

CRITICAL NONLINEAR WAVE EQUATIONS IN FRACTIONAL ORDER SOBOLEV SPACES

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In this note I describe some recent work on nonlinear wave equations, done jointly with M. Nakamura [20, 21]. We consider the nonlinear wave equations of the form

$$\partial_t^2 u - \Delta u = f(u) \quad (1)$$

where u is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbb{R}^n$, and f is a complex-valued function, a typical form of which is the single power interaction

$$f(u) = \lambda |u|^{p-1} u \quad (2)$$

with $\lambda \in \mathbb{R}$ and $1 < p < \infty$.

There is a large literature on the Cauchy problem for the equation (1) and on the asymptotic behavior in time of the global solutions [3, 4, 7-12, 14, 15, 24, 25, 28 and references therein]. The Cauchy problem for (1) has been studied mainly in the space of classical solutions and in the energy space, while there arises a new interest in the treatment of the Cauchy problem in the Sobolev spaces $H^s = (1 - \Delta)^{-s/2} L^2(\mathbb{R}^n)$ of fractional order s with $0 \leq s < n/2$. In connection with the H^s theory for (1) with (2), a homogeneity argument indicates that the power p in (2) is critical [resp. subcritical] at the level of H^s if and only if $p = 1 + 4/(n - 2s)$ [resp. $p < 1 + 4/(n - 2s)$]. Though the critical power $p = 1 + 4/(n - 2s)$ at the level of H^s is the same at that of nonlinear Schrödinger equations [2, 6, 13, 17, 18, 26], it would be natural to regard the power as $p = 1 + 4/((n - 1) - 2(s - 1/2))$ by the following reasons.

(i) In view of the sharp decay estimates for the free wave and Schrödinger equations, there is a natural shift in the corresponding space dimensions with difference

by one. This implies that results in the nonlinear wave equations should be often compatible with the corresponding results in the nonlinear Schrödinger equations by reducing the space dimension by one. The origin of the discrepancy may be traced back to the rank of the Hessian of phase functions in the oscillatory integrals for fundamental solutions.

(ii) In view of the Strichartz estimates in the diagonal case [28], there is a natural shift in the corresponding regularity requirements on the data with difference by one half.

(iii) In view of the symmetry groups acting on the associated Lagrangeans, the conformal powers of the nonlinear wave and Schrödinger equations are given respectively by $p = 1 + 4/(n - 1)$ and $p = 1 + 4/n$, while the corresponding space of data are given respectively by $H^{1/2}$ and L^2 .

By the arguments above, we could expect the H^s theory for (1) at the level compatible with that of the nonlinear Schrödinger equations in the critical case where $p = 1 + 4/(n - 2s)$ with $1/2 \leq s < n/2$. This in turn implies that $n \geq 2$ and $1 + 4/(n - 1) \leq p < \infty$ and that the critical power $p = 1 + 4/(n - 2s)$ loses its meaning at the level of $H^{n/2}$.

The purpose of this note is twofold. The first is to make the H^s theory for (1) complete with the whole admissible range $1/2 \leq s < n/2$. This means that we intend to extend the results of Lindblad and Sogge [15] to the spaces with higher regularity with the notion of criticality preserved. The second is to examine the critical phenomenon as the index s grows to $n/2$ and to construct the $H^{n/2}$ theory for (1) with critical nonlinearity of specific growth at infinity.

As regards the H^s theory with $0 \leq s < n/2$, the power behavior of the nonlinearity determines the order of the Sobolev space where smallness of the data is imposed to ensure the existence and uniqueness of global H^s solutions. This is the right phenomenon, as is usual with other nonlinear evolution equations with dilation structure, such as the nonlinear heat and Schrödinger equations with single power interaction and the Navier-Stokes equations.

In contrast, when $s > n/2$, no specific behavior of nonlinearity is required of the H^s theory for (1) at least locally in time. In fact, when $s > n/2$, for the existence and uniqueness of local H^s solutions one has only to assume that $f \in C^k(\mathbb{C}; \mathbb{C})$ with $f(0) = 0$, where differentiability refers to the real sense and k is the largest integer

less than or equal to s [19]. The proof depends on the usual Sobolev embedding $H^s \hookrightarrow L^\infty$ for $s > n/2$ in an essential way.

The case $s = n/2$ may therefore be regarded as the borderline in two aspects: (1) No power behavior of interaction amounts to the critical nonlinearity at the level of $H^{n/2}$. (2) Pointwise control of solutions falls beyond the scope of the $H^{n/2}$ theory, so that any argument similar to that of the H^s theory with $s > n/2$ breaks down even for local theory without specific behavior of interaction.

In addition to the critical phenomena described above, $H^{n/2}$ solutions deserve attention as finite energy solutions for $n = 2$ and as strong solutions for $n = 4$.

We prove the existence and uniqueness of global $H^{n/2}$ solutions to (1) with small Cauchy data under the nonlinearity of exponential type. This is reminiscent of Trudinger's inequality which replaces the Sobolev embedding in the limiting case on the basis of the exponential estimates in terms of functions in the critical order Sobolev space $H^{n/2}$ [16, 22, 23, 27, 29].

To state the results precisely, we use the following notation. For any r with $1 \leq r \leq \infty$, $L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty$, $H_r^s = (1 - \Delta)^{-s/2} L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any $s \in \mathbb{R}$ and any r, m with $1 \leq r, m \leq \infty$, $B_{r,m}^s$ denotes the Besov space defined as the space of distributions u such that $\{2^{sj} \|\phi_j * u; L^r\|\}_{j=0}^\infty \in \ell^m$, where $\{\phi_j\}$ is a dyadic decomposition on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty$, $\dot{H}_r^s$ denotes the homogeneous Sobolev space defined as the space of classes of distributions u modulo polynomials such that $(-\Delta)^{-s/2} u \in L^r$. For any $s \in \mathbb{R}$ and any r, m with $1 \leq r, m \leq \infty$, $\dot{B}_{r,m}^s$ denotes the homogeneous Besov space defined as the space of classes of distributions u modulo polynomials such that $\{2^{sj} \|\psi_j * u; L^r\|\}_{j=-\infty}^\infty \in \ell^m$, where $\{\psi_j\}$ is a dyadic decomposition on $\mathbb{R}^n \setminus \{0\}$. We refer to [1, 8, 30] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous counterparts. For simplicity, we put $H^s = H_2^s$, $\dot{H}^s = \dot{H}_2^s$, $B_r^s = B_{r,2}^s$, $\dot{B}_r^s = \dot{B}_{r,2}^s$. For any interval $I \subset \mathbb{R}$ and any Banach space X we denote by $C(I; X)$ the space of strongly continuous functions from I to X and by $L^q(I; X)$ the space of strongly measurable functions u from I to X such that $\|u(\cdot); X\| \in L^q(I)$. The Cauchy problem for the equation (1) with

data $(u(0), \partial_t u(0)) = (\phi, \psi)$ will be treated in the form of the integral equation

$$u(t) = K(t)\psi + \dot{K}(t)\phi + \int_0^t K(t-t')f(u(t'))dt', \quad (3)$$

where $K(t) = \omega^{-1} \sin t\omega$, $\dot{K}(t) = \cos t\omega$, and $\omega = (-\Delta)^{1/2}$. To treat the Cauchy problem both at finite and infinite times on the basis of the free unitary group in the generalized energy space, we formally differentiate (3) in time and introduce the following system of equations

$$\begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} = U(t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \int_{t_0}^t U(t-t') \begin{pmatrix} 0 \\ f(u(t')) \end{pmatrix} dt', \quad (4)$$

where $\begin{pmatrix} u(t_0) \\ \partial_t u(t_0) \end{pmatrix} = U(t_0) \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ is the prescribed Cauchy data at time t_0 and

$$U(t) = \begin{pmatrix} \dot{K}(t) & K(t) \\ -\omega^2 K(t) & \dot{K}(t) \end{pmatrix} = \exp t \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

is a unitary group in the Hilbert spaces $E^s \equiv \dot{H}^s \oplus \dot{H}^{s-1}$ if $1/2 \leq s < n/2$ and $E^{n/2} = (\dot{H}^{n/2} \cap \dot{H}^{1/2}) \oplus (\dot{H}^{n/2-1} \cap \dot{H}^{-1/2})$. The equation (4) will be studied in the space X^s with $1/2 \leq s \leq n/2$ defined as

$$X^s = C(\mathbb{R}; E^s) \cap \bigcap_{(1/q, 1/r, \rho) \in \Lambda^s} L^q(\mathbb{R}; \dot{B}_r^\rho \oplus \dot{B}_r^{\rho-1})$$

if $1/2 \leq s < n/2$, and

$$X^{n/2} = (C \cap L^\infty)(\mathbb{R}; E^s) \cap L^{q_0}(\mathbb{R}; (B_{q_0}^{(n-1)/2} \cap \dot{B}_{q_0}^0) \oplus (B_{q_0}^{(n-3)/2} \cap \dot{B}_{q_0}^{-1}))$$

where $q_0 = 2(n+1)/(n-1)$,

$$\Lambda^s = \{(1/q, 1/r, \rho); (1/q, 1/r) \in \Lambda_0, 0 \leq \rho \leq s, 0 \leq 1/q \leq n/2 - s\},$$

$$\Lambda_0 = \{(1/q, 1/r); 0 \leq 1/q, 1/r \leq 1/2, (1/q, 1/r) \neq (1/2, 1/2 - 1/(n-1)), \\ 1/r + 2/((n-1)q) \leq 1/2\}.$$

For the nonlinear interaction behaving as a power p at zero, we introduce the following assumption.

(A)_k $f \in C^k(\mathbb{C}; \mathbb{C})$ and $f^{(j)}(0) = 0$ for any j with $0 \leq j \leq k$. There exists a constant C such that for all $z_1, z_2 \in \mathbb{C}$

$$|f^{(k)}(z_1) - f^{(k)}(z_2)| \leq \begin{cases} C(|z_1|^{p-k-1} + |z_2|^{p-k-1})|z_1 - z_2| & \text{if } p \geq k+1, \\ C|z_1 - z_2|^{p-k} & \text{if } p < k+1. \end{cases}$$

Here $f^{(j)}$ denotes any of the j -th order derivatives of f with respect to z and $\bar{z}$ and $|f^{(j)}|$ denotes the maximum of the moduli of those derivatives. Single power interaction (2) satisfies (A)_k with $0 \leq k < p$.

For the nonlinear interaction having an exponential growth at infinity, we introduce the following assumption.

(B) $f \in C^{[n/2]}(\mathbb{C}; \mathbb{C})$, $f(0) = 0$. There exist two positive constants λ and C such that for all $z \in \mathbb{C}$

$$|f'(z)| \leq C|z|^{4/(n-1)} \exp(\lambda|z|^2).$$

Moreover,

$$|f''(z)| \leq C|z|^{1/3} \exp(\lambda|z|^2) \quad \text{if } n = 4,$$

$$\text{Max}_{2 \leq k \leq [n/2]} |f^{(k)}(z)| \leq C \exp(\lambda|z|^2) \quad \text{if } n \geq 5.$$

With the notation above we now state the main results in this paper. For any s with $1/2 \leq s \leq n/2$ and $\varepsilon > 0$, we denote by E_ε^s the closed ball in E^s with the center at zero and radius ε .

Theorem I (Critical case at the level of H^s with $1/2 \leq s < n/2$).

Let $n \geq 2$. Let s and p satisfy

$$1/2 \leq s < n/2,$$

$$[s] < p = 1 + 4/(n - 2s).$$

Let f satisfy (A)_[s]. Then there exists $\varepsilon > 0$ with the following property.

(1) For any $(\phi, \psi) \in E_\varepsilon^s$ at $t_0 = 0$ the equation (4) has a unique solution $(u, \partial_t u) \in X^s$. Moreover, there exist two pairs of asymptotic states $(\phi_\pm, \psi_\pm) \in E^s$ such that

$$\left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - U(t) \begin{pmatrix} \phi_\pm \\ \psi_\pm \end{pmatrix}; E^s \right\| \rightarrow 0 \quad (5)_\pm$$

as $t \rightarrow \pm\infty$.

(2) For any $(\phi_+, \psi_+) \in E_\varepsilon^s$ at $t_0 = +\infty$ [resp. $(\phi_-, \psi_-) \in E_\varepsilon^s$ at $t_0 = -\infty$] the equation (4) has a unique solution $(u, \partial_t u) \in X^s$ satisfying $(5)_+$ [resp. $(5)_-$].

Theorem II (Critical case at the level of $H^{n/2}$).

Let $n \geq 2$. Let f satisfy (B). Then there exists $\varepsilon > 0$ with the following property.

(1) For any $(\phi, \psi) \in E_\varepsilon^{n/2}$ at $t_0 = 0$ the equation (4) has a unique solution $(u, \partial_t u) \in X^{n/2}$. Moreover, there exist two pairs of asymptotic states $(\phi_\pm, \psi_\pm) \in E^{n/2}$ such that

$$\left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - U(t) \begin{pmatrix} \phi_\pm \\ \psi_\pm \end{pmatrix}; E^{n/2} \right\| \rightarrow 0 \quad (6)_\pm$$

as $t \rightarrow \pm\infty$.

(2) For any $(\phi_\pm, \psi_\pm) \in E_\varepsilon^{n/2}$ at $t_0 = +\infty$ [resp. $(\phi_-, \psi_-) \in E_\varepsilon^{n/2}$ satisfying $(6)_+$ [resp. $(6)_-$].

Remark 1. The theorems above shows the existence and asymptotic completeness of the wave operators $W_\pm : (\phi_\pm, \psi_\pm) \mapsto (u(0), \partial_t u(0)) = (\phi, \psi)$ defined on small asymptotic states in E^s with $1/2 \leq s \leq n/2$. The scattering operator S is then defined as $S = W_+^{-1} \circ W_-$.

Remark 2. A part of Theorem I is proved by Pecher [24] and Lindblad-Sogge [15] in the cases where $s = 1$ with $3 \leq n \leq 5$ and $1/2 \leq s \leq 3/2$ with $n \geq 2$, respectively. There are several results on the ill-posedness for (1) with $s < 1/2$ [14, 15].

Remark 3. The power $p = 1 + 4/(n - 2s)$ comes out as a critical one in $\dot{H}^s \oplus \dot{H}^{s-1}$ in the sense that $\|(u, \partial_t u); \dot{H}^s \oplus \dot{H}^{s-1}\|$ is invariant under the dilation $u \mapsto u_\lambda$ if and only if $s = n/2 - 2/(p - 1)$, where $u_\lambda(t, x) \equiv \lambda^{-2/(p-1)} u(\lambda^{-1}t, \lambda^{-1}x)$, $\lambda > 0$. We note here that $u \mapsto u_\lambda$ leaves (1) with (2) invariant in the sense that u solves (1) with (2) if and only if u_λ does.

Remark 4. The argument in Remark 3 makes sense only when $s < n/2$ and loses its meaning when $s = n/2$ for instance. In view of Trudinger's inequality the growth rate as $\exp(\lambda|z|^2)$ at infinity seems to be optimal at the level of $H^{n/2}$. We note that the L^∞ norm is out of control of the $H^{n/2}$ norm even when the latter is infinitesimally small.

We now give a sketch of the proofs. As usual the method depends on a partial contraction argument on (4) in the space X^s where the Strichartz type estimates for the free propagator fit naturally. To be more specific we use the inequality

$$\|U(\cdot) \begin{pmatrix} \phi \\ \psi \end{pmatrix}; L^q(\mathbb{R}; \dot{B}_r^\rho \oplus \dot{B}_r^{\rho-1})\| \leq C \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix}; E^s \right\|,$$

where $(1/q, 1/r) \in \Lambda_0$ and $s = \rho + n(1/2 - 1/r) - 1/q$, and its inhomogeneous version [10, 15, 28]. We combine those Strichartz estimates with the inequality in the following

Proposition 1 [17]. *Let p and s satisfy $1 \leq p < \infty$ and $0 \leq s < p$. Let ℓ, r, m satisfy $1 < \ell \leq r < \infty, 2 \leq r, m < \infty, 1/\ell = 1/r + (p-1)/m$. Let f satisfy $f \in C^{[s]}(\mathbb{C}; \mathbb{C})$. Then*

$$\|f(u); \dot{B}_\ell^s\| \leq C \|u; \dot{B}_m^0\|^{p-1} \|u; \dot{B}_r^s\|.$$

The basic estimates that the required iteration scheme goes through are completed by embedding theorems and convexity inequalities for the homogeneous Besov spaces. For the proof of Theorem II we expand the exponential nonlinearity, estimate individual L^p norms, and consider the convergence of the resulting series of norms. For that purpose we need information on the growth rate in p of the L^p estimate in terms of the $H^{n/2}$ norm. To be more specific we use the inequalities in the following

Proposition 2 [18]. *Let $1 < r < \infty$. Then there exists a constant C_0 such that for any q with $r \leq q < \infty$ the following estimates hold.*

$$\begin{aligned} \|u; L^q\| &\leq C_0 q^{1/2+(r-2)/(2q)} \|u; \dot{H}^{n/2}\|^{1-r/q} \|u; L^r\|^{r/q}, \\ \|u; \dot{B}_q^0\| &\leq C_0 q^{1/2+(r-2)/(2q)} \|u; \dot{H}^{n/2}\|^{1-r/q} \|u; \dot{B}_r^0\|^{r/q}. \end{aligned}$$

The proof of Proposition 1 follows closely that of [6, Lemma 3.4] in the sense that it depends on an equivalent norm on the homogeneous Besov spaces in terms of modulus of continuity with differences of second order, though actual proof is rather involved because of higher derivatives of functions coming from derivatives

of the composite function $f \circ u$. Proposition 2 follows from a sharp form of the Hardy-Littlewood-Sobolev inequality [22; Inequality (2.6)] and convexity inequalities between homogeneous Besov and Sobolev spaces. See [18] for details. The corresponding results have been obtained in [17, 18] for the nonlinear Schrödinger equations in the fractional order Sobolev spaces.

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