

Sharp characters and their generalizations

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1. Blichfeldt's Theorem

Let G be a finite group and χ a virtual character of G . Let L be the set of values of χ . For $l \in L$, we define the number $B(l)$ as follows:

$$B(l) = \frac{a(l)}{|G|} \prod_{l' \in L - \{l\}} (l - l'),$$

where $a(l)$ denotes the number of elements x in G with $\chi(x) = l$.

Ninety years ago, Blichfeldt [B] proved that $B(l)$ is an algebraic integer for any $l \in L$. Our first aim is to extend this result. We will show that the numbers $B(l)$ ($l \in L$) are in fact the values of a virtual character $\tilde{\chi}$ of G , constructed from χ in a definite manner. More precisely, we have the following

Theorem 1. Let $\tilde{\chi}$ be a class function on G defined by $\tilde{\chi}(x) = B(\chi(x))$ for $x \in G$. Then $\tilde{\chi}$ is a virtual character of G .

Since the value of a group character is a sum of roots of unity, it is clear that Theorem 1 implies Blichfeldt's Theorem mentioned above.

Proof of Theorem 1. (Outline) For $x \in G$, we let f_x denote the monic polynomial of least degree whose set of roots is $L - \{\chi(x)\}$. Let f be the average of f_x over G :

$$f = \frac{1}{|G|} \sum_{x \in G} f_x.$$

Then we have the following

Claim. f is a monic polynomial with integral coefficients of degree $|L| - 1$.

In fact, the coefficients of f are expressed by integral linear combinations of $(\chi^i, 1_G)$ $i = 0, 1, \dots$ and symmetric functions of the elements in L . For example, if $L = \{n, l, k\}$ then we have $f(X) = X^2 - ((n+l+k) - (\chi, 1_G))X + ((nl+lk+kn) - (n+l+k)(\chi, 1_G) + (\chi^2, 1_G))$.

Now Theorem 1 follows easily from Claim since $\widetilde{\chi} = f(\chi)$.

Remark. The above f is the polynomial of least degree with $f(l) = B(l)$ for every $l \in L$, that is, the Lagrange interpolation polynomial through the points $\{(l, B(l)) \mid l \in L\}$.

One of the typical properties of $\widetilde{\chi}$ is that it does not take the value 0. So we can define the class function $1/\widetilde{\chi}$. By direct calculation, we obtain

Proposition 2. $(\overline{\chi}^i, 1/\widetilde{\chi}) = 0$ for $i = 0, 1, \dots, |L| - 2$.

Using Proposition 2 ($i=0$), we have the following divisibility conditions.

Proposition 3. For any $l \in L$, $B(l)$ divides $a(l) \prod_{l' \in L - \{l\}} B(l')$ in the ring of algebraic integers. In particular, if χ is a character of degree n , then $B(n)$ divides $\prod_{l \in L - \{n\}} B(l)$.

2. Sharp characters of finite groups

Under the same notation as in Section 1, we will define sharp triples for group characters.

Definitions. The triple (G, χ, l) is called a sharp triple if $B(l)$ is a unit in the ring of algebraic integers. The pair (G, χ) is called a sharp pair if $(G, \chi, \chi(1))$ is a sharp triple.

The concept of sharp pairs was first introduced by Cameron and Kiyota [CK], and their definition of sharp pairs is slightly different from ours. But at least in case χ is a faithful character of G , these two definitions are the same. So the concept of sharp triples is a natural generalization of that of sharp pairs

We will give some examples of sharp triples.

Example 1. Let G be cyclic and χ be a faithful linear character of G . Then (G, χ, l) is sharp for every $l \in \text{Im } \chi$.

Example 2. Let G be a sharply t -transitive permutation group and π be the associated permutation character. Then $(G, \pi, t-2)$ is a sharp triple, and (G, π) is a sharp pair.

The following Lemmas are easy to prove. (Use Proposition 3 for Lemma 5.)

Lemma 4. If (G, χ, l) is sharp, then $a(l)$ divides $|G|$.

Lemma 5. Let χ be a character of degree n . If (G, χ, l) is sharp for all $l \in L - \{n\}$, then (G, χ) is a sharp pair.

Question 6. If (G, χ, l) is sharp with χ a faithful character, then is it true that the set $\{x \in G \mid \chi(x) = l\}$ is a single conjugacy class of G ?

Problem 7. Determine all finite groups G such that (G, χ, l) is sharp for every non-trivial irreducible character χ and for every $l \in \text{Im } \chi$. Note that abelian groups and dihedral groups of twice odd prime order are such examples.

3. Classification of sharp triples for given L

From now on we assume χ is a faithful character of G of degree n . Set

$L = \text{Im } \chi$ and $L^* = L - \{n\}$. Cameron and Kiyota [CK] posed the problem of determining all the sharp pairs (G, χ) for a given set L^* . There are many papers on this subject; see the references of [AKN]. In particular Alvis and Nozawa [AN] have given a complete classification of sharp pairs when L^* contains an irrational number.

Now we will consider the analogous problem for sharp triples (G, χ, l) . The results known to me are very few. The first one is the simplest case and easy to prove.

Result 1. Let $L^* = \{\alpha_1, \dots, \alpha_t\}$ with all α_i are algebraically conjugate. If (G, χ, α_1) is sharp, then G is cyclic of prime order.

Proof. Since all α_i are conjugate, (G, χ, α_i) are all sharp, and so (G, χ) is sharp by Lemma 5. If $t \geq 2$, then the result follows from Theorem 4.1 in [CK]. Now assume $t = 1$. Then by Lemma 4, $a(\alpha_1)$ divides $|G| = 1 + a(\alpha_1)$. Thus $a(\alpha_1) = 1$, and so G is cyclic of order two. This completes the proof.

We will state the other known results without proofs.

Result 2. Let $L^* = \{0, \alpha_1, \dots, \alpha_t\}$ with all α_i are algebraically conjugate and $t \geq 2$. If $(\chi, 1_G) = 0$ and $(G, \chi, 0)$ is sharp, then G is cyclic of order 4, dihedral of twice odd prime order, or $E_{2^v} \rtimes Z_p$, where $p = 2^v - 1$ is a Mersenne prime.

Result 3. (Matsuhisa and Yamaki [MY]) Let $L^* = \{0, \varepsilon_1, \dots, \varepsilon_t\}$ with all ε_i are roots of unity. If $(G, \chi, 0)$ is sharp, then G is a sharply 3-transitive group or a 2-transitive Frobenius group.

Result 4. Let $L^* = \{l, k\}$ with integers l, k . If $(\chi, 1_G) = 0$ and (G, χ, l) is sharp, then one of the following holds:

(i) $k=0$ and (G, χ) is sharp of type $\{0, l\}$.

(ii) $k=-1, l=0$ and G is the symmetric group of degree 3.

(iii) $k=-1, l=1$ and G is quaternion or dihedral of order 8.

Problem 8. Determine all sharp triples (G, χ, l) when L^* contains an irrational number.

References

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