

Semi-algebraic aspect of the theory of Teichmüller space

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Abstract: We expose a semi-algebraic construction of Teichmüller space due to J. Morgan and P.B. Shalen, and Brumfiel's compactification of Teichmüller space by using the real spectrum in the sense of Coste.

§1. Semi-algebraic description of Teichmüller space.

§2. Real spectrum.

§3. The real spectrum compactification of Teichmüller space
(after Brumfiel).

§ 1. Semi-algebraic description of Teichmüller space.

In this section, we review the semi-algebraic construction of Teichmüller space due to Morgan - Shalen [MS].

Let Γ be the closed surface group of genus $g (\geq 2)$:

$$\Gamma = \langle \alpha_i, \beta_i \ (1 \leq i \leq g) \mid \prod_{i=1}^g [\alpha_i, \beta_i] = \text{id} \rangle$$

We can embed $\text{Hom}(\Gamma, \text{SL}_2(\mathbb{R}))$ as an algebraic subset of $\mathbb{R}^{8g}$ by using these generators $\alpha_i, \beta_i \ (1 \leq i \leq g)$:

$$\begin{array}{ccc} \text{Hom}(\Gamma, \text{SL}_2(\mathbb{R})) & \xrightarrow{\sim} & R(\Gamma) \subset \text{SL}_2(\mathbb{R})^{2g} \subset \mathbb{R}^{8g} \\ \downarrow & & \downarrow \\ \rho & \mapsto & (\rho(\alpha_1), \rho(\beta_1), \dots, \rho(\alpha_g), \rho(\beta_g)) \end{array}$$

Let $A(R(\Gamma))$ be the affine coordinate ring of $R(\Gamma)$. For any $g \in \Gamma$, we define a function $\tau_g \in A(R(\Gamma))$ by

$$\tau_g(\rho) := \text{tr}(\rho(g)) \quad (\forall \rho \in R(\Gamma))$$

Claim. (Helling [He], Horowitz [Ho], Culler - Shalen [CS])

$\mathbb{Z}$ -subalgebra of $A(R(\Gamma))$ generated by $\tau_g \ (g \in \Gamma)$ is finitely generated
i.e. $\exists g_1, \dots, g_2 \in \Gamma$ s.t.

$$\mathbb{Z}[\tau_g \mid g \in \Gamma] = \mathbb{Z}[\tau_{g_1}, \dots, \tau_{g_2}] \quad //$$

Let $X(\Gamma)$ be the algebraic subset of $\mathbb{R}^2$ whose affine coordinate ring is $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\tau_g \mid g \in \Gamma] = \mathbb{R}[\tau_{g_1}, \dots, \tau_{g_2}]$. Then we can define

the polynomial map $t: R(\Gamma) \rightarrow X(\Gamma)$ as follows:

$$\begin{array}{ccc} t: R(\Gamma) & \rightarrow & X(\Gamma) \\ \downarrow & & \downarrow \\ \rho & \mapsto & (\tau_{g_1}(\rho), \dots, \tau_{g_2}(\rho)) = (\text{tr}(\rho g_1), \dots, \text{tr}(\rho g_2)) \end{array}$$

We call $R(\Gamma)$ a space of representations and $X(\Gamma)$ a space of characters. Next claim is due to Culler - Shalen [CS].

Claim.

1) There exists a closed algebraic subset Δ of $X(\Gamma)$ such that

$$t^{-1}(\Delta) = \left\{ \rho \in R(\Gamma) \mid \rho(\Gamma) \subset \text{SL}_2(\mathbb{R}) \text{ is an abelian subgroup or } \right. \\ \left. \text{has an invariant line in } \mathbb{R}^2 \text{ by } \rho(\Gamma) \curvearrowright \mathbb{R}^2 \right\}$$

2) For any $\rho \in R(\Gamma) \setminus t^{-1}(\Delta)$ (i.e. ρ is non-abelian irreducible rep.),

$$t^{-1}(t(\rho)) = \text{PGL}_2(\mathbb{R})\text{-conj. class of } \rho. \quad //$$

We define $DR(\Gamma), DX(\Gamma)$ as follows.

$$\begin{aligned} DR(\Gamma) &:= \{ \rho \in R(\Gamma) \mid \rho \text{ is faithful and } \rho(\Gamma) \subset \text{SL}_2(\mathbb{R}) \text{ : discrete} \} \\ &= \{ \rho \in R(\Gamma) \mid \rho \text{ is totally hyperbolic i.e. for } \forall M (\neq \text{id}) \in \rho(\Gamma), \\ &\quad |\text{tr} M| > 2 \} \end{aligned}$$

$$DX(\Gamma) := t(DR(\Gamma))$$

Then 2) of the following claim is due to Weil [W] and Jørgensen [J].

Claim.

1) $DR(\Gamma) \subset R(\Gamma) \setminus t^{-1}(\Delta)$

$$t^{-1}(DX(\Gamma)) = DR(\Gamma)$$

$DR(\Gamma)$ is a trivial $PGL_2(\mathbb{R})$ -bundle over $DX(\Gamma)$.

(i.e. $DX(\Gamma) = DR(\Gamma) / PGL_2(\mathbb{R}) = \text{Aut}(SL_2(\mathbb{R}))$.)

- 2) $DR(\Gamma)$ (resp. $DX(\Gamma)$) consists of finite many connected components of $R(\Gamma)$ (resp. $X(\Gamma)$). Therefore $DR(\Gamma), DX(\Gamma)$ are semi-algebraic sets i.e. defined by finite many polynomial equations and inequations over $\mathbb{R}$. //

Next claim which is due to Patterson, tells the relation between $DX(\Gamma)$ and Teichmüller space.

Claim (Patterson [P]).

Let $\gamma: \Gamma \rightarrow PSL_2(\mathbb{R})$ be a discrete faithful representation.

Let $A_i, B_i \in SL_2(\mathbb{R})$ be any representatives of $\gamma(\alpha_i), \gamma(\beta_i)$ of $PSL_2(\mathbb{R})$ ($1 \leq i \leq g$). Then

$$\prod_{i=1}^g [A_i, B_i] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$$

In other words, γ can be always lifted to $\rho \in DR(\Gamma)$.

$$\begin{array}{ccc} & \rho & \\ & \dashrightarrow & SL_2(\mathbb{R}) \\ & & \downarrow \\ \Gamma & \xrightarrow{\gamma} & PSL_2(\mathbb{R}) \end{array}$$

Corollary

- 1) $\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ acts on $R(\Gamma)$ as the group which changes the sign of $\rho(\alpha_i), \rho(\beta_i) \in SL_2(\mathbb{R})$ for $\rho \in R(\Gamma)$ ($1 \leq i \leq g$).

Then the action $\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}) \curvearrowright \text{DR}(\Gamma)$ induces the action $\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}) \curvearrowright \text{DX}(\Gamma)$ through the map t , and we can consider $\text{DR}(\Gamma)/\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$, $\text{DX}(\Gamma)/\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ as following sets.

$\text{DR}(\Gamma)/\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ = the set of discrete faithful $\text{PSL}_2(\mathbb{R})$ -rep. of Γ .

$\text{DX}(\Gamma)/\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ = the set of $\text{PGL}_2(\mathbb{R})$ -conj. classes of discrete faithful $\text{PSL}_2(\mathbb{R})$ -rep. of Γ .

We call $T(\Gamma) := \text{DX}(\Gamma)/\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ Teichmüller space of Γ .

2) $\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ permutes the set of connected components of $\text{DX}(\Gamma)$ freely. Therefore,

$$\# |\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})| (= 2^{2g} = 4^g) \text{ divides } \# |\text{DX}(\Gamma)|$$

//

From the above argument, $T(\Gamma)$ can be considered as some components of $\text{DX}(\Gamma)$ and therefore has a semi-algebraic structure.

$$\text{DR}(\Gamma)$$

$$t \downarrow \text{PGL}_2(\mathbb{R})\text{-trivial bundle.}$$

$$\text{DX}(\Gamma)$$

$$\downarrow \text{unramified } \text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})\text{-covering.}$$

$$T(\Gamma)$$

§2. Real spectrum.

In this section we review the theory of real spectrum due to Coste [BCR].

2.1. Real spectrum.

Let $X \subseteq \mathbb{R}^N$ be a real algebraic set and $A(X) := \mathbb{R}[x_1, \dots, x_N] / I(X)$ be the affine coordinate ring of X .

A subset $\mathcal{A} \subset A(X)$ is called prime cone if it satisfies the following conditions:

- (i) For any $a, b \in \mathcal{A}$, $a+b \in \mathcal{A}$ (i.e. $\mathcal{A} + \mathcal{A} \subset \mathcal{A}$)
- (ii) For any $a, b \in \mathcal{A}$, $a \cdot b \in \mathcal{A}$ ($\mathcal{A} \cdot \mathcal{A} \subset \mathcal{A}$)
- (iii) For any $f \in A(X)$, $f^2 \in \mathcal{A}$ ($A(X)^2 \subset \mathcal{A}$)
- (iv) $-1 \notin \mathcal{A}$
- (v) If $a \cdot b \in \mathcal{A}$ for $a, b \in A(X)$, then $a \in \mathcal{A}$ or $-b \in \mathcal{A}$.

Prime cone has the following properties

Claim.

- 1) $\mathcal{A} \cup -\mathcal{A} = A(X)$ (where $-\mathcal{A} := \{-a \mid a \in \mathcal{A}\}$)
- 2) $\text{Supp}(\mathcal{A}) := \mathcal{A} \cap -\mathcal{A}$ is a prime ideal of $A(X)$.
- 3) Let $k(\mathcal{A})$ be the quotient field of $A(X) / \text{Supp}(\mathcal{A})$. Then $P := \{ \frac{\bar{a}}{\bar{b}} \in k(\mathcal{A}) \mid a \cdot b \in \mathcal{A} \}$ is a positive cone.
(i.e. $P + P \subset P \wedge P \cdot P \subset P \wedge k(\mathcal{A})^2 \subset P \wedge -1 \notin P \wedge P \cup -P = k(\mathcal{A})$)

We define the real spectrum by the set of prime cone of $A(X)$.

$$\text{Spec}_r A(X) := \{ \mathfrak{d} \subset A(X) \mid \mathfrak{d} \text{ is a prime cone of } A(X) \}$$

Moreover we define the topology on $\text{Spec}_r A(X)$ as follows:

If we put $\mathcal{U}(f) := \{ \mathfrak{d} \in \text{Spec}_r A(X) \mid f \notin \mathfrak{d}, \text{Supp}(\mathfrak{d}) \} \quad (f \in A(X))$,

then $\bigcap_{i=1}^m \mathcal{U}(f_i) \quad (f_i \in A(X))$ is an open basis of $\text{Spec}_r A(X)$.

Claim.

- 1) With this topology, $\text{Spec}_r A(X)$ is quasi-compact.
- 2) $\text{Spec}_r^m A(X) := \{ \mathfrak{d} \in \text{Spec}_r A(X) \mid \mathfrak{d} \text{ is a closed point} \}$ is a compact Hausdorff space.
- 3) X (with induced Euclidean topology) can be embedded topologically in $\text{Spec}_r^m A(X)$:

$$\begin{array}{ccc} X & \hookrightarrow & \text{Spec}_r^m A(X) \\ \downarrow & & \downarrow \\ \vec{x} & \mapsto & \mathfrak{d}_x := \{ f \in A(X) \mid f(\vec{x}) \geq 0 \} \quad // \end{array}$$

Therefore, we can consider X as a subset of the compact Hausdorff space $\text{Spec}_r^m A(X)$.

2.2. The real spectrum compactification of closed semi-alg. sets.

A subset $S \subset X$ is called a semi-algebraic subset of X if there exist finite many $f_i, g_{ij} \in A(X)$ ($1 \leq i \leq \ell, 1 \leq j \leq m(i)$)

such that

$$S = \bigcup_{i=1}^l \{ \vec{x} \in X \mid f_i(\vec{x}) = 0 \wedge g_{i1}(\vec{x}) > 0 \wedge \dots \wedge g_{i m(i)}(\vec{x}) > 0 \}.$$

A subset $C \subset \text{Spec}_r A(X)$ is called a constructible subset of $\text{Spec}_r A(X)$ if there exist f_i, g_{ij} ($1 \leq i \leq l, 1 \leq j \leq m(i)$) such that

$$C = \bigcup_{i=1}^l \{ d \in \text{Spec}_r A(X) \mid f_i(d) = 0 \wedge g_{i1}(d) > 0 \wedge \dots \wedge g_{i m(i)}(d) > 0 \}$$

(where $f_i(d), g_{ij}(d)$ are the image of f_i, g_{ij} of the map

$A(X) \twoheadrightarrow A(X)/\text{supp}(d) \hookrightarrow k(d)$. Because $k(d)$ has a positive

cone $P = \{ \frac{a}{b} \in k(d) \mid a, b \in d \}$, we can define an order $\leq$

on $k(d)$ by $x \leq y \Leftrightarrow y - x \in P$ (for $x, y \in k(d)$))

Let $\mathcal{S}$ be the collection of semi-algebraic subsets of X and $\mathcal{C}$ be the collection of constructible subsets of $\text{Spec}_r A(X)$. Next claim tells the relation between $\mathcal{S}$ and $\mathcal{C}$.

Claim

the map $\begin{array}{ccc} \mathcal{C} & \rightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ C & \mapsto & C \cap X \end{array}$ is bijective and open (resp. closed)

constructible set goes to open (resp. closed) semi-algebraic set, //

Let $C(S) \in \mathcal{C}$ be the constructible set corresponding to $S \in \mathcal{S}$. If $W \in \mathcal{S}$ is a closed semi-aly. subset of X , then we can define the real spectrum compactification of W as the closure

$\tilde{W}$ of W in $\text{Spec}^m A(X)$ (where we assume X as a subset of $\text{Spec}^m A(X)$).

Claim (Structure of $\tilde{W}$)

- 1) $\tilde{W} = CCW \cap \text{Spec}^m A(X)$
- 2) $\tilde{B}(W) := \tilde{W} \setminus W = \{x \in \tilde{W} \mid (\sum_{i=1}^N x_i^2 - t)(x) > 0 \text{ (for } \forall t \in \mathbb{R})\}$
- 3) $W \subset \tilde{W}$: open and dense.
- 4) If $W_1, \dots, W_s$ are the connected components of W , then $\tilde{W}_1, \dots, \tilde{W}_s$ are the connected components of $\tilde{W}$. //

Next we consider the mapping between semi-alg. sets.

For $S_1, S_2 \in \mathcal{S}$, a mapping $f: S_1 \rightarrow S_2$ is called semi-algebraic if the graph of f in $S_1 \times S_2$ is a semi-alg. subset. In this case, if V_1, V_2 be semi-alg. sets of S_1, S_2 , then $f(V_1) \subset S_2$, $f^{-1}(V_2) \subset S_1$ are also semi-alg. subsets.

Claim.

If $f: S_1 \rightarrow S_2$ ($S_1, S_2 \in \mathcal{S}$) is a semi-alg. continuous map, then there exists uniquely the map $c(f): C(S_1) \rightarrow C(S_2)$ which is continuous in the real spectrum topology and satisfies the following functorial condition:

For any semi-alg. subset V of S_2

$$C(f^{-1}(V)) = c(f)^{-1}(C(V))$$

$$\begin{array}{ccc}
 C(S_1) & \xrightarrow{C(f)} & C(S_2) \\
 \uparrow & & \uparrow \\
 S_1 & \xrightarrow{f} & S_2
 \end{array}$$

In particular if f is a semi-alg. homeomorphism, then $C(f)$ is also homeomorphism and moreover if $S_1, S_2 \in \mathcal{L}$ are closed semi-alg. sets, then $C(f)$ induces the homeomorphism $f^* \tilde{S}_1 \cong \tilde{S}_2$. Therefore if f is an semi-algebraic automorphism of a closed semi-alg. set W , then f is always extended to the automorphism of its real spectrum compactification $\tilde{W}$. //

This result will be used later in the context where W is $\mathbb{D}X(\Gamma)$ and f is an element of $\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ and where W is $T(\Gamma)$ and f is an element of the mapping class group $\text{Out}^+(\Gamma)$.

§3. The real spectrum compactification of Teichmüller space (after Brumfiel [B]).

In section 1, we have seen that Teichmüller space $T(\Gamma)$ can be considered as a semi-alg. subset of $X(\Gamma)$, more exactly some components of $DX(\Gamma)$. In this section we apply the theory of the real spectrum compactification of closed semi-alg. set to $DX(\Gamma)$ or $T(\Gamma)$. Thus, $DX(\Gamma) \subset X(\Gamma)$ can be compactified as $\widetilde{DX(\Gamma)} \subset \text{Spec}_+^m A(X(\Gamma))$ and because $\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ acts on $DX(\Gamma)$ semi-algebraically, this action extends on $\widetilde{DX(\Gamma)}$, therefore we can define $\widetilde{T(\Gamma)}$ by

$$\widetilde{T(\Gamma)} := \widetilde{DX(\Gamma)} / \text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$$

3.1. Representation theoretic characterization of boundary points of $\widetilde{T(\Gamma)}$.

By using the argument in §2, the diagram

$$\begin{array}{ccc} R(\Gamma) & \xrightarrow{t} & X(\Gamma) & (A(X(\Gamma)) \xrightarrow{t^*} A(R(\Gamma))) \\ \cup & & \cup & \\ DR(\Gamma) & \twoheadrightarrow & DX(\Gamma) & \end{array}$$

induces the following maps.

$$\begin{array}{ccc} \text{Spec}_+ A(R(\Gamma)) (= C(R(\Gamma))) & \xrightarrow{C(t)} & \text{Spec}_+ A(X(\Gamma)) (= C(X(\Gamma))) \\ \cup & \downarrow \beta \mapsto t_*^{-1}(\beta) & \cup \\ C(DR(\Gamma)) & \xrightarrow{\quad} & C(DX(\Gamma)) \\ \cup & & \cup \\ \widetilde{DR(\Gamma)} & & \widetilde{DX(\Gamma)} \end{array}$$

with $C(\tau)^{-1}(C(DX(\Gamma))) = C(\tau^{-1}(DX(\Gamma))) = C(DR(\Gamma))$.

If $\alpha \in \widetilde{DX}(\Gamma) \setminus DX(\Gamma)$, then for any $\beta \in C(\tau)^{-1}(\alpha)$, there exists homomorphism from $k(\alpha)$ to $k(\beta)$ as follows.

$$\begin{array}{ccccc} A(R(\Gamma)) & \twoheadrightarrow & A(R(\Gamma))/\text{Supp}(\beta) & \hookrightarrow & k(\beta) \\ \uparrow & & \uparrow & & \uparrow \\ A(X(\Gamma)) & \twoheadrightarrow & A(X(\Gamma))/\text{Supp}(\alpha) & \hookrightarrow & k(\alpha) \end{array}$$

By using Tarski principle (this is not defined here), we can prove that there exists $\beta \in C(\tau)^{-1}(\alpha)$ such that $k(\beta)/k(\alpha)$ is algebraic. In this case, we can also prove the next claim.

Claim.

If $k(\beta)/k(\alpha)$ is algebraic, then $\beta \in \widetilde{DR}(\Gamma) \setminus DR(\Gamma)$.

Moreover, $\beta \in \text{Spec}_r A(R(\Gamma))$ induces the following map:

$$A(R(\Gamma)) \twoheadrightarrow A(R(\Gamma))/\text{Supp}(\beta) \hookrightarrow k(\beta)$$

and this means that β can be considered as $k(\beta)$ -valued point of $R(\Gamma)$

$$\text{i.e. } \beta \in \text{Hom}(A(R(\Gamma)), k(\beta)) = \text{Hom}(\Gamma, \text{SL}_2(k(\beta)))$$

Thus, β is a representation $\beta: \Gamma \rightarrow \text{SL}_2(k(\beta))$. C. Frohman proved that if $\beta \in \widetilde{PR}(\Gamma)$, then $\beta: \Gamma \rightarrow \text{SL}_2(k(\beta))$ is discrete faithful (moreover, totally hyperbolic) [B].

Summarizing,

Claim.

For any $[\alpha] \in \widetilde{T}(\Gamma) \setminus T(\Gamma)$ ($\alpha \in \widetilde{DX}(\Gamma) \setminus DX(\Gamma)$), there exists a representation $\rho: \Gamma \rightarrow SL_2(\mathbb{K}(\rho))$ over $[\alpha]$ which is discrete, faithful and belongs to $\widetilde{DR}(\Gamma) \setminus DR(\Gamma)$.

3.2. Comparison with the Thurston compactification.

Let $(F, \leq)$ be an ordered field. We call $b \in F^+ := \{x \in F \mid x > 0\}$ is a big element if for any $a \in F$, there exists $m \in \mathbb{N}$ such that $a < b^m$. (For example, any $t (> 1) \in \mathbb{R}$ is a big element of $\mathbb{R}$.)

If an ordered field $(F, \leq)$ has a big element, we can define the logarithm $\log_b: F^+ \rightarrow \mathbb{R}$ by using the Dedekind cut of $\mathbb{Q}$:

$$\frac{m'}{n} \leq \log_b(a) \leq \frac{m}{n} \quad \text{if } b^{m'} \leq a^n \leq b^m \quad \left(a, b \in F^+, m, m', n \in \mathbb{Z} \right. \\ \left. n > 0 \right)$$

This function has properties which are satisfied by the ordinary logarithm on $\mathbb{R}^+$. For example,

$$(a) \quad \log_b(b^m) = m \quad (\forall m \in \mathbb{Z})$$

$$(b) \quad \log_b(a \cdot a') = \log_b(a) + \log_b(a')$$

$$(c) \quad \text{If } 0 < a < a' \quad (a, a' \in F^+), \text{ then } \log_b(a) \leq \log_b(a')$$

$$(d) \quad \text{If } b, b' \text{ are big elements of } F \text{ and } a \in F^+, \text{ then}$$

$$\log_{b'}(a) = \log_{b'}(b) \log_b(a) \quad \text{and} \quad \log_b(b') > 0$$

Let $\mathcal{S} \subset A(X)$ be a subset which satisfies the following properties:

- (i) $\mathcal{S}$ contains generator system of $A(X)$ as $\mathbb{R}$ -algebra.
- (ii) For any $\vec{x} \in W$ and any $f \in \mathcal{S}$, $|f(\vec{x})| \geq 1$
- (iii) For any $\vec{x} \in W$, there exists $f \in \mathcal{S}$ such that $|f(\vec{x})| > 1$.

If there exists such $\mathcal{S} \subset A(X)$, we can define the continuous map θ from W to the infinite dimensional projective space $\mathbb{P}^{\mathcal{S}}$ by using logarithm:

$$\begin{array}{ccc} \theta : W & \rightarrow & \mathbb{P}^{\mathcal{S}} \\ \downarrow & & \downarrow \\ \vec{x} & \mapsto & \theta(\vec{x}) = (\log |f(\vec{x})|)_{f \in \mathcal{S}} \end{array}$$

(where θ does not depend on the base of logarithm.)

θ can be extended uniquely to the map from the real spectrum compactification of W .

Claim.

θ can be extended continuously to $\tilde{\theta}$ by the same formula.

$$\begin{array}{ccc} \tilde{\theta} : \tilde{W} & \rightarrow & \mathbb{P}^{\mathcal{S}} \\ \downarrow & & \\ \alpha & \mapsto & \tilde{\theta}(\alpha) = (\log |f(\alpha)|)_{f \in \mathcal{S}} \quad \left(\begin{array}{l} \text{where} \\ f(\alpha) \in k(\alpha) \end{array} \right) \end{array}$$

Next we apply the above consideration to $\tilde{T(\Gamma)}$.

Let S be the set of conjugacy classes of the primary elements of Γ

where primary element means that it is not a power of any other element of Γ , and put $\mathcal{S} := \{\tau_g \mid [g] \in \mathcal{S}\}$.

Then $\mathcal{S}$ satisfies the conditions (i), (ii), (iii), therefore we can consider the following map θ :

$$\begin{array}{ccc} \theta : T(\Gamma) & \longrightarrow & \mathbb{P}^{\mathcal{S}} \\ \downarrow & & \downarrow \\ [p] & \longmapsto & (\log |\tau_g(p)|)_{[g] \in \mathcal{S}} = (\log |tr(pg)|)_{[g] \in \mathcal{S}}. \end{array}$$

It is known that θ is homeomorphic and the closure of $\theta(T(\Gamma))$ in $\mathbb{P}^{\mathcal{S}}$ is essentially the Thurston compactification $\widehat{T(\Gamma)}$ of $T(\Gamma)$.

Moreover $\text{Out}^+(\Gamma)$ (subgroup of $\text{Out}(\Gamma)$ of index 2) acts on $\mathcal{S}$, therefore on $\mathbb{P}^{\mathcal{S}}$ by the change of coordinates. On the other hand, the action $\text{Out}^+(\Gamma)$ on $\text{Spec}_+ A(X(\Gamma))$ is also induced by the action of $\text{Out}^+(\Gamma)$ on $A(X(\Gamma)) = \mathbb{R}[\tau_g \mid g \in \mathcal{S}] = \mathbb{R}[\mathcal{S}]$. This leads to the last claim.

Claim.

- 1) There exists surjective continuous map $\tilde{\theta}$ from $\widehat{T(\Gamma)}$ to $\widehat{T(\Gamma)}$.

$$\begin{array}{ccc} \tilde{\theta} : \widehat{T(\Gamma)} & \longrightarrow & \widehat{T(\Gamma)} \\ \downarrow & & \downarrow \\ \lambda & \longmapsto & (\log |\tau_g(\lambda)|)_{[g] \in \mathcal{S}} \end{array}$$

- 2) $\tilde{\theta}$ is $\text{Out}^+(\Gamma)$ -equivariant. //

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