

A remark on semisimple elements in $U_q(\mathfrak{sl}(2;\mathbb{C}))$.

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In this short note we will give a remark on semisimple elements in the quantized universal enveloping algebra $U_q(\mathfrak{sl}(2;\mathbb{C}))$. Namely we will show that, even in the case of $U_q(\mathfrak{sl}(2;\mathbb{C}))$, there is a family of semisimple twisted primitive elements, analogous to the adjoint orbit $\{ \text{Ad}(g)h; g \in \text{SL}(2;\mathbb{C}) \}$ of the semisimple element h of the Lie algebra $\mathfrak{sl}(2;\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$.

Existence of semisimple twisted primitive elements in $U_q(\mathfrak{sl}(2;\mathbb{C}))$ was first recognized by T.H. Koornwinder [K]. The content of this note is a part of a joint work with Mr. Katsuhisa Mimachi on the realization of Askey-Wilson polynomials as spherical functions on the quantum group $SU_q(2)$ ([NM2]).

1. To fix the notation, we recall the definition of the quantized universal enveloping algebra $U_q(\mathfrak{sl}(2;\mathbb{C}))$. This algebra is the $\mathbb{C}$ -algebra generated by the letters t, t^{-1}, e, f subject to the fundamental relations

$$(1) \quad tt^{-1} = t^{-1}t = 1, \quad tet^{-1} = q^2e, \quad tft^{-1} = q^{-2}f, \quad ef - fe = (t - t^{-1})/(q - q^{-1}).$$

Throughout this note, the symbol q denotes a fixed non-zero

complex number (with $q^2 \neq 1$) and we always assume that q is not a root of unity. For each nonnegative integer ℓ , we denote by V_ℓ the unique $(\ell+1)$ -dimensional irreducible left $U_q(\mathfrak{sl}(2))$ -module with highest weight q^ℓ . The vector representation V_1 has a basis (v_1, v_{-1}) under which the action of $U_q(\mathfrak{sl}(2; \mathbb{C}))$ is described by

$$(2) \quad t \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad e \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We fix a Hopf algebra structure of $U_q(\mathfrak{sl}(2; \mathbb{C}))$ so that its coproduct Δ takes the following values at the generators:

$$(3) \quad \Delta(t) = t \otimes t, \quad \Delta(e) = e \otimes 1 + t \otimes e, \quad \Delta(f) = f \otimes t^{-1} + 1 \otimes f.$$

We say that an element $X \in U_q(\mathfrak{sl}(2))$ is a twisted primitive element of type $(t^{-1}, 1)$ (resp. of type $(1, t)$) if $\Delta(X) = X \otimes t^{-1} + 1 \otimes X$ (resp. $\Delta(X) = X \otimes 1 + t \otimes X$). Under the assumption that q is not a root of unity, it turns out that any twisted primitive element X of type $(t^{-1}, 1)$ is a linear combination

$$(4) \quad X = at^{-1}e + b(1-t^{-1}) + cf \quad \text{for some } a, b, c \in \mathbb{C}.$$

2. Let $X \in U_q(\mathfrak{sl}(2; \mathbb{C}))$ be a twisted primitive element of type $(t^{-1}, 1)$ and suppose that it is diagonalizable on the vector representation $V_1 = \mathbb{C}v_1 \oplus \mathbb{C}v_{-1}$. Then it is directly shown that the element X is a constant multiple of a twisted primitive element of the form

$$(5) \quad X_g = -(q-q^{-1})\alpha\beta t^{-1}e + (\alpha\delta + \beta\gamma)(1-t^{-1}) + (q-q^{-1})\gamma\delta f,$$

where $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL(2; \mathbb{C})$. Note that X_g has linearly independent eigenvectors $u_1 = v_1\alpha + v_{-1}\gamma$ and $u_{-1} = v_1\beta + v_{-1}\delta$ belonging to the

eigenvalues $(1-q^{-1})(\alpha\delta-q\beta\gamma)$ and $(1-q)(\alpha\delta-q^{-1}\beta\gamma)$, respectively.

We also remark that, in the limit as $q \rightarrow 1$, the element

$\frac{1}{(1-q^{-1})(\alpha\delta-\beta\gamma)} X_g$ gives a parametrization of the adjoint orbit of the semisimple element h in the Lie algebra $\mathfrak{sl}(2;\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$.

Theorem 1. Assume that q is not a root of unity. Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

be an element of $GL(2;\mathbb{C})$ such that $\alpha\delta - q^{2k}\beta\gamma \neq 0$ for all $k \in \mathbb{Z}$.

Then the twisted primitive element X_g defined by (5) is semisimple in the sense that it is diagonalizable on every finite dimensional $U_q(\mathfrak{sl}(2;\mathbb{C}))$ -module. Moreover, on each irreducible representation V_ℓ ($\ell \in \mathbb{N}$), the linear mapping $X_g: V_\ell \rightarrow V_\ell$ has mutually distinct eigenvalues $(1-q^{-m})(\alpha\delta - q^m\beta\gamma)$ ($m = \ell, \ell-2, \dots, \ell$).

3. We will show that, for each $\ell \in \mathbb{N}$, the linear mapping $X_g: V_\ell \rightarrow V_\ell$ has the eigenvalues as described above. Then one sees that X_g is diagonalizable on every finite dimensional representations, by the classification of finite dimensional $U_q(\mathfrak{sl}(2;\mathbb{C}))$ -modules (see Rosso [R]).

In order to study the action of X_g , we make use of the realization of the representations V_ℓ ($\ell \in \mathbb{N}$) in the coordinate ring of the quantum plane $\mathbb{C}_q^2$. Let $A(\mathbb{C}_q^2)$ be the $\mathbb{C}$ -algebra generated by the two letters z, w with commutation relation $zw = qwz$. Then it is well known that $A(\mathbb{C}_q^2)$ is a $\mathbb{C}$ -algebra with $U_q(\mathfrak{sl}(2;\mathbb{C}))$ -symmetry. Namely, it has a left $U_q(\mathfrak{sl}(2;\mathbb{C}))$ -module structure such that,

$$(6) \quad \text{if } a \in U_q(\mathfrak{sl}(2;\mathbb{C})) \text{ and } \Delta(a) = \sum_i a_i^1 \otimes a_i^2, \text{ then}$$

$$a.(\varphi\psi) = \sum_i (a_i^1 \cdot \varphi)(a_i^2 \cdot \psi) \quad \text{for any } \varphi, \psi \in A(\mathbb{C}_q^2).$$

Note that the action of $U_q(\mathfrak{sl}(2;\mathbb{C}))$ on $A(\mathbb{C}_q^2)$ is completely determined by (6) together with the following action on the generators z, w :

$$(7) \quad t.(z, w) = (zq, wq^{-1}), \quad e.(z, w) = (0, z), \quad f.(z, w) = (w, 0).$$

Furthermore, the algebra $A(\mathbb{C}_q^2)$ has the irreducible decomposition

$$(8) \quad A(\mathbb{C}_q^2) = \bigoplus_{\ell=0}^{\infty} V_{\ell}, \quad \text{where } V_{\ell} = \mathbb{C}z^{\ell} \oplus \mathbb{C}wz^{\ell-1} \oplus \dots \oplus \mathbb{C}w^{\ell}.$$

In this algebra $A(\mathbb{C}_q^2)$, we will construct the eigenvectors of X_g in an explicit manner. For any couple (a, b) of nonnegative integers, we define an element $\varphi_{a,b}$ in $A(\mathbb{C}_q^2)$ by the formula

$$(9) \quad \varphi_{a,b} = (z\beta + w\delta)(z\beta q^{-1} + w\delta) \dots (z\beta q^{-b+1} + w\delta) \\ \times (z\alpha + w\gamma q^{-b})(z\alpha + w\gamma q^{-b+1}) \dots (z\alpha + w\gamma q^{-b+a-1}).$$

Note that, in the case when $q=1$, this element corresponds to $(z\beta + w\delta)^b (z\alpha + w\gamma)^a = g \cdot w^b z^a$. For each integer $m \in \mathbb{Z}$, we set

$$(10) \quad \lambda_m = (1 - q^{-m})(\alpha\delta - q^m \beta\gamma).$$

Lemma 2. For any $a, b \in \mathbb{N}$, one has $X_g \varphi_{a,b} = \lambda_{a-b} \varphi_{a,b}$.

Proof. Note first that

$$(11) \quad \varphi_{0,b+1} = \varphi_{0,b} (z\beta q^{-b} + w\delta) \quad \text{and} \quad \varphi_{a+1,b} = \varphi_{a,b} (z\alpha + w\gamma q^{a-b})$$

for any $a, b \in \mathbb{N}$. Hence, it is enough to show that, if φ is an element of $A(\mathbb{C}_q^2)$ such that $X_g \varphi = \lambda_m \varphi$ for some $m \in \mathbb{Z}$, then

$$(12) \quad X_g(\varphi(z\alpha + w\gamma q^m)) = \lambda_{m+1} \varphi(z\alpha + w\gamma q^m) \quad \text{and}$$

$$X_g(\varphi(z\beta q^m + w\delta)) = \lambda_{m-1} \varphi(z\beta q^m + w\delta).$$

By using property (6), one can reduce formulas (12) to the equations

$$(13) \quad \lambda_{m+1} - q \lambda_m = (1-q)(\alpha\delta + \beta\gamma) + (q-q^{-1})\alpha\delta q^{-m} \quad (m \in \mathbb{Z}) \quad \text{and}$$

$$\lambda_{m+1} - q^{-1} \lambda_m = (1-q^{-1})(\alpha\delta + \beta\gamma) - (q-q^{-1})\beta\gamma q^m \quad (m \in \mathbb{Z}).$$

These can be checked directly from the definition (10) of λ_m . ■

It is clear that each $\varphi_{a,b}$ is a nonzero element in $A(\mathbb{C}_q^2)$ provided that $\alpha\delta - \beta\gamma \neq 0$. Hence, for each $\ell \in \mathbb{N}$, the elements $\varphi_{\ell,0}, \varphi_{\ell-1,1}, \dots, \varphi_{0,\ell}$ in V_ℓ are eigenvectors of X_g belonging to the eigenvalues $\lambda_\ell, \lambda_{\ell-2}, \dots, \lambda_{-\ell}$, respectively. Under the assumption $\alpha\delta - q^{2k}\beta\gamma \neq 0$ for all $k \in \mathbb{Z}$, these eigenvalues λ_m ($m = \ell, \ell-2, \dots, -\ell$) are mutually distinct; this implies that the $\ell+1$ eigenvectors $\varphi_{\ell,0}, \varphi_{\ell-1,1}, \dots, \varphi_{0,\ell}$ form a $\mathbb{C}$ -basis for V_ℓ as desired. This completes the proof of Theorem 1.

4. Remark. In the above argument, we considered only the twisted primitive elements of type $(t^{-1}, 1)$. Recall that there is an involutive algebra automorphism $\omega: U_q(\mathfrak{sl}(2; \mathbb{C})) \rightarrow U_q(\mathfrak{sl}(2; \mathbb{C}))$ such that $\omega(t) = t^{-1}$, $\omega(e) = -q^{-1}f$ and $\omega(f) = -qe$. Since ω is a coalgebra antiautomorphism, the twisted primitive elements of type $(t^{-1}, 1)$ are transformed into those of type $(1, t)$. By this involution ω , it is easy to rewrite Theorem 1 to a version for twisted primitive elements of type $(1, t)$.

5. Finally we give a remark on the construction of the

eigenvectors $\varphi_{a,b}$ ($a, b \in \mathbb{N}$) of X_g .

For a fixed element $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2; \mathbb{C})$, define the two elements Z, W in $A(\mathbb{C}_q^2)$ by the formula

$$(14) \quad Z = z\alpha + w\gamma, \quad W = z\beta + w\delta; \quad \text{namely,} \quad (Z, W) = (z, w) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

A point to be discussed is how to control this kind of "coordinate transformation" $(z, w) \mapsto (Z, W)$ induced by g . Although one cannot expect the commutation relation $ZW = qWZ$ any longer, there exists an interesting formula very close to this. Namely one has

$$(15) \quad (z\alpha + w\gamma)(z\beta q + w\delta) = q(z\beta + w\delta)(z\alpha + w\gamma q^{-1}).$$

To take this equality into the argument, regard the symbols $\alpha, \beta, \gamma, \delta$ as indeterminates and let $\mathbb{C} = \mathbb{C}[\alpha, \beta, \gamma, \delta]$ be the (commutative) polynomial ring in four variables. We define an $\mathbb{C}$ -algebra automorphism $\tau: \mathbb{C} \rightarrow \mathbb{C}$ by

$$(16) \quad \tau(\alpha) = \alpha, \quad \tau(\beta) = \beta q, \quad \tau(\gamma) = \gamma q \quad \text{and} \quad \tau(\delta) = \delta.$$

Namely, τ is the q -shift operator in the variables β and γ . Let $\mathbb{C}[\tau, \tau^{-1}]$ be the subalgebra of $\text{End}_{\mathbb{C}}(\mathbb{C})$ generated by the left multiplication of $\alpha, \beta, \gamma, \delta$ and the q -shift operators τ, τ^{-1} . We now define the elements $\tilde{Z}, \tilde{W}$ in the extension $A(\mathbb{C}_q^2) \otimes \mathbb{C}[\tau, \tau^{-1}]$ by $\tilde{Z} = Z\tau = (z\alpha + w\gamma)\tau$, $\tilde{W} = W\tau^{-1} = (z\beta + w\delta)\tau^{-1}$, namely by

$$(17) \quad (\tilde{Z}, \tilde{W}) = (Z\tau, W\tau^{-1}) = (z, w) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}.$$

Then formula (15) is equivalent to the commutation relation $\tilde{Z}\tilde{W} = q\tilde{W}\tilde{Z}$ in $A(\mathbb{C}_q^2) \otimes \mathbb{C}[\tau, \tau^{-1}]$. A surprising fact is that the eigenvectors $\varphi_{a,b}$ ($a, b \in \mathbb{N}$) we constructed above arise naturally

from this framework. In fact one has

$$(18) \quad \tilde{w}^b \tilde{z}^a = (z\beta + w\delta)\tau^{-1} \dots (z\beta + w\delta)\tau^{-1} (z\alpha + w\gamma)\tau \dots (z\alpha + w\gamma)\tau = \varphi_{a,b} \tau^{a-b}.$$

The second equality is obtained by moving τ 's and τ^{-1} 's between the linear factors to the right end.

6. In this note, we showed that there is a family of semisimple twisted primitive elements corresponding to the adjoint orbit of h in $sl(2; \mathbb{C})$ and that their eigenvectors are constructed by a sort of "coordinate transformations" on the quantum plane. These two facts are extensively used in the study of spherical functions on the quantum group $SU_q(2)$ and quantum spheres (see [NM1,2], [N]). It is also known that the connection coefficients between the two bases $(z^\ell, wz^{\ell-1}, \dots, w^\ell)$ and $(\varphi_{\ell,0}, \varphi_{\ell-1,1}, \dots, \varphi_{0,\ell})$ of V_ℓ are expressed by the q -Krawtchouk polynomials. Namely if one expresses the eigenvectors as linear combinations of $w^i z^{\ell-i}$ in the form

$$(19) \quad \varphi_{\ell-j,j} = \sum_{i=0}^{\ell} w^i z^{\ell-i} C_{ij}^{(\ell)}, \quad (C_{ij}^{(\ell)} \in \mathbb{C}),$$

then the coefficients $C_{ij}^{(\ell)}$ are polynomials in $\alpha, \beta, \gamma, \delta$ and are explicitly written in terms of q -hypergeometric series:

$$(20) \quad C_{ij}^{(\ell)} = q^{(i-j)(i+j-1)/2} \alpha^{\ell-i-j} \beta^i \gamma^j \begin{bmatrix} \ell \\ i \end{bmatrix}_q \times {}_3\phi_2 \left(\begin{matrix} q^{-2i}, q^{-2j}, q^{2(j-\ell)} \\ 0, q^{-2\ell} \end{matrix} ; q^2, q^2 \right) \alpha\delta/\beta\gamma.$$

(Cf. [NM1].)

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