

ON THE UNIVERSALITY OF A SEQUENCE OF POWERS MODULO 1

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ABSTRACT. Recently, we proved that, for any sequence of real numbers $(r_n)_{n=1}^{\infty}$ and any sequence of positive numbers $(\delta_n)_{n=1}^{\infty}$, there is an increasing sequence of positive integers $(q_n)_{n=1}^{\infty}$ and a number $\alpha > 1$ such that $\|\alpha^{q_n} - r_n\| < \delta_n$ for each $n \geq 1$. Now, we prove that there are continuum of such numbers α in any interval $I = [a, b]$, where $1 < a < b$, and give some corollaries to this statement.

1. INTRODUCTION

Throughout, we shall denote by $\{x\}$, $[x]$ and $\|x\|$ the fractional part of a real number x , the least integer which is greater than or equal to x , and the distance from x to the nearest integer, respectively.

In [1], we showed that, for any sequence of real numbers $(r_n)_{n=1}^{\infty}$ and any sequence of positive numbers $(\delta_n)_{n=1}^{\infty}$, there exist an increasing sequence of positive integers $(q_n)_{n=1}^{\infty}$ and a number $\alpha > 1$ such that $\|\alpha^{q_n} - r_n\| < \delta_n$ for each $n \geq 1$.

Now, we will show that there are continuum of such α , so at least one of them is transcendental. We also give some corollaries to this “universality property” of powers. In some sense, if $q_1 < q_2 < q_3 < \dots$ are positive integers, then the subsequence $(\alpha^{q_n})_{n=1}^{\infty}$ of the sequence of powers $(\alpha^n)_{n=1}^{\infty}$ represents the sequence $(r_n)_{n=1}^{\infty}$ modulo 1 with any prescribed “precision”. In addition, we relax the condition on q_n . These numbers need not be integers. They can be any positive numbers with “large” gaps between them.

Theorem 1. *Let $(\delta_n)_{n=1}^{\infty}$ be a sequence of positive numbers, where $\delta_n \leq 1/2$, and let $(r_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that $I = [a, b]$ is an interval with $1 < a < b$, and suppose M is the least positive integer satisfying $a^{M-1}(a-1) \geq \max(10, 2a/(b-a))$. If $(q_n)_{n=1}^{\infty}$ is a sequence of real numbers satisfying $q_1 \geq M$ and*

$$q_{n+1} - q_n \geq M + 1 + \max(0, \log_a(2.22/(\delta_n(a-1))))$$

for each $n \geq 1$, then the interval I contains continuum of numbers α such that the inequality

$$\|\alpha^{q_n} - r_n\| < \delta_n$$

holds for each positive integer n .

This theorem will be proved in the next section. In Section 3, we give some corollaries.

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2. PROOF OF THEOREM 1

Without loss of generality we may assume that $r_n \in [0, 1)$ for each $n \geq 1$. Let $w = (w_n)_{n=1}^\infty$ be an arbitrary sequence consisting of two numbers 0 and $1/2$. Consider the sequence $(\theta_n)_{n=1}^\infty$ defined as $\theta_{2n-1} = r_n$ and $\theta_{2n} = w_n$ for each positive integer n , namely,

$$(\theta_n)_{n=1}^\infty = r_1, w_1, r_2, w_2, r_3, w_3, \dots$$

Let also $\ell_{2n-1} = q_n$ and $\ell_{2n} = q_{n+1} - M$ for each integer $n \geq 1$. The inequalities $q_{n+1} - q_n \geq M + 1$ and $q_1 \geq M$ imply that $M \leq \ell_1 < \ell_2 < \ell_3 < \dots$ is a sequence of positive numbers satisfying $\ell_{n+1} - \ell_n \geq 1$ for each $n \geq 1$.

Put $y_0 = a$ and

$$y_n = (\lceil y_{n-1}^{\ell_n} \rceil + \theta_n)^{1/\ell_n}$$

for $n \geq 1$. Since $\theta_n \geq 0$ and $\lceil y_{n-1}^{\ell_n} \rceil \geq y_{n-1}^{\ell_n}$, we have $y_n \geq y_{n-1}$. Thus the sequence $(y_n)_{n=0}^\infty$ is non-decreasing. Furthermore, $y_n^{\ell_n} - \theta_n$ is an integer, so $\{y_n^{\ell_n}\} = \{\theta_n\} = \theta_n$ for every $n \in \mathbb{N}$.

From $\lceil y_{n-1}^{\ell_n} \rceil < y_{n-1}^{\ell_n} + 1$ and $\theta_n < 1$, we deduce that $y_n^{\ell_n} = \lceil y_{n-1}^{\ell_n} \rceil + \theta_n < y_{n-1}^{\ell_n} + 2$. Hence $(y_n/y_{n-1})^{\ell_n} < 1 + 2y_{n-1}^{-\ell_n}$. Since $\ell_n > 1$ for every $n \geq 1$, we have $y_n/y_{n-1} < 1 + 2y_{n-1}^{-\ell_n}/\ell_n$. This implies that $y_n - y_{n-1} < 2/(\ell_n y_{n-1}^{\ell_n-1})$. Since $y_n \geq y_{n-1} \geq \dots \geq y_0$ and $\ell_n - \ell_{n-1} \geq 1$ for $n \geq 2$, by adding n such inequalities (for $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$), we obtain

$$y_n - a = y_n - y_0 = \sum_{k=1}^n (y_k - y_{k-1}) < \frac{2}{\ell_1} \sum_{k=\ell_1-1}^\infty y_0^{-k} = \frac{2}{\ell_1 y_0^{\ell_1-2} (y_0 - 1)} = \frac{2}{\ell_1 a^{\ell_1-2} (a - 1)}.$$

Using $a^{M-1}(a-1) \geq 2a/(b-a)$ and $\ell_1 = q_1 \geq M \geq 1$, we deduce that

$$y_n - a < \frac{2}{\ell_1 a^{\ell_1-2} (a - 1)} \leq \frac{2}{a^{\ell_1-2} (a - 1)} \leq \frac{2a}{a^{M-1} (a - 1)} \leq \frac{2a}{2a/(b-a)} = b - a.$$

Hence $y_n < b$ for every n . Thus the limit $\alpha = \lim_{n \rightarrow \infty} y_n$ exists and belongs to the interval $[a, b]$. (Of course, $\alpha = \alpha(w)$ depends on the sequence w .)

Next, we shall estimate the quotient $(y_{k+1}/y_k)^{\ell_n}$ for $k \geq n$. Since $(y_{k+1}/y_k)^{\ell_{k+1}} < 1 + 2y_k^{-\ell_{k+1}}$ and $\ell_n/\ell_{k+1} < 1$, we have $(y_{k+1}/y_k)^{\ell_n} < (1 + 2y_k^{-\ell_{k+1}})^{\ell_n/\ell_{k+1}} < 1 + 2y_k^{-\ell_{k+1}}$. It follows that

$$(\alpha/y_n)^{\ell_n} = \prod_{k=n}^\infty (y_{k+1}/y_k)^{\ell_n} < \prod_{k=n}^\infty (1 + 2y_k^{-\ell_{k+1}})$$

for every fixed positive integer n .

In order to estimate the product $\prod_{k=n}^\infty (1 + \tau_k)$, where $\tau_k = 2y_k^{-\ell_{k+1}}$, we shall first bound this product from above by $\exp(\sum_{k=n}^\infty \tau_k)$ and then use the inequality $\exp(\tau) < 1 + 1.11\tau$, because the sum $\tau = \sum_{k=n}^\infty \tau_k$ is less than $1/5$. Indeed, using the inequalities $y_k \geq y_n \geq a$ and $\ell_n - \ell_{n-1} \geq 1$, where the inequality is strict for infinitely many n 's, we derive that

$$\tau = \sum_{k=n}^\infty 2y_k^{-\ell_{k+1}} < \frac{2}{y_n^{\ell_{n+1}-1} (y_n - 1)} \leq \frac{2}{a^{\ell_{n+1}-1} (a - 1)} \leq \frac{2}{a^{\ell_2-1} (a - 1)}$$

is at most $1/5$, because $a^{\ell_2-1}(a-1) \geq a^{M-1}(a-1) \geq 10$. Consequently,

$$(\alpha/y_n)^{\ell_n} < 1 + 1.11\tau < 1 + 2.22/(y_n^{\ell_{n+1}-1}(y_n - 1)).$$

Multiplying both sides by $y_n^{\ell_n}$ and subtracting $y_n^{\ell_n}$ from both sides, we find that

$$0 \leq \alpha^{\ell_n} - y_n^{\ell_n} < 2.22/(y_n^{\ell_{n+1}-\ell_n-1}(y_n - 1)) \leq 2.22/(a^{\ell_{n+1}-\ell_n-1}(a - 1)).$$

From this, using $\{y_n^{\ell_n}\} = \theta_n$, we deduce that

$$\|\alpha^{\ell_n} - \theta_n\| < 2.22a^{-\ell_{n+1}+\ell_n+1}/(a - 1)$$

for each $n \in \mathbb{N}$.

For n odd, the last inequality $\|\alpha^{\ell_{2n-1}} - \theta_{2n-1}\| < 2.22a^{-\ell_{2n}+\ell_{2n-1}+1}/(a - 1)$ becomes $\|\alpha^{q_n} - r_n\| < 2.22a^{-q_{n+1}+q_n+M+1}/(a - 1)$. The right hand side is less than or equal to δ_n , because $q_{n+1} - q_n \geq M + 1 + \log_a(2.22/(\delta_n(a - 1)))$. Thus $\|\alpha^{q_n} - r_n\| < \delta_n$ for each $n \in \mathbb{N}$, as claimed.

For n even, the inequality on $\|\alpha^{\ell_n} - \theta_n\|$ becomes $\|\alpha^{\ell_{2n}} - \theta_{2n}\| < 2.22a^{-\ell_{2n+1}+\ell_{2n}+1}/(a - 1)$. Using $\ell_{2n+1} = q_{n+1}$, $\ell_{2n} = q_{n+1} - M$, $\theta_{2n} = w_n$ and $a^{M-1}(a - 1) \geq 10$, we derive that the inequality

$$\|\alpha^{q_{n+1}-M} - w_n\| < 2.22a^{-\ell_{2n+1}+\ell_{2n}+1}/(a - 1) = 2.22a^{-M+1}/(a - 1) \leq 0.222$$

holds for each positive integer n .

We shall use this inequality in order to show that all of the numbers $\alpha = \alpha(w) \in I$ corresponding to distinct sequences $w = (w_n)_{n=1}^{\infty}$ of 0 and $1/2$ are distinct. Indeed, suppose that $\alpha(w) = \alpha(w')$, although $w_n \neq w'_n$ for some positive integer n . Without loss of generality, we may assume that $w_n = 0$ and $w'_n = 1/2$. Then the inequality $\|\alpha(w)^{q_{n+1}-M} - w_n\| < 0.222$ implies that

$$\{\alpha(w)^{q_{n+1}-M}\} \in [0, 0.222) \cup (0.788, 1),$$

whereas the inequality $\|\alpha(w')^{q_{n+1}-M} - w'_n\| < 0.222$ implies that

$$\{\alpha(w')^{q_{n+1}-M}\} \in (0.288, 0.722).$$

Consequently, $\alpha(w) \neq \alpha(w')$, as claimed. Since there are continuum of infinite sequences w of two symbols 0, $1/2$, there is continuum of distinct numbers $\alpha(w) \in I$ such that the inequality $\|\alpha(w)^n - r_n\| < \delta_n$ holds for each positive integer n . This completes the proof of Theorem 1.

3. APPLICATIONS OF THE MAIN THEOREM

It is well known that there exist many numbers $\alpha > 1$ such that $\lim_{n \rightarrow \infty} \|\alpha^n\| = 0$ and, more generally, $\lim_{n \rightarrow \infty} \|\xi \alpha^n\| = 0$ for some $\xi \neq 0$. Such α must be a Pisot-Vijayaraghavan number, namely, an algebraic integer whose conjugates over $\mathbb{Q}$ (if any) are all of moduli strictly smaller than 1. (See [3], [4], [5], [6] and also [2].) However, it is not known whether there is at least one transcendental number $\alpha > 1$ such that $\lim_{n \rightarrow \infty} \|\alpha^n\| = 0$ (see [7]). From Theorem 1 we shall derive the following:

Corollary 2. *Let $(q_n)_{n=1}^{\infty}$ be a sequence of positive numbers satisfying $\lim_{n \rightarrow \infty} (q_{n+1} - q_n) = \infty$. Then there is a transcendental number $\alpha > 1$ such that $\lim_{n \rightarrow \infty} \|\alpha^{q_n}\| = 0$.*

Proof: Let us take $a = 11$ and $b = 13.2$ in Theorem 1. Then $M = 1$. Select $\delta_n = 0.222 \cdot 11^{2+q_n-q_{n+1}}$. Clearly, $q_{n+1} - q_n = 2 + \log_{11}(0.222/\delta_n)$, so the condition of the theorem is satisfied. Thus Theorem 1 with $r_1 = r_2 = r_3 = \dots = 0$ implies that there exists a transcendental number $\alpha \in [11, 13.2]$ such that $\|\alpha^{q_n}\| < 0.222 \cdot 11^{2+q_n-q_{n+1}}$ for every positive integer n such that $q_n \geq 1$. The condition $\lim_{n \rightarrow \infty} (q_{n+1} - q_n) = \infty$ implies that $q_n \geq 1$ for all sufficiently large n , and $\lim_{n \rightarrow \infty} 0.222 \cdot 11^{2+q_n-q_{n+1}} = 0$. Hence $\lim_{n \rightarrow \infty} \|\alpha^{q_n}\| = 0$, as claimed.

Corollary 3. *Let $(r_n)_{n=1}^{\infty}$ be a sequence of real numbers, and let $s_1, s_2, s_3, \dots \in \{1, \dots, L\}$, where L is a positive integer. Then, for any $\varepsilon > 0$, there is a transcendental number $\alpha > 1$ such that $\|s_n \alpha^n - r_n\| < \varepsilon$ for each positive integer n .*

Proof: This time, let us take in the theorem $a = 2$, $b = 3$, $M = 5$, $\delta_n = \varepsilon/s_n$ and $q_n = nT$ for each $n \geq 1$. Here, T is an integer satisfying $T \geq M + 1 + \log_2(1.11\varepsilon^{-1}L)$. The theorem with each r_n replaced by r_n/s_n implies that there is a transcendental number $\beta \in [2, 3]$ such that $\|\beta^{Tn} - r_n/s_n\| < \varepsilon/s_n$ for each positive integer n . Multiplying by the integer s_n and setting $\alpha = \beta^T$, we get that $\|s_n \alpha^n - r_n\| < \varepsilon$ for each $n \geq 1$, as claimed.

In particular, by Corollary 3, for any real numbers $a \geq 0$ and $\varepsilon > 0$ satisfying $0 \leq a < a + \varepsilon \leq 1$, there is a transcendental number $\alpha > 1$ such that $\{\alpha^n\} \in (a, a + \varepsilon)$ for each positive integer n .

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