

Gravitational perturbations as $T\bar{T}$ -deformations in 2D dilaton gravity systems

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Received 6 October 2019; received in revised form 2 December 2019; accepted 14 December 2019

Available online 19 December 2019

Editor: Clay Córdova

Abstract

We consider gravitational perturbations of 2D dilaton gravity systems and show that these can be recast into $T\bar{T}$ -deformations (at least) under certain conditions, where T means the energy-momentum tensor of the matter field coupled to a dilaton gravity. In particular, the class of theories under this condition includes a Jackiw-Teitelboim (JT) theory with a negative cosmological constant including conformal matter fields. This is a generalization of the preceding work on the flat-space JT gravity by S. Dubovsky, V. Gorbenko and M. Mirbabayi [arXiv:1706.06604].

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Contents

1. Introduction	2
2. 2D dilaton gravity systems coupled with matter field	3

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<https://doi.org/10.1016/j.nucphysb.2019.114901>

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2.1.	Comment on the flat-space JT gravity	4
3.	Perturbing 2D dilaton gravity systems	5
3.1.	The quadratic action	5
3.2.	Equations of motion for the fluctuations	6
3.3.	Simplification of the quadratic action	6
4.	Gravitational perturbations as $T\bar{T}$ -deformations	7
4.1.	The case of the flat-space JT gravity	7
4.2.	The $U'(\phi) \neq 0$ and $U''(\phi) = 0$ case	9
5.	Conclusion and discussion	14
	Acknowledgements	14
	Appendix A. Useful formulae	15
	Appendix B. A derivation of the quadratic action	16
	References	18

1. Introduction

A significant subject is to study integrable deformations of 2D integrable quantum field theories (IQFTs) like sine-Gordon models and $O(N)$ vector models.¹ An example that has been investigated vigorously in recent years is specified by the energy-momentum tensor T and often called the $T\bar{T}$ -deformation [1,2]. This deformation is actually given by a determinant of T , not $T\bar{T}$, but it is conventionally called the $T\bar{T}$ -deformation. The origin of the name would be that the deformation of conformal field theory (CFT) is eventually described by $T\bar{T}$. A peculiar feature of $T\bar{T}$ -operator has been originally realized in [3]. Then it has been presented in closed form and studied systematically [4]. For a concise review, see [5].

In 2D IQFTs, any N -body S-matrix is factorized into a product of 2-body S-matrices. This factorization property is the onset of the quantum integrability. The 2-body S-matrix is a representation of quantum R -matrix satisfying the Yang-Baxter equation and is determined by Lorentz symmetry, crossing symmetry, and unitarity, up to the Castillejo-Dalitz-Dyson (CDD) factor [6]. The CDD factor has to be determined case by case, depending on the models we are concerned with. This is the usual S-matrix bootstrap program in 2D IQFTs. In this context, the $T\bar{T}$ -deformations are irrelevant deformations and then modify only the CDD factor. Hence this factorization property is preserved, and thus the $T\bar{T}$ -deformations may be called integrable deformations in the usual sense.

An amazing observation is that one may consider the $T\bar{T}$ -deformation of general QFTs apart from the integrability. This was pointed out in [3] and further elaborated in [1,2]. The deformation effect appears only in the modification of the CDD factor in the case of 2D IQFTs. Then it would be reasonable to anticipate that the modification due to the $T\bar{T}$ -deformation appears only as some multiplicative factor to the S-matrix even for the general (non-integrable) QFT cases, where the S-matrix factorization does not occur any more. One would be able to see works by S. Dubovsky et al. [7–9] as a support for this anticipation.

One may find out an interesting application of $T\bar{T}$ -deformation in the context of 2D dilaton gravity called the Jackiw-Teitelboim (JT) gravity [10,11] (for comprehensive reviews, see [12, 13]). In the pioneering work [9], a gravitational perturbation around a solution in the flat-space JT gravity has been considered. The perturbation can be reinterpreted as a $T\bar{T}$ -deformation of

¹ In the following, we will assume the Lorentz symmetry and consider relativistic unitary theories only.

the original matter action. Then the matter theory gets a gravitational dressing factor in front of the S-matrix due to the perturbation. This result indicates that the (classical) gravitational perturbation can be seen as a non-perturbative quantum effect to the matter sector, and the deformation effect can be computed rigorously while the S-matrix of the original theory cannot be evaluated exactly in general. However, this intriguing result has been shown only in the flat-space JT gravity with a simple dilaton potential. It should be significant to figure out to what extent this result should be valid.

In this paper, we study gravitational perturbations of 2D dilaton gravity systems with matter fields in a more general setup. Then we show that these perturbations can be seen as $T\bar{T}$ -deformations (at least) under certain conditions, where T means the energy-momentum tensor of the matter field coupled to the dilaton gravity. In particular, the class of theories under this condition includes a JT gravity with a negative cosmological constant with conformal matter fields. This is a generalization of the work [9] and has potential applications in the context of the AdS_2 holography [14–17].

This paper is organized as follows. In section 2, we introduce 2D dilaton-gravity systems coupled with an arbitrary matter field. In section 3, gravitational perturbations in the systems are considered and the equations of motion for the fluctuations are derived. In section 4, (at least) for some cases, it is shown that the gravitational perturbations can be regarded as $T\bar{T}$ -deformations of the original matter Lagrangian. Section 5 is devoted to conclusion and discussion. In Appendix A, we list some useful formulae to compute gravitational perturbations in a covariant way. Appendix B explains how to derive the quadratic action in terms of the fluctuation in detail.

2. 2D dilaton gravity systems coupled with matter field

In the following, we will consider a 2D dilaton gravity system coupled with an arbitrary matter field ψ . We will work with the Lorentzian signature, and the coordinates are described as $x^\mu = (x^0, x^1) = (t, x)$. The metric field and dilaton are given by $g_{\mu\nu}(x^\mu)$ and $\phi(x^\mu)$, respectively.

The classical action is given by

$$S[g_{\mu\nu}, \phi, \psi] = S_{\text{dg}}[g_{\mu\nu}, \phi] + S_m[\psi, g_{\mu\nu}, \phi], \quad (2.1)$$

$$S_{\text{dg}}[g_{\mu\nu}, \phi] = \frac{1}{16\pi G_N} \int d^2x \sqrt{-g} [\phi R - U(\phi)], \quad (2.2)$$

$$S_m[\psi, g_{\mu\nu}, \phi] = \int d^2x \sqrt{-g} F(\phi) \mathcal{L}_m[g_{\mu\nu}, \psi],$$

where G_N is the Newton constant in two dimensions and $U(\phi)$ is a dilaton potential. The matter action S_m may include a non-trivial dilaton coupling $F(\phi)$ in general in front of the matter Lagrangian $\mathcal{L}_m$. In the following, we assume that $F(\phi)$ is constant and normalized as $F(\phi) = 1$, for simplicity.

The equations of motion of this system are given by

$$R - U'(\phi) = 0, \quad (2.3)$$

$$\frac{1}{2} g_{\mu\nu} U(\phi) - (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi) = 8\pi G_N T_{\mu\nu}, \quad (2.4)$$

where we have defined the energy-momentum tensor $T_{\mu\nu}$ for the matter field ψ as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \quad (2.5)$$

We do not discuss the dynamics of the matter field itself (nor the backreaction of the metric and dilaton to the matter field). Therefore the equation of motion for ψ is not included. In deriving the equations of motion (2.4) for the metric, we have used the fact that the Einstein tensor $G_{\mu\nu}$ in two dimensions vanishes:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \quad (2.6)$$

In our later discussion, we are interested in studying gravitational perturbations around a vacuum solution (i.e., a solution obtained when $T_{\mu\nu} = 0$).² Hence, it would be useful to write down some relations for the dilaton ϕ in an arbitrary vacuum solution. When $T_{\mu\nu} = 0$, the equation of motion (2.4) for the metric takes a simple form

$$\nabla_\mu \nabla_\nu \phi = g_{\mu\nu} \left(\nabla^2 \phi + \frac{1}{2}U(\phi) \right). \quad (2.7)$$

The trace of (2.7) is given by

$$\nabla^2 \phi + U(\phi) = 0. \quad (2.8)$$

By using (2.8) and (2.7), the dilaton potential $U(\phi)$ can be deleted. The resulting expression is

$$\nabla_\mu \nabla_\nu \phi = \frac{1}{2}g_{\mu\nu} \nabla^2 \phi. \quad (2.9)$$

2.1. Comment on the flat-space JT gravity

In the work [9], S. Dubovsky et al. considered a special case called the flat-space JT gravity. This case corresponds to the following dilaton potential

$$U(\phi) = \Lambda, \quad (2.10)$$

where Λ is a constant. The vacuum solution is uniquely determined (up to trivial ambiguities) as

$$d^2s = -2dx^+ dx^-, \quad \phi = \frac{\Lambda}{2} x^+ x^-, \quad (2.11)$$

where the light-cone coordinates are defined as

$$x^\pm \equiv \frac{1}{\sqrt{2}}(t \pm x).$$

In [9], the dilaton gravity system coupled with an arbitrary matter field has been expanded around this vacuum solution and the quadratic fluctuations have been recast into a form of $T\bar{T}$ -deformation. We will return to this point as a special example later after we carry out general computation.

² Note here that the dilaton is not regarded as a matter field but a part of the metric. This viewpoint would be rather natural as some dilaton gravities are obtained by dimensional reduction of higher-dimensional theories.

3. Perturbing 2D dilaton gravity systems

By starting from the classical action (2.1), let us consider a gravitational perturbation around a vacuum solution. In the following, we will slightly change our notation. The original metric, dilaton and matter field are described as $g_{\mu\nu}$, ϕ and ψ , respectively. The vacuum solution, which is taken as an expansion point, is specified by $\bar{g}_{\mu\nu}$ and $\bar{\phi}$. Since we assume that the expansion point is a vacuum solution with $T_{\mu\nu} = 0$, the matter field ψ should be regarded as a fluctuation (i.e., ψ should be expanded around zero). In summary, a gravitational perturbation around a vacuum solution is described as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \phi = \bar{\phi} + \sigma, \quad \psi = 0 + \psi, \quad (3.1)$$

where $h_{\mu\nu}$ and σ are fluctuations of metric and dilaton, respectively, and ψ in the right hand side is treated as a fluctuation with a slight abuse of notations. Note here that since $\bar{g}_{\mu\nu}$ and $\bar{\phi}$ describe a vacuum solution, the equations of motion (2.7) should be satisfied.

It is an easy practice to derive a vacuum solution explicitly by specifying a dilaton potential at the beginning. However, we will not do that here and keep an abstract form of the vacuum solution so as to argue in a *covariant* way. If we use a concrete expression of the vacuum solution, covariance of the expression is not manifest like in [9].

3.1. The quadratic action

Let us expand the classical action $S[g_{\mu\nu}, \phi, \psi]$ in (2.1) by the fluctuations (3.1). The classical action can be expanded as

$$\begin{aligned} S[g_{\mu\nu}, \phi, \psi] &= S^{(0)} + S^{(1)} + S^{(2)} + \dots \\ &= S_{\text{dg}}^{(0)}[\bar{g}_{\mu\nu}, \bar{\phi}] + S_{\text{dg}}^{(1)}[\bar{g}_{\mu\nu}, \bar{\phi}; h_{\mu\nu}, \sigma] + S_{\text{dg}}^{(2)}[\bar{g}_{\mu\nu}, \bar{\phi}; h_{\mu\nu}, \sigma] \\ &\quad + S_{\text{m}}^{(1)}[\bar{g}_{\mu\nu}; \psi] + S_{\text{m}}^{(2)}[\bar{g}_{\mu\nu}; \psi, h_{\mu\nu}] + \dots, \end{aligned} \quad (3.2)$$

where the superscript of $S^{(n)}$ denotes the order of fluctuations. The zeroth order part $S_{\text{dg}}^{(0)}$ is the classical value of S_{dg} with the vacuum configuration. It is just a constant in the case of [9] but in general depends on the coordinates as we will see later. Then the first order action $S_{\text{dg}}^{(1)}$ should vanish since the vacuum solution satisfies the equations of motion with $\bar{\psi} = 0$. For the matter sector, $S_{\text{m}}^{(1)}$ describes the matter field action on the classical background specified by the metric of the vacuum solution. The second-order contribution $S_{\text{m}}^{(2)}$ is evaluated as

$$S_{\text{m}}^{(2)} = \delta g^{\mu\nu} \left. \frac{\delta S_{\text{m}}}{\delta g^{\mu\nu}} \right|_{g_{\mu\nu} = \bar{g}_{\mu\nu}} = \frac{1}{2} \int d^2x \sqrt{-\bar{g}} h^{\mu\nu} t_{\mu\nu}, \quad (3.3)$$

where $t_{\mu\nu}$ is the energy-momentum tensor for the matter theory described by $S_{\text{m}}^{(1)}$. Note here that

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2),$$

where the indices in the perturbations are raised, lowered, and contracted with the background metric $\bar{g}_{\mu\nu}$: $h^{\mu\nu} \equiv \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} h_{\rho\sigma}$.

After carrying out a lengthy calculation, we obtain the explicit expression of the quadratic action $S^{(2)} \equiv S_{\text{dg}}^{(2)} + S_{\text{m}}^{(2)}$. By ignoring total derivative terms, this is given by

$$\begin{aligned}
S^{(2)} = & \frac{1}{16\pi G_N} \int d^2x \sqrt{-\bar{g}} \left(\left[\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2} h U'(\bar{\phi}) - \frac{1}{2} U''(\bar{\phi}) \sigma \right] \sigma \right. \\
& - \frac{1}{8} \bar{\nabla}^2 \bar{\phi} h_{\mu\nu} h^{\mu\nu} - \bar{\nabla}^\rho \bar{\phi} \left[-\frac{1}{2} h_{\rho\sigma} \bar{\nabla}_\mu h^{\mu\sigma} + \frac{1}{4} h \bar{\nabla}^\mu h_{\mu\rho} + \frac{3}{4} h_{\rho\mu} \bar{\nabla}^\mu h \right] \\
& \left. + \frac{1}{2} \int d^2x \sqrt{-\bar{g}} h^{\mu\nu} t_{\mu\nu} \right), \quad (3.4)
\end{aligned}$$

where $h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$. The derivation of the above expression is given in Appendix B.

3.2. Equations of motion for the fluctuations

Taking the variation of the quadratic action (B.1), or expanding the equations of motion (2.3) and (2.4), we obtain the equations of motion for the fluctuations as

$$\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2} U'(\bar{\phi}) h - U''(\bar{\phi}) \sigma = 0, \quad (3.5)$$

$$\begin{aligned}
& \left(-\bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \bar{g}_{\mu\nu} \bar{\nabla}^2 \sigma + \frac{1}{2} \bar{g}_{\mu\nu} U'(\bar{\phi}) \sigma \right) + \frac{1}{2} \bar{\nabla}^2 \bar{\phi} (h_{\mu\nu} - \bar{g}_{\mu\nu} h) \\
& + \frac{1}{2} \bar{\nabla}^\rho \bar{\phi} \left[(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}) - 2\bar{g}_{\mu\nu} \left(\bar{\nabla}^\sigma h_{\rho\sigma} - \frac{1}{2} \bar{\nabla}_\rho h \right) \right] = 8\pi G_N t_{\mu\nu}. \quad (3.6)
\end{aligned}$$

Taking the trace of (3.6) leads to

$$\left(\bar{\nabla}^2 + U'(\bar{\phi}) \right) \sigma + \frac{1}{2} U(\bar{\phi}) h - \bar{\nabla}^\rho \bar{\phi} \left(\bar{\nabla}^\sigma h_{\rho\sigma} - \frac{1}{2} \bar{\nabla}_\rho h \right) = 8\pi G_N t_\mu^\mu. \quad (3.7)$$

Subtracting the trace part (3.7) from (3.6), we obtain

$$\begin{aligned}
& \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \frac{1}{2} \bar{g}_{\mu\nu} U'(\bar{\phi}) \sigma \\
& = -8\pi G_N (t_{\mu\nu} - \bar{g}_{\mu\nu} t_\rho^\rho) - \frac{1}{2} U(\bar{\phi}) h_{\mu\nu} + \frac{1}{2} \bar{\nabla}^\rho \bar{\phi} (\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}), \quad (3.8)
\end{aligned}$$

which may also be useful in our discussion later.

Finally, it would be worth noting why we employed (B.1) rather than (3.4). As already pointed out, the Einstein tensor vanishes in two dimensions. Hence there may be some non-trivial relations among fluctuations. If such relations are utilized before taking variations with the fluctuations, then the forms of equations of motion become different. Eventually, if we derive the equations of motion from (3.4), then the resulting expression is different from the ones obtained by expanding the equations of motion (2.3) and (2.4), though these are equivalent under the relations intrinsic to two dimensions. However, if we start from (B.1), the result is the same as the one obtained from (2.3) and (2.4), without using any special relations.

3.3. Simplification of the quadratic action

It is worth noting that the quadratic action (3.4) can be further simplified by using the equations of motion obtained in Sec. 3.2.

Let us assume that the metric fluctuation takes a covariant expression of the one employed in the flat-space JT case [9] as follows:

$$h_{\mu\nu} = -16\pi G_N (t_{\mu\nu} - \bar{g}_{\mu\nu} t^\rho_\rho) k. \quad (3.9)$$

Then this ansatz leads to the relation $\bar{\nabla}^\mu h_{\mu\nu} = \bar{\nabla}_\nu h$ and

$$\begin{aligned} -\frac{1}{2} h_{\rho\sigma} \bar{\nabla}_\mu h^{\mu\sigma} + \frac{1}{4} h \bar{\nabla}^\sigma h_{\sigma\rho} + \frac{3}{4} h_{\rho\mu} \bar{\nabla}^\mu h &= \frac{1}{4} h_{\rho\mu} \bar{\nabla}^\mu h + \frac{1}{4} h \bar{\nabla}_\rho h \\ &= \frac{1}{4} \bar{\nabla}_\mu (h^\mu_\rho h). \end{aligned} \quad (3.10)$$

Thus, by using this relation and (3.5), the quadratic action (3.4) can be simplified as

$$\begin{aligned} S^{(2)} &= \frac{1}{2\kappa} \int d^2x \sqrt{-\bar{g}} \left[\frac{1}{2} U''(\bar{\phi}) \sigma^2 - \frac{1}{8} \bar{\nabla}^2 \bar{\phi} h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} (\bar{\nabla}^\rho \bar{\nabla}^\sigma \bar{\phi}) h_{\rho\sigma} h + \kappa h^{\mu\nu} t_{\mu\nu} \right] \\ &= \frac{1}{2\kappa} \int d^2x \sqrt{-\bar{g}} \left[\frac{1}{2} U''(\bar{\phi}) \sigma^2 + \frac{1}{8} U(\bar{\phi}) (h_{\mu\nu} h^{\mu\nu} - h^2) + \kappa h^{\mu\nu} t_{\mu\nu} \right] \\ &= \int d^2x \sqrt{-\bar{g}} \left[\frac{1}{4\kappa} U''(\bar{\phi}) \sigma^2 - \kappa \left(k - \frac{k^2}{4} U(\bar{\phi}) \right) (t_{\mu\nu} t^{\mu\nu} - t^2) \right]. \end{aligned} \quad (3.11)$$

The second term is proportional to the $T\bar{T}$ operator, though the coefficient depends on the background dilaton $\bar{\phi}$ and in general has space-time coordinate dependence.

In conclusion, if $U''(\bar{\phi}) = 0$ and the metric fluctuation $h_{\mu\nu}$ satisfies the ansatz (3.9), the quadratic action can be regarded as a $T\bar{T}$ deformation of the original matter action, up to the background dilaton dependence.

However, we still need to check the existence of a solution to the equations of motion. This remaining task will be discussed in the next section.

4. Gravitational perturbations as $T\bar{T}$ -deformations

In the previous section, we have shown the gravitational perturbations can be seen as $T\bar{T}$ -deformations under some conditions. Here, let us check the consistency of these conditions with the equations of motion. A general treatment seems difficult, so we will consider some simple cases of the dilation potential like flat space, AdS, dS, and then construct the explicit solutions of the metric and dilaton fluctuations.

4.1. The case of the flat-space JT gravity

As the first example, let us revisit the case of the flat-space JT gravity considered in [9]. This case is realized by taking a constant dilaton potential

$$U'(\phi) = 0, \quad U(\phi) = \Lambda, \quad (4.1)$$

where Λ is a constant. The background dilaton $\bar{\phi}$ should satisfy the following conditions

$$\bar{R} = 0, \quad \bar{\nabla}^2 \bar{\phi} = -U(\bar{\phi}) = -\Lambda, \quad (4.2)$$

which follow from (2.3) and (2.7).

The first is to solve the equation of motion (3.5), which in the present case is simplified to

$$\partial^\nu (\partial^\mu h_{\mu\nu} - \partial_\nu h) = 0. \quad (4.3)$$

A possible solution to the equation (4.3) is given by

$$h_{\mu\nu} = -16\pi G_N (t_{\mu\nu} - \bar{g}_{\mu\nu} t^\rho_\rho) k, \quad (4.4)$$

where k is an overall constant. It is easy to see that the above $h_{\mu\nu}$ indeed solves the equation (4.3) by noting that the conservation law of the energy-momentum tensor $t_{\mu\nu}$, $\bar{\nabla}_\mu t^{\mu\nu} = 0$, leads to the relation

$$\partial^\mu h_{\mu\nu} = \partial_\nu h. \quad (4.5)$$

The next is to construct an explicit solution of the dilaton fluctuation σ under the metric solution (4.4). By using the conservation law of the energy-momentum tensor, the equations of motion (3.8) can be rewritten as

$$\begin{aligned} \partial_\mu \partial_\nu \sigma &= -8\pi G_N (t_{\mu\nu} - \bar{g}_{\mu\nu} t^\rho_\rho) - \frac{1}{2} U(\bar{\phi}) h_{\mu\nu} + \frac{1}{2} \partial^\rho \bar{\phi} (\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}) \\ &= -8\pi G_N \left[(1 - k\Lambda) - \frac{k\Lambda}{2} x^\rho \partial_\rho \right] (t_{\mu\nu} - \bar{g}_{\mu\nu} t^\sigma_\sigma). \end{aligned} \quad (4.6)$$

It is possible to construct explicitly a non-local solution to the equations (4.6). To see this, let us first decompose the dilaton into two parts as

$$\sigma(x^+, x^-) = \sigma_0(x^+, x^-) + \sigma_{\text{non-local}}(x^+, x^-). \quad (4.7)$$

Here the first term $\sigma_0(x^+, x^-)$ corresponds to the sourceless part,

$$\sigma_0(x^+, x^-) = a_1 + a_2 x^+ + a_3 x^-, \quad a_i \ (i = 1, 2, 3): \text{arbitrary real consts.}, \quad (4.8)$$

and obviously satisfies $\partial_\mu \partial_\nu \sigma = 0$. The second term $\sigma_{\text{non-local}}(x^+, x^-)$ describes the non-local part,

$$\begin{aligned} \sigma_{\text{non-local}} &= 4\pi G_N \left[k\Lambda \int_0^{x^+} ds s t_{++}(s, x^-) + k\Lambda \int_0^{x^-} ds s t_{--}(x^+, s) \right. \\ &\quad \left. - 2(k\Lambda - 1) \int_0^{x^+} ds \int_0^{x^-} ds' t_{+-}(s, s') \right. \\ &\quad \left. + (k\Lambda - 2) \left(\int_{u_1^+}^{x^+} ds \int_{u_2^+}^s ds' t_{++}(s', 0) + \int_{u_1^-}^{x^-} ds \int_{u_2^-}^s ds' t_{--}(0, s') \right) \right], \end{aligned} \quad (4.9)$$

where $u_{1,2}^\pm$ are arbitrary constants.³ It is easy to check that the non-local solution (4.7) satisfies the equations of motion (4.6) by using the conservation law of the energy-momentum tensor. Note here that the sign of the deformation depends on the values of Λ and k .

³ The domain of integration may change due to the shift symmetry of the background, $x^+ \rightarrow x^+ - a^+$ and $x^- \rightarrow x^- - a^-$. After making this shift, the background dilaton is transformed like

$$\bar{\phi} = \frac{\Lambda}{2} (x^+ - a^+)(x^- - a^-). \quad (4.10)$$

Then the non-local part (4.9) is modified as

After substituting the solutions (4.4) and (4.7) into (3.4), the resulting quadratic action is given by (up to the total derivative terms)

$$\begin{aligned} S^{(2)} &= \frac{1}{16\pi G_N} \int d^2x \left(-\frac{1}{8} \partial^2 \bar{\phi} h_{\mu\nu} h^{\mu\nu} - \frac{1}{4} \partial^\rho \bar{\phi} (h \partial_\rho h + h_{\rho\mu} \partial^\mu h) + 8\pi G_N h^{\mu\nu} t_{\mu\nu} \right) \\ &= \frac{1}{16\pi G_N} \int d^2x \left(-\frac{1}{8} \partial^2 \bar{\phi} h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} \partial^\mu \partial^\rho \bar{\phi} h_{\rho\mu} h + 8\pi G_N h^{\mu\nu} t_{\mu\nu} \right) \\ &= -16\pi G_N \left(\frac{1}{2k} - \frac{\Lambda}{8} \right) k^2 \int d^2x \left[t_{\mu\nu} t^{\mu\nu} - (t_\mu^\mu)^2 \right]. \end{aligned} \quad (4.12)$$

Thus the quadratic action can be regarded as a $T\bar{T}$ deformation of $S_m^{(1)}$. It should be remarked here that the quadratic action (4.12) can be derived by using only the expression of $h_{\mu\nu}$, without using the explicit expression of σ . It is because the first line proportional to σ in the action (3.4) vanishes identically under the condition (4.5) and then the action is independent of σ , though the existence of σ as a consistent solution to (4.6) is crucial as carefully discussed in [9].

If we set $k = \frac{2}{\Lambda}$ as in [9], then the resulting action is simplified to

$$S^{(2)} = -\frac{8\pi G_N}{\Lambda} \int d^2x \left[t_{\mu\nu} t^{\mu\nu} - (t_\mu^\mu)^2 \right]. \quad (4.13)$$

This is nothing but the result obtained in [9].

Finally, we should argue the signature of (4.12). It depends on the value of k . If this signature is negative, it is well known that some pathologies like complex energies appear or closed time-like curve may appear as the onset of the breakdown of gravitational physics as discussed in [18]. In our current analysis, there is no argument to determine the signature of (4.12). It would be possible to fix it or obtain a bound for k by considering another property like causality or S-matrix. We will leave it as a future problem.

4.2. The $U'(\phi) \neq 0$ and $U''(\phi) = 0$ case

The next case is a more general class of 2D dilaton gravity systems with dilaton potentials satisfying the following conditions:

$$U'(\phi) \neq 0, \quad U''(\phi) = 0. \quad (4.14)$$

Under these conditions, the equations of motion for the fluctuations are simplified to

$$\begin{aligned} \sigma_{\text{non-local}} &= 4\pi G_N \left[k\Lambda \int_{a^+}^{x^+} ds (s - a^+) t_{++}(s, x^-) + k\Lambda \int_{a^-}^{x^-} ds (s - a^-) t_{--}(x^+, s) \right. \\ &\quad - 2(k\Lambda - 1) \int_{a^+}^{x^+} ds \int_{a^-}^{x^-} ds' t_{+-}(s, s') \\ &\quad \left. + (k\Lambda - 2) \left(\int_{u_1^+}^{x^+} ds \int_{u_2^+}^s ds' t_{++}(s', a^-) + \int_{u_1^-}^{x^-} ds \int_{u_2^-}^s ds' t_{--}(a^+, s') \right) \right]. \end{aligned} \quad (4.11)$$

$$\begin{aligned}
& \bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2} U'(\bar{\phi}) h = 0, \\
& \left(-\bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \bar{g}_{\mu\nu} \bar{\nabla}^2 \sigma + \frac{1}{2} \bar{g}_{\mu\nu} U'(\bar{\phi}) \sigma \right) + \frac{1}{2} \bar{\nabla}^2 \bar{\phi} (h_{\mu\nu} - \bar{g}_{\mu\nu} h) \\
& + \frac{1}{2} \bar{\nabla}^\rho \bar{\phi} \left[(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}) - 2 \bar{g}_{\mu\nu} \left(\bar{\nabla}^\sigma h_{\rho\sigma} - \frac{1}{2} \bar{\nabla}_\rho h \right) \right] = 8\pi G_N t_{\mu\nu}.
\end{aligned} \tag{4.15}$$

$$\tag{4.16}$$

As in the previous case, let us solve the first equation (4.15). Suppose that the matter ψ is taken to be a conformal matter i.e. $t_\mu^\mu = 0$. Then it is easy to find out a solution to the equation (4.15),

$$h_{\mu\nu} = -k \cdot 16\pi G_N t_{\mu\nu}, \quad h = -k \cdot 16\pi G_N t_\mu^\mu = 0, \tag{4.17}$$

with the help of the conservation law of the energy-momentum tensor. Finally, the quadratic action (3.4) can be rewritten as

$$S^{(2)} = (16\pi G_N) \frac{k^2}{8} \int d^2x \sqrt{-\bar{g}} \left(U(\bar{\phi}) - \frac{4}{k} \right) t_{\mu\nu} t^{\mu\nu}. \tag{4.18}$$

Thus this may also be regarded as a $T\bar{T}$ deformation.

However, we should make some comments here for the interpretation of (4.18). First of all, the coefficient of $t_{\mu\nu} t^{\mu\nu}$ depends on the background dilaton $\bar{\phi}$ and especially on the space-time coordinates in general. Hence we need to consider the physical interpretation of this coefficient. Then we have imposed the conformal matter condition, so we cannot interpret our result as a $T\bar{T}$ -flow, but a traditional RG flow. We need to devise to remove the conformal matter condition somehow in order to apply the $T\bar{T}$ -flow interpretation as in the flat-space JT gravity. However, our result can be understood as the exact solution to the linear-order perturbation of the AP model, and there should be some potential application in the context of $\text{AdS}_2/\text{CFT}_1$.

4.2.1. Concrete examples

In the following, we show two examples for the present case.

(i) The Almheiri-Polchinski model

An interesting example of 2D dilaton gravity systems satisfying the condition (4.14) is the Almheiri-Polchinski (AP) model [14].⁴ This model has the dilaton potential⁵

$$U(\phi) = \Lambda - \frac{2}{L^2} \phi. \tag{4.19}$$

Let us construct the explicit form of σ . In the following, we will employ conformal gauge and the metric is given by

$$d^2s = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -2e^{2\bar{\omega}} dx^+ dx^-. \tag{4.20}$$

The general vacuum solution incorporates the AdS_2 metric and a non-constant dilaton

⁴ Here, we dare to call the JT gravity with a conformal matter field as the AP model so as to respect the analysis on the conformal matter in [14] which plays a crucial role in our analysis here.

⁵ As for the notation, note that our $\bar{\phi}$ corresponds to Φ^2 in [14].

$$e^{2\bar{\omega}} = \frac{2L^2}{(x^+ - x^-)^2}, \quad \bar{\phi} = \frac{L^2}{2} \left(\Lambda + \frac{a + b(x^+ + x^-) + c x^+ x^-}{x^+ - x^-} \right), \quad (4.21)$$

where L is the AdS radius and a, b and c is an arbitrary constant.

Let us solve the equations of motion (3.8). The $(++)$ and $(--)$ components of (3.8) are evaluated as

$$e^{2\bar{\omega}} \partial_+ \left(e^{-2\bar{\omega}} \partial_+ \sigma \right) = -8\pi G_N \mathcal{T}_+(x^+), \quad (4.22)$$

$$e^{2\bar{\omega}} \partial_- \left(e^{-2\bar{\omega}} \partial_- \sigma \right) = -8\pi G_N \mathcal{T}_-(x^-), \quad (4.23)$$

where $\mathcal{T}_\pm(x^\pm)$ are defined as

$$\mathcal{T}_\pm(x^\pm) \equiv (1 \mp b k \mp c k x^\pm) t_{\pm\pm} \mp \frac{k}{4} \left(a + 2b x^\pm + c (x^\pm)^2 \right) \partial_\pm t_{\pm\pm}. \quad (4.24)$$

By following [14], it is useful to express σ with a scalar function $M(x^+, x^-)$ as

$$\sigma(x^+, x^-) \equiv \frac{M(x^+, x^-)}{x^+ - x^-}. \quad (4.25)$$

Then the left-hand sides of (4.22) and (4.23) can be written as

$$e^{2\bar{\omega}} \partial_\pm \left(e^{-2\bar{\omega}} \partial_\pm \sigma \right) = \frac{\partial_\pm \partial_\pm M(x^+, x^-)}{x^+ - x^-}. \quad (4.26)$$

By integrating (4.22) and (4.23), the general solution can be derived as

$$M(x^+, x^-) = I_0(x^+, x^-) + I^+(x^+, x^-) - I^-(x^+, x^-). \quad (4.27)$$

Here I_0 is the sourceless solution,

$$I_0(x^+, x^-) \equiv A + B(x^+ + x^-) + C x^+ x^-, \quad A, B, C: \text{arbitrary real consts.},$$

and $I^\pm(x^+, x^-)$ correspond to the non-local parts of dilaton and are given by

$$I^\pm(x^+, x^-) \equiv 8\pi G_N \int_{u^\pm}^{x^\pm} ds (s - x^+)(s - x^-) \mathcal{T}_\pm(s). \quad (4.28)$$

The $(+-)$ component of (3.8) is drastically simplified due to the traceless condition $t_{+-} = 0$ and is given by

$$\partial_+ \partial_- \sigma + \frac{2\sigma}{(x^+ - x^-)^2} = 0. \quad (4.29)$$

It is an easy task to see that σ with (4.27) satisfies the above condition (4.29).

It should be remarked that this *non-local* solution to (4.29) might be epochal. One would usually try to employ hypergeometric functions or Gegenbauer polynomials to solve it by assuming that the solution is local. But this solution is non-local and it has not been presented at least as far as we know. This non-local solution may play an important role in resolving the long-standing issue of the AdS₂/CFT₁ correspondence.

4.2.2. A flat-space limit

It is intriguing to consider a flat-space limit of the vacuum solution in the AP model (see also Appendix B of [9] for the limit with the embedding coordinates).

Let us first take a constant shift of $x^\pm$ and introduce new coordinates $X^\pm$ defined as

$$X^\pm \equiv x^\pm \mp \frac{L}{\sqrt{2}}. \quad (4.30)$$

By taking the large radius limit $L \rightarrow \infty$, the AdS_2 metric goes to the Minkowski metric,

$$ds^2 = \frac{-4L^2 dx^+ dx^-}{(x^+ - x^-)^2} \longrightarrow -2dX^+ dX^-. \quad (4.31)$$

For the background dilaton $\bar{\phi}$, it is helpful to take a particular choice of a , b and c as

$$a = -\frac{\Lambda L}{\sqrt{2}}, \quad b = 0, \quad c = \frac{\sqrt{2}\Lambda}{L}. \quad (4.32)$$

Then the dilaton is rewritten as

$$\bar{\phi} = \frac{\Lambda L^2}{2} \left(1 - \frac{1}{\sqrt{2}L} \frac{L^2 - 2x^+ x^-}{x^+ - x^-} \right). \quad (4.33)$$

After taking the limit $L \rightarrow \infty$, the dilaton reduces to the one (2.11) in the flat-space JT gravity:

$$\bar{\phi} \longrightarrow \frac{\Lambda}{2} X^+ X^-. \quad (4.34)$$

It may be worth noting that the choice (4.32) corresponds to a black hole solution discussed in [14]. In particular, the parameter c is basically associated with the Hawking temperature and eventually this part gives rise to the dilaton in the flat-space JT gravity. It would be intriguing to try to get much deeper understanding for this connection.

(ii) 2D de Sitter space

Another interesting example is a de Sitter version of the AP model. This case is realized by taking the following dilaton potential,

$$U(\phi) = \Lambda + \frac{2}{L^2} \phi. \quad (4.35)$$

For the recent progress on the dS_2 in the JT gravity, see [19,20]. There might be a potential application in the context of dS/dS correspondence [21].

The general vacuum solution incorporates the dS_2 metric and a non-constant dilaton

$$e^{2\bar{\omega}} = \frac{2L^2}{(x^+ + x^-)^2}, \quad \bar{\phi} = \frac{L^2}{2} \left(-\Lambda + \frac{a + b(x^+ - x^-) + c x^+ x^-}{x^+ + x^-} \right), \quad (4.36)$$

where L is the curvature radius and a , b and c are arbitrary constants.

Again, let us examine the equations of motion (3.8). The $(++)$ and $(--)$ components of (3.8) are evaluated as

$$e^{2\bar{\omega}} \partial_+ \left(e^{-2\bar{\omega}} \partial_+ \sigma \right) = -8\pi G_N \mathcal{T}_{dS+}(x^+), \quad (4.37)$$

$$e^{2\bar{\omega}} \partial_- \left(e^{-2\bar{\omega}} \partial_- \sigma \right) = -8\pi G_N \mathcal{T}_{dS-}(x^-), \quad (4.38)$$

where $\mathcal{T}_{\text{dS}\pm}(x^\pm)$ are defined as

$$\mathcal{T}_{\text{dS}\pm}(x^\pm) \equiv (1 \pm bk - ckx^\pm)t_{\pm\pm} + \frac{k}{4} \left(a \pm 2bx^\pm - c(x^\pm)^2 \right) \partial_\pm t_{\pm\pm}. \quad (4.39)$$

Similarly to the AdS case, it is useful to represent σ by using a scalar function $M_{\text{dS}}(x^+, x^-)$:

$$\sigma(x^+, x^-) \equiv \frac{M_{\text{dS}}(x^+, x^-)}{x^+ + x^-}. \quad (4.40)$$

By integrating (4.37) and (4.38), the general solution can be derived as

$$M_{\text{dS}}(x^+, x^-) = J_0(x^+, x^-) + J^+(x^+, x^-) + J^-(x^+, x^-). \quad (4.41)$$

Here J_0 is the sourceless solution

$$J_0(x^+, x^-) \equiv A + B(x^+ - x^-) + Cx^+x^-, \quad A, B, C: \text{arbitrary real consts.},$$

and $J^\pm(x^+, x^-)$ correspond to the non-local part of dilaton and are given by

$$J^\pm(x^+, x^-) \equiv 8\pi G_N \int_{u^\pm}^{x^\pm} ds (s \mp x^+)(s \pm x^-) \mathcal{T}_{\text{dS}\pm}(s). \quad (4.42)$$

The $(+-)$ component of (3.8) is drastically simplified due to the traceless condition $t_{+-} = 0$ and is given by

$$\partial_+ \partial_- \sigma - \frac{2\sigma}{(x^+ + x^-)^2} = 0. \quad (4.43)$$

σ with (4.41) also satisfies the above condition (4.43).

4.2.3. A flat-space limit

In a similar way to the AdS₂ case, it is easy to consider a flat-space limit for the vacuum solution in the dS model.

Let us take a constant shift of $x^\pm$ and introduce new coordinates $X^\pm$ defined as

$$X^\pm \equiv x^\pm \pm \frac{L}{\sqrt{2}}. \quad (4.44)$$

By taking the large radius limit $L \rightarrow \infty$, the dS₂ metric goes to the Minkowski metric,

$$ds^2 = \frac{-4L^2 dx^+ dx^-}{(x^+ + x^-)^2} \longrightarrow -2dX^+ dX^-. \quad (4.45)$$

For the background dilaton $\bar{\phi}$, it is helpful to take a particular choice of a, b and c as

$$a = \frac{\Lambda L}{\sqrt{2}}, \quad b = 0, \quad c = \frac{\sqrt{2}\Lambda}{L}. \quad (4.46)$$

Then the dilaton is rewritten as

$$\bar{\phi} = \frac{\Lambda L^2}{2} \left(-1 + \frac{1}{\sqrt{2}L} \frac{L^2 + 2x^+x^-}{x^+ + x^-} \right). \quad (4.47)$$

After taking the limit $L \rightarrow \infty$, the dilaton reduces to the one (2.11) in the flat-space JT gravity:

$$\bar{\phi} \longrightarrow \frac{\Lambda}{2} X^+ X^-. \quad (4.48)$$

5. Conclusion and discussion

In this paper, we have discussed gravitational perturbations in 2D dilaton-gravity systems with some dilaton potentials, motivated by the pioneering work [9]. It has been shown that at least under some conditions the perturbations can be regarded as $T\bar{T}$ -deformations of the original matter action. In particular, the class of theories that this identification applies to include the JT gravity with a cosmological constant, and there would be potential applications in the context of the AdS_2 or dS_2 holography. It is significant to figure out to what extent this identification should be valid, but we will leave it as a future problem.

There are some open problems and future directions. Here we have discussed the gravitational perturbation in the case of the JT gravity including a negative cosmological constant. It should be important to study the AdS_2 holography from the viewpoint of the $T\bar{T}$ -deformation of the bulk geometry. Along this direction, the Gibbons-Hawking term, which has not been taken into account here, should be considered carefully so that one can seek for the CFT_1 interpretation of this gravitational $T\bar{T}$ -deformation. In addition, it will also be important to compute the partition function by following [22]. In this evaluation, the spacetime has to be compactified so as to obtain the finite-volume spectrum. While in the flat-space JT gravity the spacetime is compactified to a torus (with the Wick rotation), what should we do for the AdS_2 case? A possible solution seems to be the cut-off AdS geometry proposed in [23]. However, the cut-off AdS geometry corresponds to the $T\bar{T}$ -deformation with a “bad sign” parameter and it has some pathologies like the presence of complex energies in the finite-volume spectrum of the field theory (for example, see [18,24]). Hence there are a number of challenges to be overcome. We hope to report on some progress along this direction in the near future.

The most significant issue is to investigate gravitationally dressed S-matrix in the matter field theory. In the case of the flat-space JT gravity, a dressing factor to the S-matrix caused by the $T\bar{T}$ -deformation has been evaluated [9]. At least in principle, it should be possible to carry out similar analysis and examine the dressing factor, though the treatment of the S-matrix on a curved spacetime would be quite complicated. As a matter of course, the 2D de Sitter case would be intriguing. It is also interesting to consider a relation to the random geometry by following the work [25].

It should also be interesting to try to relax our condition so as to support non-conformal matter fields, more general types of dilaton potential, more general ansatz for the metric fluctuation and general forms of dilaton coupling to matter fields. In particular, it has been shown that Yang-Baxter deformations [26–28] are closely related to $T\bar{T}$ -deformations [29,30] (for related works, see [31,32] as well). Hence Yang-Baxter deformations of the JT gravity [33–35] may also be related to $T\bar{T}$ -deformations. It would also be nice to generalize the relation between the deformed solutions and the unperturbed ones via a specific field-dependent local change of coordinates, which has been revealed in [36,37], from flat space to curved spaces like AdS_2 or dS_2 . As another direction, it may be interesting to try to include supersymmetries by following the works [38–41].

We hope that our result would open up a new route to the AdS_2 holography.

Acknowledgements

We thank H.Y. Chen, D. Orlando, S. Reffert and R. Tateo for useful discussions. In particular, K.Y. is really grateful to R. Tateo for drawing our attention to this subject and also would like to thank discussions with participants during the OIST workshop on “Recent developments in

AdS/CFT,” organized by N. Iizuka and T. Ugajin. The work of T.I. was supported in part by JSPS Grant-in-Aid for Scientific Research (C) No. 19K03871. The work of S.O. was supported by the Japan Society for the Promotion of Science (JSPS) No. 18J21605. The work of J.S. was supported in part by JSPS Grant-in-Aid for Scientific Research (B) No. 16H03979 and Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics). The works of T.I. and K.Y. were supported by the Supporting Program for Interaction-based Initiative Team Studies (SPIRITS) from Kyoto University, and JSPS Grant-in-Aid for Scientific Research (B) No. 18H01214. This work is also supported in part by the JSPS Japan-Russia Research Cooperative Program.

Appendix A. Useful formulae

It would be helpful for readers to summarize formulae useful in computing some quantities in this paper.

We consider a small perturbation around a given metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (\text{A.1})$$

and expand some geometric quantities in terms of the metric fluctuation up to and including the second-order in $h_{\mu\nu}$. In the following we will drop the higher order terms. The explicit expressions of the perturbed quantities are useful in deriving the quadratic action (3.4).

We start by expanding the inverse and the determinant of the perturbed metric $g_{\mu\nu}$. These are given by

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^\mu_\rho h^{\rho\nu}, \quad (\text{A.2})$$

$$\sqrt{-g} = \sqrt{-\bar{g}} \left(1 + \frac{1}{2} h - \frac{1}{4} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) \right), \quad (\text{A.3})$$

where $h^{\mu\nu} = \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} h_{\rho\sigma}$, $h = \bar{g}^{\mu\nu} h_{\mu\nu} = h^\mu_\mu$.

The Christoffel symbol is defined as

$$\Gamma^\rho_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}), \quad (\text{A.4})$$

and can be expanded as

$$\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + \Gamma^{(1)\rho}_{\mu\nu} + \Gamma^{(2)\rho}_{\mu\nu}, \quad (\text{A.5})$$

where the first and second order terms in the fluctuation are

$$\Gamma^{(1)\rho}_{\mu\nu} = \frac{1}{2} \bar{g}^{\rho\sigma} (\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}), \quad (\text{A.6})$$

$$\Gamma^{(2)\rho}_{\mu\nu} = -\frac{1}{2} h^{\rho\sigma} (\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}). \quad (\text{A.7})$$

The Riemann tensor and the Ricci tensor are defined as

$$R^\mu_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\rho\alpha} \Gamma^\rho_{\nu\beta} - \Gamma^\mu_{\rho\beta} \Gamma^\rho_{\nu\alpha}, \quad (\text{A.8})$$

$$R_{\mu\nu} \equiv R^\rho_{\mu\rho\nu}. \quad (\text{A.9})$$

The Ricci tensor can be expanded as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}, \quad (\text{A.10})$$

where $R_{\mu\nu}^{(1)}$ and $R_{\mu\nu}^{(2)}$ are

$$R_{\mu\nu}^{(1)} = \frac{1}{2} \bar{\nabla}^\rho (\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}) - \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\nu h, \quad (\text{A.11})$$

$$\begin{aligned} R_{\mu\nu}^{(2)} = & \frac{1}{2} \bar{\nabla}_\nu (h^{\rho\sigma} \bar{\nabla}_\mu h_{\rho\sigma}) - \frac{1}{2} \bar{\nabla}_\rho [h^{\rho\sigma} (\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu})] \\ & + \frac{1}{4} \bar{\nabla}^\rho h (\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}) \\ & - \frac{1}{4} \bar{g}^{\alpha\beta} \bar{g}^{\rho\sigma} (\bar{\nabla}_\mu h_{\alpha\rho} + \bar{\nabla}_\alpha h_{\rho\mu} - \bar{\nabla}_\rho h_{\alpha\mu}) (\bar{\nabla}_\sigma h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\sigma} - \bar{\nabla}_\beta h_{\sigma\nu}). \end{aligned} \quad (\text{A.12})$$

The Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ can be expanded as

$$R = \bar{R} + R^{(1)} + R^{(2)}, \quad (\text{A.13})$$

where $R^{(1)}$ and $R^{(2)}$ are given by

$$\begin{aligned} R^{(1)} = & \bar{g}^{\mu\nu} R_{\mu\nu}^{(1)} - h^{\mu\nu} \bar{R}_{\mu\nu} \\ = & \bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - h^{\mu\nu} \bar{R}_{\mu\nu}, \end{aligned} \quad (\text{A.14})$$

$$R^{(2)} = \bar{g}^{\mu\nu} R_{\mu\nu}^{(2)} - h^{\mu\nu} R_{\mu\nu}^{(1)} + h^\mu h^{\rho\nu} \bar{R}_{\mu\nu}, \quad (\text{A.15})$$

with

$$\begin{aligned} \bar{g}^{\mu\nu} R_{\mu\nu}^{(2)} = & \frac{1}{2} \bar{\nabla}^\mu (h^{\rho\sigma} \bar{\nabla}_\mu h_{\rho\sigma}) - \bar{\nabla}_\rho \left[h^{\rho\sigma} \left(\bar{\nabla}^\mu h_{\sigma\mu} - \frac{1}{2} \bar{\nabla}_\sigma h \right) \right] \\ & + \frac{1}{2} \bar{\nabla}^\rho h \left(\bar{\nabla}^\mu h_{\rho\mu} - \frac{1}{2} \bar{\nabla}_\rho h \right) \\ & - \frac{1}{4} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} \bar{g}^{\rho\sigma} (\bar{\nabla}_\mu h_{\alpha\rho} + \bar{\nabla}_\alpha h_{\rho\mu} - \bar{\nabla}_\rho h_{\alpha\mu}) (\bar{\nabla}_\sigma h_{\beta\nu} + \bar{\nabla}_\nu h_{\beta\sigma} - \bar{\nabla}_\beta h_{\sigma\nu}) \\ = & \bar{\nabla}^\rho \left(\frac{3}{4} h^{\mu\nu} \bar{\nabla}_\rho h_{\mu\nu} - h_\rho^\sigma \left(\bar{\nabla}^\mu h_{\sigma\mu} - \frac{1}{2} \bar{\nabla}_\sigma h \right) \right) + \frac{1}{2} \bar{\nabla}^\rho h \left(\bar{\nabla}^\mu h_{\rho\mu} - \frac{1}{2} \bar{\nabla}_\rho h \right) \\ & - \frac{1}{2} \bar{\nabla}_\mu h_{\alpha\beta} \bar{\nabla}^\alpha h^{\beta\mu} - \frac{1}{4} h^{\mu\nu} \bar{\nabla}^2 h_{\mu\nu}. \end{aligned} \quad (\text{A.16})$$

Appendix B. A derivation of the quadratic action

We explain how to derive the quadratic action (3.4) in detail.

By using (A.2) and (A.3), it is straightforward to derive the following quadratic action:

$$\begin{aligned} S^{(2)} = & \frac{1}{16\pi G_N} \int d^2x \sqrt{-\bar{g}} \left(\left[\frac{1}{2} h (\bar{R} - U'(\bar{\phi})) + R^{(1)} - \frac{1}{2} U''(\bar{\phi}) \sigma \right] \sigma \right. \\ & + \frac{1}{4} U(\bar{\phi}) \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) + \bar{\phi} \left[R^{(2)} + \frac{1}{2} h R^{(1)} - \frac{1}{4} \bar{R} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) \right] \Big) \\ & + \frac{1}{2} \int d^2x \sqrt{-\bar{g}} h^{\mu\nu} t_{\mu\nu}. \end{aligned} \quad (\text{B.1})$$

By using the explicit expression (A.14) of $R^{(1)}$ and the vanishing of the two dimensional Einstein tensor (2.6), this can be rewritten as

$$\begin{aligned}
S^{(2)} = & \frac{1}{16\pi G_N} \int d^2x \sqrt{-\bar{g}} \left(\left[\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2} h U'(\bar{\phi}) - \frac{1}{2} U''(\bar{\phi}) \sigma \right] \sigma \right. \\
& + \frac{1}{4} U(\bar{\phi}) \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) + \bar{\phi} \left[R^{(2)} + \frac{1}{2} h R^{(1)} - \frac{1}{4} \bar{R} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) \right] \Big) \\
& + \frac{1}{2} \int d^2x \sqrt{-\bar{g}} h^{\mu\nu} t_{\mu\nu}.
\end{aligned} \tag{B.2}$$

We then rewrite the terms proportional to the background dilaton $\bar{\phi}$ in (B.2). For this purpose, it is helpful to employ the following identity:

$$0 = h^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right). \tag{B.3}$$

The identity (B.3) gives at $\mathcal{O}(h_{\mu\nu}^2)$

$$-\frac{1}{4} \bar{R} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) = -\frac{1}{2} h^{\mu\nu} R_{\mu\nu}^{(1)} + \frac{1}{4} h \left(\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h \right). \tag{B.4}$$

By using the identity (B.4) and the formula (A.15) together with the fact that the Einstein tensor vanishes in two dimensions (2.6), the part proportional to the background dilaton $\bar{\phi}$ in (B.2) can be rewritten as

$$\begin{aligned}
& \bar{\phi} \left[R^{(2)} + \frac{1}{2} h R^{(1)} - \frac{1}{4} \bar{R} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) \right] \\
& = \bar{\phi} \left[\bar{g}^{\mu\nu} R_{\mu\nu}^{(2)} - \frac{1}{2} h^{\mu\nu} R_{\mu\nu}^{(1)} + \frac{1}{4} h \left(\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h \right) \right].
\end{aligned} \tag{B.5}$$

Substituting (A.11) and (A.16) into (B.5), the above expression becomes

$$\begin{aligned}
(\text{B.5}) = & \bar{\phi} \bar{\nabla}^\rho \left(\frac{3}{4} h^{\mu\nu} \bar{\nabla}_\rho h_{\mu\nu} - \frac{1}{4} h \bar{\nabla}_\rho h - h_{\rho\sigma} \bar{\nabla}_\mu h^{\mu\sigma} \right. \\
& \left. - \frac{1}{2} h^{\mu\nu} \bar{\nabla}_\mu h_{\rho\nu} + \frac{1}{4} h \bar{\nabla}^\sigma h_{\sigma\rho} + \frac{3}{4} h_{\rho\mu} \bar{\nabla}^\mu h \right) \\
& = \frac{1}{4} \bar{\nabla}^2 \bar{\phi} \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) + \frac{1}{8} \bar{\nabla}^2 \bar{\phi} h_{\mu\nu} h^{\mu\nu} \\
& + \bar{\phi} \bar{\nabla}^\rho \left(-h_{\rho\sigma} \bar{\nabla}_\mu h^{\mu\sigma} - \frac{1}{2} h^{\mu\nu} \bar{\nabla}_\mu h_{\rho\nu} + \frac{1}{4} h \bar{\nabla}^\sigma h_{\sigma\rho} + \frac{3}{4} h_{\rho\mu} \bar{\nabla}^\mu h \right) \\
& + \bar{\nabla}^\rho \left(\bar{\phi} \left[\frac{3}{4} h^{\mu\nu} \bar{\nabla}_\rho h_{\mu\nu} - \frac{1}{4} h \bar{\nabla}_\rho h \right] - \bar{\nabla}_\rho \bar{\phi} \left[\frac{3}{8} h^{\mu\nu} - \frac{1}{8} h^2 \right] \right).
\end{aligned} \tag{B.6}$$

Furthermore, it should be useful to rewrite the second line from the end in (B.6) so that it contains only the terms containing $\bar{\nabla}_\mu h$ and $\bar{\nabla}^\mu h_{\mu\nu}$. By using the fact that the background dilaton $\bar{\phi}$ satisfies (2.9), the second term in that line can be rewritten as

$$\begin{aligned}
-\frac{1}{2} \bar{\phi} \bar{\nabla}^\rho (h^{\mu\nu} \bar{\nabla}_\mu h_{\rho\nu}) & = -\frac{1}{2} (\bar{\nabla}_\mu \bar{\nabla}^\rho \bar{\phi}) h^{\mu\nu} h_{\rho\nu} - \frac{1}{2} \bar{\nabla}^\rho \bar{\phi} h_{\rho\nu} \bar{\nabla}_\mu h^{\mu\nu} \\
& - \frac{1}{2} \bar{\nabla}^\rho (\bar{\phi} h^{\mu\nu} \bar{\nabla}_\mu h_{\rho\nu}) + \frac{1}{2} \bar{\nabla}_\mu (\bar{\nabla}^\rho \bar{\phi} h^{\mu\nu} h_{\rho\nu}) \\
& = -\frac{1}{4} \bar{\nabla}^2 \bar{\phi} h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} \bar{\nabla}^\rho \bar{\phi} h_{\rho\nu} \bar{\nabla}_\mu h^{\mu\nu}
\end{aligned}$$

$$-\frac{1}{2}\bar{\nabla}^\rho(\bar{\phi}h^{\mu\nu}\bar{\nabla}_\mu h_{\rho\nu})+\frac{1}{2}\bar{\nabla}_\mu(\bar{\nabla}^\rho\bar{\phi}h^{\mu\nu}h_{\rho\nu}). \quad (\text{B.7})$$

As a result, we obtain

$$\begin{aligned} & \bar{\phi} \left[R^{(2)} + \frac{1}{2}hR^{(1)} - \frac{1}{4}\bar{R} \left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2 \right) \right] \\ &= \frac{1}{4}\bar{\nabla}^2\bar{\phi} \left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2 \right) - \frac{1}{8}\bar{\nabla}^2\bar{\phi}h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}\bar{\nabla}^\rho\bar{\phi}h_{\nu\rho}\bar{\nabla}_\mu h^{\mu\nu} \\ &+ \bar{\phi}\bar{\nabla}^\rho \left(-h_{\rho\sigma}\bar{\nabla}_\mu h^{\mu\sigma} + \frac{1}{4}h\bar{\nabla}^\sigma h_{\sigma\rho} + \frac{3}{4}h_{\rho\mu}\bar{\nabla}^\mu h \right) - \frac{1}{2}\bar{\nabla}^\rho\bar{\phi}h_{\rho\nu}\bar{\nabla}_\mu h^{\mu\nu} \\ &+ \bar{\nabla}^\rho \left(\bar{\phi} \left[\frac{3}{4}h^{\mu\nu}\bar{\nabla}_\rho h_{\mu\nu} - \frac{1}{4}h\bar{\nabla}_\rho h - \frac{1}{2}h^{\mu\nu}\bar{\nabla}_\mu h_{\nu\rho} \right] - \bar{\nabla}_\rho\bar{\phi} \left[\frac{3}{8}h^{\mu\nu} - \frac{1}{8}h^2 \right] \right) \\ &+ \frac{1}{2}\bar{\nabla}_\mu(\bar{\nabla}^\rho\bar{\phi}h^{\mu\nu}h_{\nu\rho}). \end{aligned} \quad (\text{B.8})$$

Finally, by using (B.8) and the on-shell condition of the background dilaton (2.9) and doing partial integration, the quadratic action $S^{(2)}$ becomes

$$\begin{aligned} S^{(2)} = & \frac{1}{16\pi G_N} \int d^2x \sqrt{-\bar{g}} \left(\left[\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2}hU'(\bar{\phi}) - \frac{1}{2}U''(\bar{\phi})\sigma \right] \sigma \right. \\ & \left. - \frac{1}{8}\bar{\nabla}^2\bar{\phi}h_{\mu\nu}h^{\mu\nu} - \bar{\nabla}^\rho\bar{\phi} \left[-\frac{1}{2}h_{\rho\sigma}\bar{\nabla}_\mu h^{\mu\sigma} + \frac{1}{4}h\bar{\nabla}^\sigma h_{\sigma\rho} + \frac{3}{4}h_{\rho\mu}\bar{\nabla}^\mu h \right] \right) \\ & + \frac{1}{2} \int d^2x \sqrt{-\bar{g}} h^{\mu\nu} t_{\mu\nu}, \end{aligned} \quad (\text{B.9})$$

where we have ignored the total derivative terms. This is the quadratic action in (3.4).

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