

ON CERTAIN P-VALENTLY STARLIKENESS CONDITIONS FOR ANALYTIC FUNCTIONS

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1. Introduction.

Let $A(p)$ denote the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

which are analytic in the open unit disk $E = \{z: |z| < 1\}$. A function $f(z) \in A(p)$ is called p -valently starlike with respect to the origin if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E.$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are p -valently starlike in E . Krzyz [1] showed by a counterexample that if $f(z) \in A(1)$, the condition $\operatorname{Re} f'(z) > 0$ in E does not imply $f(z) \in S(1)$. In [3], Mocanu proved the following theorem.

Theorem. If $f(z) \in A(1)$ and

$$|\arg f'(z)| < \alpha_0 \frac{\pi}{2} = 0.968... \quad \text{in } E,$$

where $\alpha_0 = 0.6165...$ is the unique root of the equation

$$2 \tan^{-1}(1 - \alpha) + \pi(1 - 2\alpha) = 0,$$

then $f(z) \in S(1)$.

Definition. Let $F(z)$ be analytic and univalent in E and suppose that $F(E) = D$. If $f(z)$ is analytic in E , $f(0) = F(0)$, and $f(E) \subset D$, then we say that $f(z)$ is subordinate to $F(z)$ in E , and we write

$$f(z) \prec F(z).$$

2. Preliminaries.

We shall use the following lemma to prove our results.

Lemma 1. Let $\beta^* = 1.218\dots$ be the solution of $\pi\beta = 3\pi/2 - \tan^{-1}\beta$ and let $\alpha = \alpha(\beta) = \beta + (2/\pi)\tan^{-1}\beta$, for $0 < \beta < \beta^*$. If $p(z)$ is analytic in E , with $p(0) = 1$, then

$$p(z) + zp'(z) < \left(\frac{1+z}{1-z}\right)^\alpha \implies p(z) < \left(\frac{1+z}{1-z}\right)^\beta.$$

We owe this lemma to [2, Theorem 5].

3. Main theorem.

Theorem 1. Let $p \geq 2$. If $f(z) \in A(p)$ and

$$(1) \quad |\arg f^{(p)}(z)| < \frac{\pi}{2} \alpha_1 = 1.249\dots \quad \text{in } E,$$

where $\alpha_1 = \frac{1}{2} + \frac{2}{\pi} \tan^{-1} \frac{1}{2} = 0.795\dots$, then $f(z) \in S(p)$ or $f(z)$ is p -valently starlike in E .

Proof. If we put

$$p(z) = \frac{f^{(p-1)}(z)}{p!z},$$

then we have

$$p(z) + zp'(z) = \frac{1}{p!} f^{(p)}(z)$$

and $p(0) = 1$.

From the assumption (1), we have

$$|\arg(p(z) + zp'(z))| = |\arg f^{(p)}(z)| < \frac{\pi}{2} \alpha_1 \quad \text{in } E.$$

Then, from Lemma 1, we have

$$(2) \quad \left| \arg \frac{f^{(p-1)}(z)}{p!z} \right| = \left| \arg \frac{f^{(p-1)}(z)}{z} \right| < \frac{\pi}{4} \quad \text{in } E.$$

On the other hand, we easily have

$$\begin{aligned} (3) \quad \frac{f^{(p-2)}(z)}{z^2} &= \frac{1}{z^2} \int_0^z f^{(p-1)}(t) dt \\ &= \frac{1}{r^2} \int_0^r \frac{f^{(p-1)}(t)}{t} \rho d\rho \end{aligned}$$

where $z = re^{i\theta}$, $0 < r < 1$, $t = \rho e^{i\theta}$ and $0 \leq \rho \leq r$.

From (2), we easily have

$$\left| \arg \frac{f^{(p-1)}(z)}{z} \right| = \left| \arg \frac{f^{(p-1)}(z)}{z} \right| < \frac{\pi}{4} \quad \text{in } E,$$

and the same is true of its integral mean of (3) (see e.g. [5, Lemma 1]).

Therefore we have

$$\begin{aligned} (4) \quad \left| \arg \frac{f^{(p-2)}(z)}{z^2} \right| &= \left| \arg \frac{1}{r^2} \int_0^r \frac{f^{(p-1)}(t)}{t} \rho \, d\rho \right| \\ &= \left| \arg \int_0^r \frac{f^{(p-1)}(t)}{t} \rho \, d\rho \right| \\ &< \frac{\pi}{4} \quad \text{in } E. \end{aligned}$$

From (2) and (4), we have

$$\begin{aligned} \left| \arg \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} \right| &= \left| \arg \frac{f^{(p-1)}(z)}{z} \cdot \frac{z^2}{f^{(p-2)}(z)} \right| \\ &\leq \left| \arg \frac{f^{(p-1)}(z)}{z} \right| + \left| \arg \frac{f^{(p-2)}(z)}{z^2} \right| \\ &< \frac{\pi}{2} \quad \text{in } E. \end{aligned}$$

This shows that

$$\operatorname{Re} \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} > 0 \quad \text{in } E.$$

Applying the same method as in the proof of [4, Theorem 5], we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E.$$

This shows that $f(z)$ is p -valently starlike in E .

Theorem 2. Let $p \geq 2$. If $f(z) \in A(p)$ and

$$(5) \quad \operatorname{Re} f^{(p)}(z) > 0 \quad \text{in } E,$$

then we have

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} 2\alpha_2 \quad \text{in } E$$

where $\alpha_2 = 0.638\dots$ is the solution of the equation

$$1 = \beta + \frac{2}{\pi} \tan^{-1} \beta.$$

Proof. Applying the same method as in the proof of Theorem 1 and from the assumption (5), we have

$$(6) \quad \left| \arg \frac{f^{(p-1)}(z)}{z} \right| < \frac{\pi}{2} \alpha_2 \quad \text{in } E.$$

Applying the same method as in the proof of Theorem 1 and from (6), we have

$$\left| \arg \frac{f^{(p-2)}(z)}{z^2} \right| < \frac{\pi}{2} \alpha_2 \quad \text{in } E.$$

Repeating the same method as the above, we have

$$(7) \quad \left| \arg \frac{f'(z)}{z^{p-1}} \right| < \frac{\pi}{2} \alpha_2 \quad \text{in } E$$

and

$$(8) \quad \left| \arg \frac{f(z)}{z^p} \right| < \frac{\pi}{2} \alpha_2 \quad \text{in } E.$$

Then from (7) and (8), we have

$$\begin{aligned} \left| \arg \frac{zf'(z)}{f(z)} \right| &\leq \left| \arg \frac{f'(z)}{z^{p-1}} \right| + \left| \arg \frac{f(z)}{z^p} \right| \\ &< \frac{\pi}{2} 2\alpha_2 \quad \text{in } E. \end{aligned}$$

This completes our proof.

References

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