Q^V$ and $D \supset Q$. This completes the proof of step 1.

Step 2. Let ℓ be a block of $C_G(Q)$ covered by B . We may assume that $T(\ell) = G$, $T(\ell) = \{x \in G \mid \ell^x = \ell\}$

Proof. By step 1, Q is normal in G and $C_G(Q)$ is normal in G . Let B_1 be a block of $T(\ell)$ which covers ℓ and satisfies $B_1^G = B$. ($B_1 = \ell$ as elements of FG). Since D is abelian, we may assume that D is a defect group of ℓ and B_1 . Let ℓ_0 be a Brauer correspondent of ℓ , a block of $C_G(Q) \cap N_G(D)$, and $(B_1)_0$ be a Brauer correspondent of B_1 , a block of $T(\ell) \cap N_G(D)$. Since there is a one to one correspondence between the indecomposable FG -modules in B and the indecomposable $FT(\ell)$ -modules in B_1 by induction, we can show

$$(7) \quad |Ind(B|L)| = \sum_{y \in T(L) \setminus G/T(\ell)} |Ind(B_1|L^y)|.$$

On the other hand $T(\ell_0) = T(\ell) \cap N_G(D)$, $(B_1)_0$ covers ℓ_0 and $B_0 = ((B_1)_0)^{N_G(D)}$. So we have as in (7)

$$(8) \quad |Ind(B_0|L^x)| = \sum_{z \in T(L^x) \cap N_G(D) \setminus N_G(D)/T(\ell_0)} |Ind((B_1)_0|L^{xz})|.$$

Here we assume that Proposition is true for $T(\ell)$. Since Q is normal in G , the following holds

$$|Ind(B_1|L^y)| = \sum_{w \in (T(L^y) \cap T(\ell)) \setminus T(\ell)/(T(\ell) \cap N_G(D))} |Ind((B_1)_0|L^{yw})|.$$

Therefore we have from (7) and (8)

$$\begin{aligned}
 & | \text{Ind}(B|L) | \\
 = & \sum_{y \in T(L) \setminus G/T(\mathfrak{L})} \sum_{w \in (T(L^y) \cap T(\mathfrak{L})) \setminus T(\mathfrak{L}) / (T(\mathfrak{L}) \cap N_G(D))} | \text{Ind}((B_1)_0 | L^{yw}) | \\
 = & \sum_{u \in T(L) \setminus G / (T(\mathfrak{L}) \cap N_G(D))} | \text{Ind}((B_1)_0 | L^u) | \\
 = & \sum_{x \in T(L) \setminus G / N_G(D)} \sum_{z \in (T(L^x) \cap N_G(D)) \setminus N_G(D) / (T(\mathfrak{L}) \cap N_G(D))} | \text{Ind}((B_1)_0 | L^{xz}) | \\
 = & \sum_{x \in T(L) \setminus G / N_G(D)} | \text{Ind}(B_0 | L^x) |.
 \end{aligned}$$

This completes the proof of step 2 and we may assume that $B = \mathfrak{L}$.

Step 3. We may assume that L is G -invariant.

Proof. By step 2, B is a block of $C_G(Q)$ and hence B is a block of $T(L)$. Similarly B_0 is a block of $T(L) \cap N_G(D)$ and B_0 is a Brauer correspondent of B as a block of $T(L)$. Let N be an indecomposable FG -module with source module L . Then there exists an indecomposable $FT(L)$ -module N_0 with source module L such that $N = N_0^G$. N_0 is uniquely (up to isomorphism) determined by N , and $N = NB$ if and only if $N_0 = N_0 B$. Conversely if N_0 is an indecomposable $FT(L)$ -module with source module L , then N_0^G is an indecomposable FG -module with source module L . Hence the number of isomorphism classes of indecomposable $(FG)B$ -modules with source module L is equal to that of isomorphism classes of indecomposable $(FT(L))B$ -modules with source module L . Similarly, the number of isomorphism classes of indecomposable $(FN_G(D))B_0$ -modu-

les with source module L is equal to that of isomorphism classes of indecomposable $(F(T(L) \cap N_G(D))B_0)$ -modules with source module L . So if Proposition is true for $T(L)$, we have $|\text{Ind}(B|L)| = |\text{Ind}(B_0|L)|$. By the way, since $T(\mathfrak{G}) = G$ and D is a defect group of $\mathfrak{G}$ (see step 2), we have $G = C_G(Q)N_G(D) = T(L)N_G(D)$. Hence Proposition holds for B . So we may assume that $T(L) = G$.

Step 4. Conclusion.

Proof. Under the assumption that Q is normal in G , B is a block of $C_G(Q)$ and that L is G -invariant, we will show $|\text{Ind}(B|L)| = |\text{Ind}(B_0|L)|$, using results and notations in §1. Since any indecomposable FG -module with source module L is a component of L^G and an indecomposable component of L^G has L as a source module by the assumption, we have

$$|\text{Ind}(B|L)| = |\text{Irr}(F(\bar{G}, \alpha)\bar{B})|,$$

where $\text{Irr}(F(\bar{G}, \alpha)\bar{B})$ is the set of isomorphism classes of irreducible $F(\bar{G}, \alpha)\bar{B}$ -modules and $\bar{B}$ is a block of $\overline{FC_G(Q)}$ corresponding to B .

Similarly we have

$$|\text{Ind}(B_0|L)| = |\text{Irr}(F(\overline{N_G(D)}, \alpha)\bar{B}_0)|,$$

where $\bar{B}_0$ is a block of $F(\overline{C_G(Q) \cap N_G(D)})$ corresponding to B_0 . We note B_0 is a Brauer correspondent of B as a block of $C_G(Q)$.

Let $\tilde{B} = \theta(\bar{B})$. If $\bar{B} = \sum_{\bar{x} \in C_G(Q)} a_{\bar{x}} \bar{x}$, $a_{\bar{x}} \in F$, then $\tilde{B} =$

$\sum_{\bar{x} \in C_G(Q)} a_{\bar{x}} \delta(\bar{x}) r(\bar{x}) e_{\lambda}$. Since $\bar{D}$ is a defect group of $\bar{B}$ as a block of

$\overline{C_G(Q)}$ and A is a central p' -subgroup of $\tilde{G}$, a defect group $\tilde{D}$ of $\tilde{B}$ as a block of $C_G^{\sim}(Q)$ is a p -subgroup of $C_G^{\sim}(Q)$ such that $\sigma(\bar{D}) = \tilde{D}A/A$. We see $N_{\tilde{G}}(\tilde{D})/A = N_G^{\sim}(D)$ and $((C_G^{\sim}(Q) \cap N_{\tilde{G}}(\tilde{D}))/A = C_G(Q) \cap N_G(D)$. Let $\tilde{B}_0 =$

$\theta(\bar{B}_0)$. $\tilde{B}_0$ is a block of $C_G(Q) \widetilde{\cap N_G(D)}$ and $\tilde{D}$ is a defect group of $\tilde{B}_0$. Since $\bar{B}_0$ is a Brauer correspondent of $\bar{B}$ as a block of $\overline{C_G(Q)}$, we can show that $\tilde{B}_0$ is a Brauer correspondent of $\tilde{B}$ as a block of $\tilde{C_G(Q)}$.

Let $\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_n$ be the blocks of $\tilde{G}$ which cover $\tilde{B}$. Since $\bar{B}$ belongs to $Z(F(\bar{G}, \alpha))$, and $\tilde{B}$ belongs to $Z(F\tilde{G})$, the following holds.

$$(9) \quad \tilde{B} = \tilde{B}_1 + \tilde{B}_2 + \dots + \tilde{B}_n.$$

Therefore we have

$$(10) \quad \begin{aligned} |\text{Ind}(B|L)| &= |\text{Irr}(F(\bar{G}, \alpha)\bar{B})| = |\text{Irr}((F\tilde{G})\tilde{B})| \\ &= \sum_{i=1}^n |\text{Irr}((F\tilde{G})\tilde{B}_i)|, \end{aligned}$$

because $F(\bar{G}, \alpha)\bar{B}$ and $(F\tilde{G})\tilde{B}$ are isomorphic by θ . By the assumption that B is a block of $C_G(Q)$ and since D is abelian, $p \nmid |G : C_G(Q)|$ and hence $p \nmid |\tilde{G} : \tilde{C_G(Q)}|$. So $\tilde{D}$ is a defect group of $\tilde{B}_i$. Let $(\tilde{B}_i)_0$ be a Brauer correspondent of $\tilde{B}_i$ which is a block of $N_{\tilde{G}}(\tilde{D})$. From (9),

$$\tilde{B}_0 = (\tilde{B}_1)_0 + (\tilde{B}_2)_0 + \dots + (\tilde{B}_n)_0.$$

Applying (10) for $N_{\tilde{G}}(\tilde{D})$ and B_0 , we have

$$(11) \quad |\text{Ind}(B_0|L)| = \sum_{i=1}^n |\text{Irr}((FN_{\tilde{G}}(\tilde{D}))(\tilde{B}_i)_0)|.$$

Now if Alperin's conjecture is true, then $|\text{Irr}((F\tilde{G})\tilde{B}_i)| = |\text{Irr}(FN_{\tilde{G}}(\tilde{D}))(\tilde{B}_i)_0|$, hence from (10) and (11), $|\text{Ind}(B|L)| = |\text{Ind}(B_0|L)|$. This completes the proof of Proposition.

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