

Lebesgue Piecewise Affine Approximation of Nonlinear Systems

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This paper addresses a piecewise affine (PWA) approximation problem, i.e., a problem of finding a PWA system model which approximates a given nonlinear system. First, we propose a new class of PWA systems, called the *Lebesgue PWA approximation systems*, as a model to approximate nonlinear systems. Next, we derive an error bound of the PWA approximation model, and provide a technique for constructing the approximation model with specified accuracy. Finally, the proposed method is applied to a gene regulatory network with nonlinear dynamics, which shows that the method is a useful approximation tool.

1. Introduction

When strong nonlinearities are involved in a controlled plant, it is often hard to analyze and control the plant in a direct way. In fact, as is well-known, chemical plants, computer networks, and biosystems are governed by very complex dynamics, while the first-order linearization, which is the standard technique in the control community, is not always practical. So an alternative possible approach is to approximately express the complex dynamics by a simple system model.

Motivated by the above background, this paper discusses a *piecewise affine (PWA) approximation problem* for nonlinear systems, i.e., a problem of finding a PWA system model which approximates a given nonlinear system. Using the solution, many tools developed for hybrid systems may be applied to analysis and control of complex (nonlinear) systems. For instance, the following results (and them in progress) will be promising: the clustering based identification method [1], the MLD model based approach [2], the safety verification technique [3], the symbolic reachability analysis method [4], and the probabilistic controllability/observability analysis method [5, 6]. In fact, a PWA approximation system model (with three continuous states and nine modes) has been used for model predictive control of a production system [7]. Also, to provide any useful clues on plans for experiments performed by biologists, a PWA model (with eight continuous states and eight modes) has been utilized for the controllability analysis of a biosystem [8], where the probabilistic technique [5] for PWA systems has given new insightful observations for biosystems control. Development of hybrid tools has been continued to handle a larger class of problems,

and the class of solvable problems will be extended. So the PWA approximation will become an important technique.

For this approximation problem, various results have been obtained so far in [9–14], where nonlinear systems are approximately expressed by hybrid system models with the partitioned continuous state spaces. However, the existing methods are not satisfactory for the purpose to obtain an approximate PWA system with a computationally tractable form. For example, although Asarin *et al.* [12] have provided some error bounds for the approximation, they have not satisfactorily discussed how state spaces are reasonably partitioned. As a result, the evenly sized partition of nonlinear vector fields (as shown in Fig. 1(a)) may be employed in spite of the following practical limitation: for a given nonlinear system, the partition size has to be sufficiently fine if the variation at some state is large, while such fine partitioning implies that the obtained PWA approximation system has a large number of discrete state values (modes), i.e., the computational complexity of its analysis and control problems will be large. Thus, in many cases, the partition size should be determined depending upon the variation of the vector field at each state, as shown in Fig. 1(b).

This paper thus proposes a new technique, called the *Lebesgue piecewise affine approximation*, for obtaining a PWA system model which approximates a given nonlinear system. Based on a simple partition technique inspired by the idea of the Lebesgue integral and the Lebesgue sampling [15], the proposed approach divides the state space depending upon the variation of the vector field. As the result, it provides the advantage that the PWA approximation systems can be expressed in a

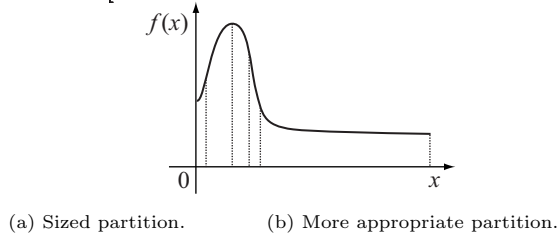


Figure 1. Examples of partition.

compact form and have a relatively smaller number of discrete state values. After the proposal of the Lebesgue PWA approximation, a method for constructing a practical PWA approximation system, which has a smaller number of discrete state values, is presented. Next, the proposed method is applied to the hybrid system modeling of a gene regulatory network.

Finally, it is remarked that the main purposes of this paper are (i) to propose the new technique “Lebesgue PWA approximation” for approximating nonlinear systems, and (ii) to present a prototype method for obtaining PWA approximation systems which (approximately) satisfy a given error specification. Although a somewhat limited class of nonlinear systems will be considered in this paper, the results for the purposes will be an important first step to establish the framework of the Lebesgue PWA approximation.

Notation: let $\mathbf{R}$, $\mathbf{R}_{0+}$, $\mathbf{R}_+$ denote the real number field, the set of nonnegative real numbers, and the set of positive real numbers, respectively, and let $\mathbf{N}_+$ be the set of positive integers. For the positive scalar $a \in \mathbf{R}_+$ and the vector $y \in \mathbf{R}^n$, we use $\lceil a \rceil$ to express the ceil of a , and $\lceil a \rceil_y$ to express the element of y which is the nearest to a and not smaller than a . Similarly, for the discrete set $\mathbf{H} \subset \mathbf{R}$, we denote by $\lceil a \rceil_{\mathbf{H}}$ the element of $\mathbf{H}$ which is the nearest to a and not larger than a , and we define $\lceil h \rceil_{\mathbf{H}}$ for the vector $h := [h_1 \ h_2 \ \cdots \ h_n]^T \in \mathbf{R}^n$ as $[\lceil h_1 \rceil_{\mathbf{H}} \ \lceil h_2 \rceil_{\mathbf{H}} \ \cdots \ \lceil h_n \rceil_{\mathbf{H}}]^T$. For the measurable set $\mathbf{X}$ of the vector x , let $\text{vol}(\mathbf{X})$ be the measure of the set $\mathbf{X}$, i.e., $\text{vol}(\mathbf{X}) := \int_{\mathbf{X}} dx$. For the vector x , the matrix M , and the scalar function f on the closed set $\mathbf{X}$, the symbols $\|x\|$, $\|M\|$, and $\|f\|$ express their ∞ -norms. Finally, the symbols $\mathbf{Z}$ and z are used to express an interval and a point on $\mathbf{R}$ in this paper (though, in traditional mathematics books, $\mathbf{Z}$ and z have been often used to express the set of integers and a complex number).

2. Problem Formulation

Consider the nonlinear system

$$\Sigma : \dot{x}(t) = f(x(t)) \quad (1)$$

on the set $\mathbf{X} := \mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n$ for the closed interval $\mathbf{X}_i \subset \mathbf{R}$, where $x \in \mathbf{X}$ is the state. The function $f : \mathbf{X} \rightarrow \mathbf{R}^n$ is Lipschitz continuous and is in the form

$$f(x) := \sum_{k=1}^N f_{1k}(x_1) f_{2k}(x_2) \cdots f_{nk}(x_n) \phi_k \quad (2)$$

where x_i is the i -th element of x , $f_{ik} : \mathbf{X}_i \rightarrow \mathbf{R}$ ($i = 1, 2, \dots, n$) are Lipschitz continuous functions, $\phi_k \in \mathbf{R}^n$ is a unit vector ($\|\phi_k\| = 1$), and $N \in \mathbf{N}_+$ is the number of terms. Note that even when a given nonlinear system is not in the form, it will be (approximately) expressed as (1) and (2) by using the Taylor expansion.

On the other hand, the PWA systems considered here are given by the following form:

$$\Sigma_p : \dot{x}(t) = A_I x(t) + a_I \quad \text{if } x(t) \in \mathbf{C}_I \quad (3)$$

where $x \in \mathbf{X}$ is the continuous state, $I \in \mathbf{I}$ is the discrete state (the *mode*), $\mathbf{I} := \{0, 1, \dots, \xi - 1\}$ is the set of the discrete state values, and $A_I \in \mathbf{R}^{n \times n}$ and $a_I \in \mathbf{R}^n$ are a constant matrix and a vector for mode I . In addition, $\mathbf{C}_I$ is the subregion of the continuous state assigned to I , given by $\mathbf{C}_I := \{x \in \mathbf{X} \mid C_I x + c_I \leq 0\}$ for $C_I \in \mathbf{R}^{l_I \times n}$ and $c_I \in \mathbf{R}^{l_I}$.

Then the following problem is considered.

Problem 1 For the nonlinear system Σ , suppose that the approximation time period $T \in \mathbf{R}_+$ and the tolerance $\varepsilon \in \mathbf{R}_+$ are given. Then, find a PWA system Σ_p satisfying

(C1) the condition

$$\|x(t, x_0) - x_p(t, x_0)\| \leq \varepsilon, \quad \forall t \in [0, T] \quad (4)$$

holds for every $x_0 \in \mathbf{X}$

where $x(t, x_0)$ and $x_p(t, x_0)$ respectively express the state of Σ and the continuous state of Σ_p at time t for the initial state $x_0 \in \mathbf{X}$. $\square$

For Problem 1, several remarks are given.

First, (C1) implies that the PWA system approximates the nonlinear system Σ in terms of the state behavior on the time interval $[0, T]$. Such a PWA approximation system is useful, e.g., for the safety verification, i.e., checking whether the state reaches the undesirable set or not on a finite-time interval, based on the hybrid systems theory, e.g., [3].

Second, it might be easy to find a solution to Problem 1, because a given Σ can be approximately expressed by some PWA system with a huge number of fine subregions $\mathbf{C}_I$ (a huge number of discrete state values). However, the computational complexities for analysis and control of PWA systems exponentially grow with the number of discrete state values (e.g., see [2, 16] for the controllability/observability analysis and optimal control). Thus it is important to obtain a PWA system with a practically small number of discrete state values. From this viewpoint, Problem 1 is addressed in the following sections.

Third, a solution to Problem 1 is not always well-posed, i.e., for the PWA system, a unique solution does not necessarily exist. However, after a solution is obtained, the well-posedness can be checked by the existing results in [17]. In order to focus on essential topics of the PWA approximation problem (as the first step), we do not deal with it in this paper. It will be discussed in a future work.

Fourth, Problem 1 is formulated in the continuous-time domain. Thus when one uses analysis/control tools developed for discrete-time PWA systems, continuous-time PWA systems obtained from Problem 1 have to be discretized by applying the standard discretization method to the state equation assigned to each I . There may exist some discrepancy between the continuous-time and discrete-time systems, while it must be true that, with a sufficiently small sampling period, the discrete-time systems capture the outline of the continuous-time original dynamics. So, in many practical situations, such tools can be applied to continuous-time systems.

Finally, for simplicity of discussion, the system (1) does not have the external input. However, the part of the proposed method, in Sections 3 and 4, is straightforwardly extended to the case that the external input is considered; see [18].

3. Lebesgue Piecewise Affine Approximation

To give a solution to Problem 1, this section proposes the Lebesgue piecewise affine approximation technique.

Let us consider a smooth function $\psi : \mathbf{Z} \rightarrow \mathbf{R}$ ($y = \psi(z)$) on a closed interval $\mathbf{Z} \subset \mathbf{R}$. For ψ , suppose that a scalar $h \in \mathbf{H} := \{w, \frac{w}{2}, \frac{w}{3}, \dots\}$ is given, where $w := \max_{z \in \mathbf{Z}} \psi(z) - \min_{z \in \mathbf{Z}} \psi(z)$. Then as shown in Fig. 2 (a), let $y_0, y_1, \dots, y_{\nu_y} \in \mathbf{R}$ be the mesh points on the y -axis (called the *y-axis mesh points*), satisfying $y_0 = \min_{z \in \mathbf{Z}} \psi(z)$ and $y_j = y_{j-1} + h$ ($j = 1, 2, \dots, \nu_y$), and let $z_0, z_1, \dots, z_{\nu_z} \in \mathbf{Z}$ be the mesh points on the z -axis (*z-axis mesh points*) determined by the function ψ and the y -axis mesh points. Here, $\nu_y \in \mathbf{N}_+$ is the num-

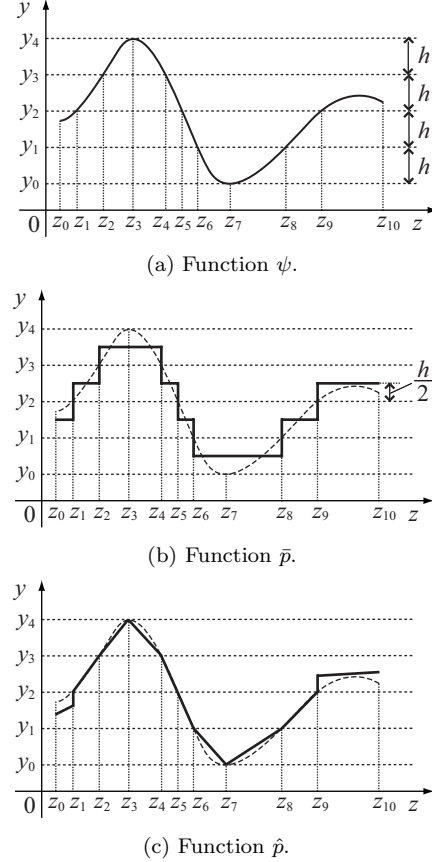


Figure 2. Example of functions ψ , $\bar{p}$, and $\hat{p}$.

ber of the y -axis mesh points, that is, $\nu_y = (w/h) + 1$. The z -axis mesh points z_0 and z_{ν_z} satisfy $\mathbf{Z} = [z_0, z_{\nu_z}]$ and $z_1, z_2, \dots, z_{\nu_z-1}$ are defined as the scalars such that (a) there does not exist an $r \in \mathbf{R}_+$ satisfying $\psi(z_i) = \psi(\tilde{z})$ for every $\tilde{z} \in [z_i - r, z_i + r]$, (b) there exists a $j \in \{0, 1, \dots, \nu_y\}$ satisfying $y_j = \psi(z_i)$, where $z_i < z_{i+1}$ holds, $\nu_z \in \mathbf{N}_+$ is uniquely determined from the above definition. Condition (a) implies that each interval $[z_i, z_{i+1}]$ is a set of non-zero measure and (b) means that each z_i is a solution of the nonlinear equation $y_j = \psi(z)$ for some y_j . Then we introduce the two functions $\bar{p} : \mathbf{Z} \rightarrow \mathbf{R}$ and $\hat{p} : \mathbf{Z} \rightarrow \mathbf{R}$ in (5) and (7).

The function $\bar{p}$ is given by the piecewise *constant* form

$$\bar{p}(z) := \left[\psi \left(\frac{z_i + z_{i+1}}{2} \right) \right]_y - \frac{h}{2} \quad \text{if } z \in \mathbf{Z}_i, \quad (5)$$

for $y := [y_0 \ y_1 \ \dots \ y_{\nu_y}]^\top$ and $\mathbf{Z}_i := [z_i, z_{i+1}]$ ($i = 0, 1, \dots, \nu_z - 1$), which can be considered as an approximate function of ψ with the accuracy

$$\| \psi - \bar{p} \| \leq \frac{h}{2}. \quad (6)$$

An example of $\bar{p}$ is shown in Fig. 2 (b).

On the other hand, the piecewise *affine* function $\hat{p}$, for which an example is provided in Fig. 2 (c), is defined as an improved version of $\bar{p}$:

$$\hat{p}(z) := s_i^* \left(z - \frac{z_i + z_{i+1}}{2} \right) + \left\lceil \psi \left(\frac{z_i + z_{i+1}}{2} \right) \right\rceil_y - \frac{h}{2} \quad \text{if } z \in \mathbf{Z}_i \quad (7)$$

where $s_i^* \in \mathbf{R}$ is the optimal slope of the affine function on $\mathbf{Z}_i$, given as an optimal solution of s_i to the minimax problem $\min_{s_i \in \mathbf{S}_i} \max_{z \in \mathbf{Z}_i} |d(s_i, z)|$ for

$$d(s_i, z) := \psi(z) - \left\{ s_i \left(z - \frac{z_i + z_{i+1}}{2} \right) + \left\lceil \psi \left(\frac{z_i + z_{i+1}}{2} \right) \right\rceil_y - \frac{h}{2} \right\},$$

$$\mathbf{S}_i := \begin{cases} \left[0, \frac{\psi(z_{i+1}) - \psi(z_i)}{z_{i+1} - z_i} \right] & \text{if } \frac{\psi(z_{i+1}) - \psi(z_i)}{z_{i+1} - z_i} \geq 0, \\ \left[\frac{\psi(z_{i+1}) - \psi(z_i)}{z_{i+1} - z_i}, 0 \right] & \text{otherwise.} \end{cases}$$

The scalar $d(s_i, z)$ expresses the deviation on $\mathbf{Z}_i$ between ψ and the affine function $s_i(z - \frac{z_i + z_{i+1}}{2}) + \lceil \psi(\frac{z_i + z_{i+1}}{2}) \rceil_y - \frac{h}{2}$. The function $\hat{p}$, in general, provides a better approximation than $\bar{p}$ because $\|\psi - \hat{p}\| = \min_{s_i \in \mathbf{S}_i} \max_{z \in \mathbf{Z}_i} |d(s_i, z)| \leq \max_{z \in \mathbf{Z}_i} |d(0, z)| \leq \frac{h}{2}$, where the first and second inequalities are derived from the relation $d(0, z) = \psi(z) - \bar{p}(z)$ and (6).

We respectively call the functions $\bar{p}$ and $\hat{p}$ the *Lebesgue piecewise constant function* (or simply, *L-PWC function*) and the *Lebesgue piecewise affine function* (*L-PWA function*) of ψ for the *Lebesgue mesh size* h (*L-mesh size*). Note here that we have

$$\|\psi - \bar{p}\| = 0 \quad (8)$$

for the L-PWC function $\bar{p}$ of a constant function ψ , and

$$\|\psi - \hat{p}\| = 0 \quad (9)$$

for the L-PWA function $\hat{p}$ of a affine function ψ . This will be used in the following sections.

It is remarked that the above Lebesgue based method produces a partition, depending on the variation of ψ , in a simple way. As a result, compared with the z -axis based method, i.e., the method that approximates a nonlinear function with the evenly-sized partition on the z -axis, it gives an approximate piecewise affine (constant) function with a relatively smaller number of partitions. More precisely, in order to obtain an approximate function with the accuracy (6) by the z -axis based method, the z -axis will be partitioned at even intervals whose length is less than or equal to the *smallest* z -axis partition size obtained by the Lebesgue based method.

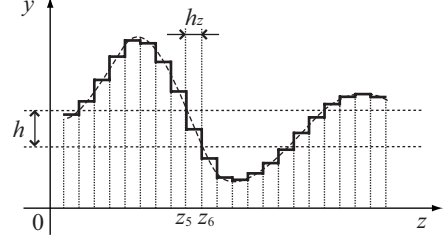


Figure 3. Example of piecewise constant function by the z -axis based method.

For example, if, in Fig. 2 (b), the interval $[z_5, z_6]$ is the smallest and the size is denoted by h_z , the z -axis based method derives the piecewise constant function shown in Fig. 3 for achieving the same accuracy (cf. Fig. 2 (b)).

Note also that, for a class of nonlinear functions (e.g., affine functions), there is a better partition than the proposed equidistant partition of the y -axis. In such a case, by handling $y_1, y_2, \dots, y_{\mu_y-1}$ as design parameters, a better PWC/PLA function can be derived. For example, in Fig. 2 (b) and (c), the approximations of $\bar{p}$ and $\hat{p}$ are more accurate by designing $y_1, y_2, \dots, y_{\mu_y-1}$. But, in this paper, we employ the equidistant partition to simply show the basic idea of the “Lebesgue PWA approximation”.

Now, let us introduce a new class of PWA system models. Using the above notation, we define $p_{ik} : \mathbf{X}_i \rightarrow \mathbf{R}$ as the L-PWC function or L-PWA function of f_{ik} in (2) with the L-mesh size $h_{ik} \in \mathbf{H}$. Furthermore, let $\mathbf{h} \in \mathbf{H}^{Nn}$, called the *L-mesh size vector*, denote the vector composed of h_{ik} ($i = 1, 2, \dots, n, k = 1, 2, \dots, N$). Then for the system

$$\Sigma_L(\mathbf{h}) : \dot{x}(t) = p(x(t)) \quad (10)$$

where

$$p(x) := \sum_{k=1}^N p_{1k}(x_1) p_{2k}(x_2) \cdots p_{nk}(x_n) \phi_k, \quad (11)$$

the following result is straightforwardly obtained.

Lemma 1 *For the system $\Sigma_L(\mathbf{h})$, suppose that $\mathbf{h} \in \mathbf{H}^{Nn}$ is given. Then if*

(C2) *the number of L-PWA functions is at most one in the function set $\{p_{1k}, p_{2k}, \dots, p_{nk}\}$,*

holds for every $k \in \{1, 2, \dots, N\}$, $\Sigma_L(\mathbf{h})$ is a PWA system, i.e., $\Sigma_L(\mathbf{h})$ can be expressed in the form of Σ_p . □

When (C2) is satisfied for every $k \in \{1, 2, \dots, N\}$, the system $\Sigma_L(\mathbf{h})$ is in a PWA system form for approx-

imating the nonlinear system Σ . We call here the system $\Sigma_L(\mathbf{h})$ the *Lebesgue PWA approximation system* (or simply, *L-PWA approximation system*).

The L-PWA approximation systems are in a compact form. In fact, the systems are specified by the L-mesh size vector $\mathbf{h}$ and the assignment of the L-PWC and L-PWA functions to p_{ik} . So the computational space complexity for expressing $\Sigma_L(\mathbf{h})$ is practically small, which is one of its advantages. This paper considers such a class of PWA system models.

In the following sections, we denote by $\Sigma_{Lp}(\mathbf{h})$ the system $\Sigma_L(\mathbf{h})$ satisfying (C2) for every $k \in \{1, 2, \dots, N\}$, and by $x_p(t, x_0)$ its continuous state at time t for the initial state $x_0 \in \mathbf{X}$.

Remark 1 As stated, the L-PWA function to f_{ik} provides a better approximation than the L-PWC function (with the same h_{ik}). On the other hand, Lemma 1 shows that an L-PWA approximation system cannot be obtained by using only the former. Hence, both the L-PWC function and the L-PWA function are introduced.

Remark 2 It is feasible to efficiently and approximately compute (i) the z -axis mesh points, (ii) the two y -axis mesh points y_0, y_{ν_y} (i.e., $\max_{z \in \mathbf{Z}} \psi(z)$, $\min_{z \in \mathbf{Z}} \psi(z)$), and (iii) the optimal slope s_i^* in (7), because they are solutions to one-dimensional nonlinear equations and nonlinear programming problems. For example, each z -axis mesh point is obtained by solving the one-dimensional nonlinear equation $y_j = \psi(z)$ with the standard techniques such as the bisection method.

Note for (iii) that, in order to easily obtain the L-PWA function $\hat{p}$ from a one-dimensional problem, the intercept of $\hat{p}$ is fixed (as shown in (7)) and the only slope is optimized.

Remark 3 In the proposed approach, the idea of the Lebesgue approximation is introduced for the “scalar” nonlinear functions p_{ik} , which gives the Lebesgue PWA approximation system in (10). On the other hand, the same idea can be applied to the approximation of “multivariable” nonlinear functions. This gives a Lebesgue PWA approximation system with a smaller number of discrete state values. However, for multivariable nonlinear functions, it is hard to compute the corresponding L-PWA and L-PWA functions, since multidimensional nonlinear equations and optimization problems have to be solved for this purpose. Furthermore, the subregions assigned to each discrete state value are neither always hyper rectangles nor convex polyhedra. For example, as easily imagined, if $f(x) := x^\top x$ for $x \in \mathbf{R}^2$, each subregion is between two concentric circles. Since many analysis/control tools for hybrid systems are based on

polyhedral manipulations, such a multivariable approximation will not be matched for the existing tools. This is the reason why the scalar approximation is employed in this paper.

4. An Error Bound of L-PWA Approximation

In this section, we derive an error bound between the original system Σ and the L-PWA approximation system $\Sigma_{Lp}(\mathbf{h})$. The proof of all the results in this section are given in Appendix A.

First, the following result is prepared.

Lemma 2 Consider the systems Σ and $\Sigma_{Lp}(\mathbf{h})$, and suppose that $\mathbf{h} \in \mathbf{H}^{Nn}$ is given. Then

$$|p_{1k}(x_1)p_{2k}(x_2) \cdots p_{nk}(x_n) - f_{1k}(x_1)f_{2k}(x_2) \cdots f_{nk}(x_n)| \leq M_k(\|f_{1k}\|, \|f_{2k}\|, \dots, \|f_{nk}\|)\mathbf{h}_k$$

holds for every $x \in \mathbf{X}$ and $k \in \{1, 2, \dots, N\}$, where $\mathbf{h}_k := [h_{1k} \ h_{2k} \ \cdots \ h_{nk}]^\top$ (which is a part of $\mathbf{h}$) and $M_k \in \mathbf{R}^{1 \times n}$ is a row vector satisfying $\eta_k(n) = M_k \mathbf{h}_k$ for $\eta_k(n) \in \mathbf{R}_{0+}$ given by the recurrence formula

$$\eta_k(i) = \|f_{ik}\|\eta_k(i-1) + \|f_{1k}\| \|f_{2k}\| \cdots \|f_{(i-1)k}\| \frac{h_{ik}}{2},$$

$$\eta_k(1) := \frac{h_{1k}}{2}.$$

□

Lemma 2 presents an inequality to estimate the difference between the vector fields of Σ and $\Sigma_{Lp}(\mathbf{h})$. From Lemma 2, the following result is obtained.

Theorem 1 Consider the systems Σ and $\Sigma_{Lp}(\mathbf{h})$, and suppose that $\mathbf{h} \in \mathbf{H}^{Nn}$ is given. Let $L \in \mathbf{R}_+$ be a Lipschitz constant of the function $f(x)$ on $\mathbf{X}$, and $M := [M_1 \ M_2 \ \cdots \ M_n]$ (M_k is defined in Lemma 2) and

$$R(t) := \frac{M}{L}(e^{Lt} - 1). \quad (12)$$

Then

$$\|x(t, x_0) - x_p(t, x_0)\| \leq R(t)\mathbf{h}, \quad \forall t \in [0, \infty) \quad (13)$$

holds for every $x_0 \in \mathbf{X}$. □

Theorem 1 allows us to estimate the error of the L-PWA approximation system $\Sigma_{Lp}(\mathbf{h})$ to the original nonlinear system Σ .

In addition, this result gives a solution to Problem 1 in the following way. Consider the system $\Sigma_{Lp}(\mathbf{h})$ with a parameter $\mathbf{h} \in \mathbf{H}^{Nn}$ to be determined. Let $\bar{\mathbf{h}} \in \mathbf{R}_{0+}^{Nn}$ be a solution to the linear inequality

$$R(T)\bar{\mathbf{h}} \leq \varepsilon. \quad (14)$$

Then each element of $R(t)$ is positive and monotonically nondecreasing for $t \in [0, \infty)$, and the relation $R(T)[\bar{\mathbf{h}}]_{\mathbf{H}} \leq R(T)\bar{\mathbf{h}}$ holds. Thus it follows that $\Sigma_{Lp}(\mathbf{h})$ with $\mathbf{h} := [\bar{\mathbf{h}}]_{\mathbf{H}}$ satisfies (C1), which means that this $\Sigma_{Lp}(\mathbf{h})$ is a solution to Problem 1.

Note that when f_{ik} is a constant (affine) function approximated by the L-PWC (L-PWA) function p_{ik} , the relation

$$\|f_{ik} - p_{ik}\| = 0 \quad (\text{i.e., } f_{ik} = p_{ik}) \quad (15)$$

holds for every $h_{ik} \in \mathbf{H}$, as shown in (8) (in (9)). Then the error bound (13) can be rewritten as

$$\|x(t, x_0) - x_p(t, x_0)\| \leq \tilde{R}(t)\tilde{\mathbf{h}}, \quad \forall t \in [0, \infty) \quad (16)$$

where $\tilde{\mathbf{h}} \in \mathbf{H}^\lambda$ is the vector composed of the elements of $\mathbf{h}$ for whose index f_{ik} does not satisfy (15) (λ is the dimension of $\tilde{\mathbf{h}}$), and $\tilde{R}(t)$ is the row vector composed of the corresponding elements of $R(t)$. For example, if $\mathbf{h} := [h_{11} \ h_{21} \ h_{31}]^\top$, $R(t) := [8 \ 7 \ 3]$, and (15) holds only for f_{21} , then (16) holds for $\tilde{\mathbf{h}} = [h_{11} \ h_{31}]^\top$ and $\tilde{R}(t) = [8 \ 3]$. So (16) implies that the error depends on the only $\tilde{\mathbf{h}}$. In the next section, $\tilde{\mathbf{h}}$ will be designed for solving Problem 1. Then the vector $\tilde{\mathbf{h}}$ is also referred as the L-mesh size vector, and the L-PWA approximation system with $\tilde{\mathbf{h}}$ is simply denoted by $\Sigma_{Lp}(\tilde{\mathbf{h}})$.

5. Design of L-PWA Approximation Systems with Guaranteed Tolerance

This section provides a practical solution to Problem 1. As seen in Section 4, a solution to the problem is derived by (13) and (14). However, due to the conservativeness of (13), the L-PWA approximation system will have a large number of the discrete state values. So we address here a method to construct an L-PWA approximation system having a relatively smaller number of the discrete state values.

Based on the L-PWA approximation systems, Problem 1 is divided into the two issues:

- (D1) determine the assignment of the L-PWC and L-PWA functions to p_{ik} for Σ_L to satisfy (C2).
- (D2) determine the L-mesh size vector $\tilde{\mathbf{h}} \in \mathbf{H}^\lambda$ for $\Sigma_{Lp}(\tilde{\mathbf{h}})$ (derived in (D1)) to satisfy (C1) and have a relatively small number of the discrete state values. Here, the L-mesh size parameters excluded from $\tilde{\mathbf{h}}$ are set as their maximum values, e.g., $h_{ik}^p := \max_{x_i \in \mathbf{X}_i} f_{ik}(x_i) - \min_{x_i \in \mathbf{X}_i} f_{ik}(x_i)$, for reducing the number of discrete state values in Σ_{Lp} .

The former can be easily solved by enumerating all assignments of L-PWC and L-PWA functions to p_{ik} , since the number of all assignments is at most n^N and is not

so large in this problem setting. In contrast, (D2) is a central issue in the design of L-PWA approximation systems, and thus we focus on (D2) hereafter.

Typically, the problem (D2) is formulated as the problem of minimizing the number of the discrete state values under the error constraint (C1), that is,

$$(P1): \min_{\tilde{\mathbf{h}} \in \mathbf{H}^\lambda} D(\tilde{\mathbf{h}}) \quad \text{s.t. (C1) holds for } \Sigma_{Lp}(\tilde{\mathbf{h}})$$

where $D(\tilde{\mathbf{h}}) \in \mathbf{N}_+$ is the number of the discrete state values in the L-PWA system. However, to this problem, any typical methods for (combinatorial) optimization cannot be applied. In fact,

- The function $D(\tilde{\mathbf{h}})$ cannot be expressed in an explicit form, since the number of the z -axis mesh points is known only after the z -axis mesh points are numerically computed (see Remark 2). In addition, $D(\tilde{\mathbf{h}})$ is not necessarily convex with respect to $\tilde{\mathbf{h}}$.
- The feasible set, i.e., the set of $\tilde{\mathbf{h}} \in \mathbf{H}^\lambda$ for which $\Sigma_{Lp}(\tilde{\mathbf{h}})$ satisfies (C1), cannot be explicitly represented, because $x(t, x_0)$ and $x_p(t, x_0)$ are solutions of the nonlinear differential equations (1) and (3). Also, it is hard to verify (C1) even for a fixed $\tilde{\mathbf{h}}$.

Taking this fact into account, we propose a method based on “one”-dimensional optimization and Monte-Carlo tests. More concretely, an L-mesh size vector $\tilde{\mathbf{h}}$ is derived from the following optimization problem:

$$(P2): \min_{\tilde{\mathbf{h}} \in \mathbf{L}^\lambda} D(\tilde{\mathbf{h}}) \quad \text{s.t. (C1')} \text{ holds for } \Sigma_{Lp}(\tilde{\mathbf{h}}), \quad (17)$$

where $\mathbf{L}^\lambda$ is a subset of $\mathbf{H}^\lambda$ and (C1') is a relaxed version of (C1). The set $\mathbf{L}^\lambda$ is defined as $\mathbf{L}^\lambda := \{\tilde{\mathbf{h}} \in \mathbf{H}^\lambda | \exists \theta \in [0, \infty) \text{ s.t. } \tilde{\mathbf{h}} = [\Pi\theta]_{\mathbf{H}}\}$ where $\Pi := [1/\tilde{R}_1(T) \ 1/\tilde{R}_2(T) \ \cdots \ 1/\tilde{R}_\lambda(T)]^\top$ and $\tilde{R}_i(T)$ is the i -th element of the vector $\tilde{R}(T)$ in (16). This set corresponds to the elements of $\mathbf{H}^\lambda$ around the line $\Pi\theta$ for the parameter $\theta \in [0, \infty)$. On the other hand, (C1') is given as

- (C1') for each i.i.d. random sample x_0^j ($j = 1, 2, \dots, S$) from the uniform distribution on $\mathbf{X}$, (4) holds for $x_0 := x_0^j$.

Problem (P2) is derived by substituting $\mathbf{L}^\lambda$ and (C1') for $\mathbf{H}^\lambda$ and (C1) in the original problem (P1).

As easily confirmed, (P2) is reduced into a one-dimensional optimization problem (See Remark 4), which can be easily solved. So it is a more practical to obtain an L-mesh size vector from (P2).

Problem (P2) is introduced according to the following idea.

First, due to the hardness of (P1), it is acceptable to approximately solve the problem by reducing into a one-dimensional optimization problem. Then the error bound $\tilde{R}(T)\tilde{\mathbf{h}}$ in (16) specifies a kind of the variation of the error to $\tilde{\mathbf{h}}$, and so it is reasonable to find an $\tilde{\mathbf{h}}$ according to $\tilde{R}(T)\tilde{\mathbf{h}}$. For example, in the case of $\tilde{R}(T) = [2 \ 1]$, the error is more affected by the first element than the second element of $\tilde{\mathbf{h}}$, which may implies that the second element should be two times larger than the first element. From the generalization of this idea, the set $\mathbf{L}^\lambda$ is employed. Note that this substitution allows us to easily obtain a practical solution to (P1) (see Remark 4).

Second, when we cannot check whether a condition with the parameter α holds for every $\alpha \in \mathbf{A}$, it is practical to verify whether the condition holds for almost all $\alpha \in \mathbf{A}$. Such an idea is formulated by [19] with a probabilistic statement for the robustness analysis of control systems, where α is an uncertainty and the satisfaction of a performance level is considered. Then if α is regarded as x_0 and the satisfaction of (4) is discussed, this is exactly our case. More precisely, it is formalized as follows: Suppose that $\omega, \delta \in (0, 1)$ are arbitrarily given, and let $\mathbf{X}_v$ be the set of $x_0 \in \mathbf{X}$ for which (4) is violated. Then if (C1') holds for $S := \lceil \ln \frac{1}{\delta} / \ln \frac{1}{1-\omega} \rceil$, the relation

$$\text{Prob}\{\text{vol}(\mathbf{X}_v) / \text{vol}(\mathbf{X}) \leq \omega\} \geq 1 - \delta \quad (18)$$

is satisfied. Here, (C1) implies (18), and (18) with $\omega \rightarrow 0$ and $\delta \rightarrow 0$ is approximately equivalent to (C1) in a generic sense; so (C1') with $\omega \rightarrow 0$ and $\delta \rightarrow 0$ is approximately equivalent to (C1). Then it follows from (18) that if (C1') holds for sufficiently small ω and δ , it is guaranteed with sufficiently high probability $1 - \delta$ that for almost all $x_0 \in \mathbf{X}$, (4) holds, i.e., (C1) is approximately satisfied with the probabilistic accuracy (18) specified by users. In this sense, (C1') can be considered as a kind of relaxed condition of (C1).

In this way, (P2) is employed. By solving this problem, a practical solution to (P1) will be given, as shown in the next section.

Remark 4 A solution to (P2) is given by $\tilde{\mathbf{h}}^* = \lfloor \Pi \theta^* \rfloor_{\mathbf{H}}$ for an optimal solution θ^* to the one-dimensional nonlinear programming problem

$$(P3) : \min_{\theta \in [0, \infty)} D(\lfloor \Pi \theta \rfloor_{\mathbf{H}}) \text{ s.t. (C1') holds for } \Sigma_{LP}(\lfloor \Pi \theta \rfloor_{\mathbf{H}}).$$

This can be solved by the direct search for the one-dimensional parameter θ . For example, for each θ given as an element of a discrete set $\{\theta_1, \theta_2, \dots\}$ (a grid of the interval $[0, \infty)$), the value of $D(\lfloor \Pi \theta \rfloor_{\mathbf{H}})$ is computed and (C1') is verified. The condition (C1') is checked by numerically computing the solution behaviors of $\Sigma_{LP}(\lfloor \Pi \theta \rfloor_{\mathbf{H}})$ for each sample x_0^j .

6. Application to Hybrid System Modeling of Gene Regulatory Networks

Let us apply the proposed method to the hybrid system modeling of a gene regulatory network. Through an example, it is shown that a practical PWA system model can be derived by the method developed in Section 5.

Fig. 4 shows a typical gene regulatory network model. This is composed of the genes A and B interconnected each other, and the expression level of each gene is controlled by the protein which is produced by the other gene. The genes will be transcribed into the messenger RNAs (mRNAs) and the mRNAs are next translated into the corresponding proteins.

A mathematical model of this regulatory network is given as follows [20]:

$$\begin{cases} \dot{x}_1(t) = \alpha_1 \sigma_n(x_4(t)) - \beta_1 x_1(t), \\ \dot{x}_2(t) = \alpha_2 x_1(t) - \beta_2 x_2(t), \\ \dot{x}_3(t) = \alpha_3 \sigma_p(x_2(t)) - \beta_3 x_3(t), \\ \dot{x}_4(t) = \alpha_4 x_3(t) - \beta_4 x_4(t), \end{cases} \quad (19)$$

where $x_1, x_3 \in [0, 1]$ and $x_2, x_4 \in [0, 1]$ are the concentrations of the mRNAs and the proteins produced by the genes A and B , respectively, $\alpha_i, \beta_i \in \mathbf{R}_{0+}$ ($i = 1, 2, 3, 4$) are the production and degradation rate constants, and σ_n and σ_p are the nonlinear functions defined as $\sigma_n(x_4) := \theta_n^\kappa / (\theta_n^\kappa + x_4^\kappa)$, $\sigma_p(x_2) := x_2^\kappa / (\theta_p^\kappa + x_2^\kappa)$ for $\kappa, \theta_n, \theta_p \in \mathbf{R}$. The functions σ_n and σ_p are a kind of switch functions, which express the inhibition/activation of the gene expression.

For this nonlinear system, we consider Problem 1 with $\mathbf{X} := [0, 1]^4$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) := (0.95, 1.23, 0.86, 1.43, 0.90, 1.31, 0.83, 2.60)$, $\theta_n := 0.20$, $\theta_p := 0.19$, $\kappa := 4$, $T := 1$, and $\varepsilon := 0.1$. The system in (19) is expressed in the form of Σ , where $N := 6$, $f_{41}(x_4) := \alpha_1 \sigma_n(x_4)$, $f_{12}(x_1) := \Lambda_{12} x_1$, $f_{23}(x_2) := -\beta_2 x_2$, $f_{24}(x_2) := \alpha_3 \sigma_p(x_2)$, $f_{35}(x_3) := \Lambda_{34} x_3$, $f_{46}(x_4) := -\beta_4 x_4$, $\phi_1 := [1 \ 0 \ 0 \ 0]^\top$, $\phi_2 := [-\beta_1/\Lambda_{12} \ \alpha_2/\Lambda_{12} \ 0 \ 0]^\top$, $\phi_3 := [0 \ 1 \ 0 \ 0]^\top$, $\phi_4 := [0 \ 0 \ 1 \ 0]^\top$, $\phi_5 := [0 \ 0 \ -\beta_3/\Lambda_{34} \ \alpha_4/\Lambda_{34}]^\top$, $\phi_6 := [0 \ 0 \ 0 \ 1]^\top$ for $\Lambda_{12} := \max\{\beta_1, \alpha_2\}$, $\Lambda_{34} := \max\{\beta_3, \alpha_4\}$, the other $f_{ik}(x_i)$ are given by $f_{ik}(x_i) := 1$.

Then the L-PWA approximation system is derived according to (D1) and (D2) in the beginning of Section 5, where the accuracy parameters are set as $\omega := 10^{-3}$ and $\delta := 10^{-3}$ (corresponding to $S := 6905$). The L-PWA functions are assigned to $p_{41}(x_4)$, $p_{12}(x_1)$, $p_{23}(x_2)$, $p_{24}(x_2)$, $p_{35}(x_3)$, and $p_{46}(x_4)$, and the L-PWC functions are assigned to the other $p_{ik}(x_i)$. By the proposed method, the L-mesh size vector $\mathbf{h} := [h_{41} \ h_{24}]^\top$ is given by $[\alpha_1/4 \ \alpha_3/4]^\top$ from $\tilde{R}(T) \simeq [30.8 \ 27.9]$, and the other L-mesh size parameters are set as their maximum values (as explained in (D2)). In this case, the L-PWA

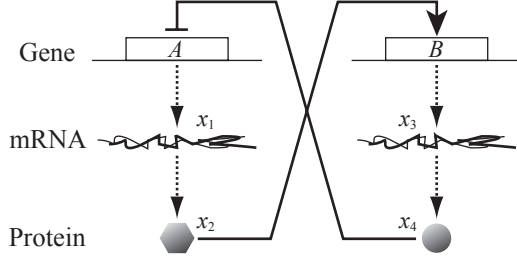
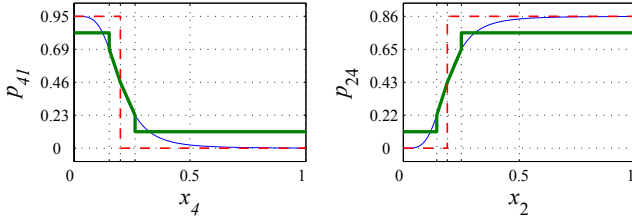


Figure 4. Two-genes regulatory network model.

Figure 5. L-PWA functions (thick line) for f_{41} and f_{24} (thin line) and primitive approximate functions (dash line).

system has 16 ($= 4^2$) discrete state values and satisfies the relaxed condition (C1').

Fig. 5 shows the L-PWA functions $p_{41}(x_4)$ and $p_{24}(x_2)$, obtained by the proposed approach, by thick lines. Fig. 6 demonstrates the solution behaviors of the original system in (19), of $\Sigma_{Lp}(\mathbf{h})$ derived here, and of the PWA system obtained by the well-used approximation $\sigma_n(x_4) \simeq -0.5 \text{sign}(x_4 - \theta_n) + 0.5$ and $\sigma_p(x_2) \simeq 0.5 \text{sign}(x_2 - \theta_p) + 0.5$ (in which the number of the discrete state values is four) for the initial state $x_0 := [0.21 \ 0.19 \ 0.17 \ 0.22]^\top$. The primitive approximate functions for $f_{41}(x_4)$ and $f_{24}(x_2)$ are also shown in Fig. 5 by dash lines. It turns out that the solution behaviors of the original system and $\Sigma_{Lp}(\mathbf{h})$ are quite similar, and $\Sigma_{Lp}(\mathbf{h})$ gives a much better approximation than the PWA system by the primitive approximation. Furthermore, when Problem 1 is solved from the error bound (16), a PWA system with more than 100,000 discrete state values is obtained, which implies that the proposed method in Section 5 gives a PWA system with a practically small number of discrete state values. Next, the computation time for deriving the L-PWA system was 3285 seconds with Intel Pentium D 3GHz and 2GB RAM, which is a practical computation cost in the modeling process of dynamical systems.

On the other hand, if α_3 is reset at the 10 percent-value, i.e., $\alpha_3 := 0.086$, we obtain $\hat{R}(T) \simeq [30.8 \ 2.8]$ and $\mathbf{h} \simeq [\alpha_1/4 \ \alpha_3]^\top$ for which the corresponding L-PWA system has 4 ($= 4 \times 1$) discrete state values. In

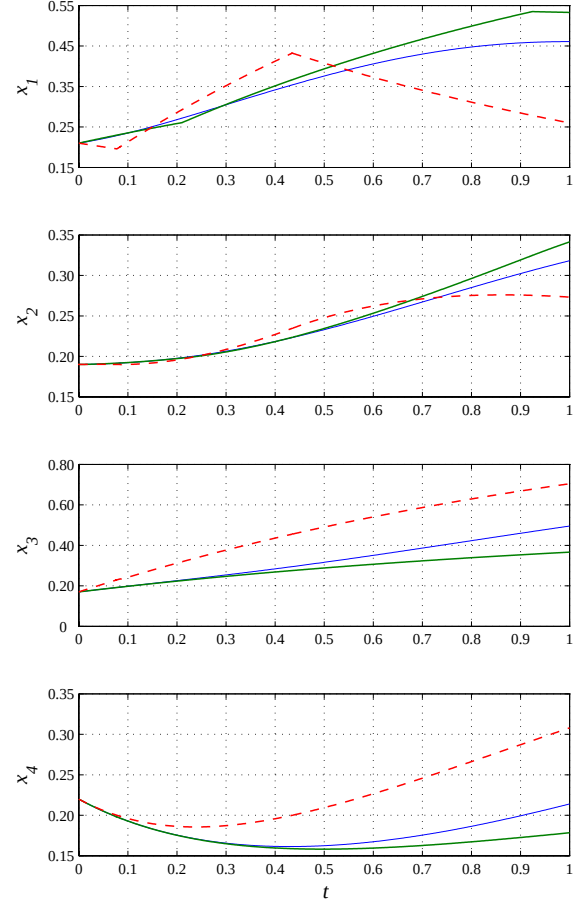


Figure 6. Solution behaviors of the original system (thin line), the L-PWA approximation system (thick line), and the PWA system by the primitive approximation (dash line).

this case, the influence of the nonlinear term $\sigma_p(x_2)$ on the state trajectory is relatively smaller than the former case. Hence, the proposed method provides an L-PWA system which approximates a given nonlinear system with the consideration of the dependence of each nonlinear term on the original system behaviors.

7. Conclusion

A piecewise affine approximation technique for nonlinear systems has been presented. The proposed method is based on the Lebesgue piecewise affine approximation scheme which partitions the state spaces depending upon the variation of the vector fields in a simple way. In addition to the proposal, an error bound on the state trajectory has been derived, and a method for constructing a PWA approximation system with a guaranteed tolerance has been presented. Furthermore, the

technique has been applied to a gene regulatory network with nonlinear dynamics, and it has been shown that the Lebesgue piecewise affine approximation is useful.

A. Proofs of Lemma 2 and Theorem 1

At first, the following result [21] is prepared.

Consider the systems

$$\Sigma_1 : \dot{x}(t) = f(t, x(t)), \quad \Sigma_2 : \dot{x}(t) = f(t, x(t)) + g(t, x(t))$$

where $f : [0, T] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a function which is piecewise continuous for $t \in [0, T]$ and L -Lipschitz on $\mathbf{X} \subseteq \mathbf{R}^n$, and $g : [0, T] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a function satisfying $\|g(t, x)\| \leq G$ for every $x \in \mathbf{X}$ and $t \in [0, T]$ ($G \in \mathbf{R}_{0+}$, $T \in [0, \infty)$). Let $x_i(t, x_0)$ be the state of the system Σ_i under $x(0) = x_0 \in \mathbf{X}$ ($i = 1, 2$).

Lemma 3 For the systems Σ_1 and Σ_2 , suppose that $T \in [0, \infty)$ and $x_0 \in \mathbf{X}$ are given. Then

$$\|x_1(t, x_0) - x_2(t, x_0)\| \leq \frac{G}{L}(e^{Lt} - 1) \quad (20)$$

holds for every $t \in [0, T]$. $\square$

Now, we prove Lemma 2 and Theorem 1.

(Lemma 2) For every $x \in \mathbf{X}$ and $k \in \{1, 2, \dots, N\}$, the following inequalities hold from (6).

$$\begin{aligned} |p_{1k}(x_1) - f_{1k}(x_1)| &\leq \eta_k(1), \\ |p_{1k}(x_1)p_{2k}(x_2) - f_{1k}(x_1)f_{2k}(x_2)| \\ &\leq |(p_{1k}(x_1) - f_{1k}(x_1))p_{2k}(x_2) + f_{1k}(x_1)(p_{2k}(x_2) - f_{2k}(x_2))| \\ &\leq \eta_k(1)\|p_{2k}\| + \|f_{1k}\|\frac{h_{2k}}{2} \leq \eta_k(2), \\ &\vdots \end{aligned}$$

$$\begin{aligned} |p_{1k}(x_1)p_{2k}(x_2) \cdots p_{nk}(x_n) - f_{1k}(x_1)f_{2k}(x_2) \cdots f_{nk}(x_n)| \\ \leq \eta_k(n) = M_k(\|f_{1k}\|, \|f_{2k}\|, \dots, \|f_{nk}\|)\mathbf{h}_k. \end{aligned}$$

This completes the proof.

(Theorem 1) From (11) and Lemma 2, we derive

$$\begin{aligned} \|p(x) - f(x)\| &\leq \sum_{k=1}^N |p_{1k}(x_1)p_{2k}(x_2) \cdots p_{nk}(x_n) \\ &\quad - f_{1k}(x_1)f_{2k}(x_2) \cdots f_{nk}(x_n)| \|\phi_k\| \\ &\leq \sum_{k=1}^N M_k(\|f_{1k}\|, \|f_{2k}\|, \dots, \|f_{nk}\|)\mathbf{h}_k \\ &\leq M[\mathbf{h}_1^\top \mathbf{h}_2^\top \cdots \mathbf{h}_n^\top]^\top. \end{aligned}$$

On the other hand, (10) is rewritten as $\dot{x} = f(x) + (p(x) - f(x))$. Therefore, it follows from Lemma 3 that

$$\begin{aligned} \|x(t, x_0) - x_p(t, x_0)\| &\leq \frac{\|p(x) - f(x)\|}{L}(e^{Lt} - 1) \\ &\leq \frac{M\mathbf{h}}{L}(e^{Lt} - 1) = R(t)\mathbf{h} \quad (21) \end{aligned}$$

holds for every $t \in [0, \infty)$ and $x_0 \in \mathbf{X}$.

REFERENCES

1. G. Ferrari-Trecate, M. Muselli, D. Liberati, and M. Morari: A clustering technique for the identification of piecewise affine systems, *Automatica*, Vol. 39, No. 2, pp. 205–217 (2003)
2. A. Bemporad and M. Morari: Control of systems integrating logic, dynamics, and constraints, *Automatica*, Vol. 35, No. 3, pp. 407–427 (1999)
3. C. Tomlin, I. Mitchell, and R. Ghosh: Safety verification of conflict resolution maneuvers, *IEEE Transactions on Intelligent Transportation Systems*, Vol. 2, No. 2, pp. 110–120 (2001)
4. R. Ghosh, A. Tiwari, and C. Tomlin: Automated symbolic reachability analysis; with application to Delta-Notch signaling automata, *Hybrid Systems: Computation and Control, LNCS 1790*, pp. 233–248 (2003)
5. S. Azuma and J. Imura: Polynomial-time probabilistic controllability analysis of discrete-time piecewise affine systems, *IEEE Transactions on Automatic Control*, Vol. 52, No. 11, pp. 2029–2046 (2007)
6. S. Azuma and J. Imura: Polynomial-time probabilistic observability analysis of sampled-data piecewise affine systems, *Systems & Control Letters*, Vol. 56, No. 11–12, pp. 685–694 (2007)
7. A. Bemporad, S. Di Cairano, E. Henriksson, and K. H. Johansson: Hybrid model predictive control based on wireless sensor feedback: an experimental study, *Proceedings of 46th IEEE Conference on Decision and Control*, pp. 5062–5067 (2006)
8. S. Azuma, E. Yanagisawa, and J. Imura: Controllability Analysis of Biosystems Based on Piecewise Affine Systems Approach, *IEEE Transactions on Automatic Control & IEEE Transactions on Circuits and Systems I: Regular Papers, Joint Special Issue on Systems Biology*, pp. 139–152 (2008)
9. E.D. Sontag: Nonlinear regulation: The piecewise linear approach, *IEEE Transactions on Automatic Control*, Vol. 26, No. 2, pp. 346–358 (1981)
10. T.A. Henzinger, P.-H. Ho, and H. Wong-Toi: Algorithmic analysis of nonlinear hybrid systems, *IEEE Transactions on Automatic Control*, Vol. 43, No. 4, pp. 540–554 (1998)
11. O. Stursberg and S. Kowalewski: Approximating Switched Continuous Systems by Rectangular Automata *Proceedings of European Control Conference 1999*, F1014-4 (1999)
12. E. Asarin, T. Dang, and A. Girard: Reachability analysis of nonlinear systems using conservative approximation, *Hybrid Systems: Computation and Control, LNCS 2623*, pp. 20–35 (2003)
13. A. Puri, V. Borkarand, and P. Varaiya: ϵ -Approximation of differential inclusions, *Hybrid Systems III: Verification and Control, LNCS 1066*, pp. 362–376 (1996)
14. A. Girard: Approximate solutions of ODEs using piece-

- wise linear vector fields, *Proceedings of the 5th International Workshop on Computer Algebra in Scientific Computing*, pp. 107–120 (2002)
15. K.J. Åström and B.M. Bernhardsson: Comparison of Riemann and Lebesgue sampling for first order stochastic systems, *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 2011–2016 (2002)
 16. A. Bemporad, G. Ferrari-Trecate, and M. Morari: Observability and controllability of piecewise affine and hybrid systems, *IEEE Transactions on Automatic Control*, Vol. 45, No. 10, pp. 1864–1876 (2000)
 17. J. Imura and A. J. Van der Schaft: Characterization of well-posedness of piecewise-linear systems, *IEEE Transactions on Automatic Control*, Vol. 45, No. 9, pp. 1600–1619 (2000)
 18. S. Azuma, J. Imura, and T. Sugie: Lebesgue piecewise affine approximation of nonlinear systems and its application to hybrid system modeling of biosystems, *Proceedings of 45th IEEE Conference on Decision and Control*, pp. 2128–2133 (2006)
 19. R. Tempo, E. W. Bai, and F. Dabbene: Probabilistic robustness analysis: Explicit bounds for the minimum number of samples, *Systems & Control Letters*, Vol. 30, No. 5, pp. 237–242 (1997)
 20. A. Mochizuki: An analytical study of the number of steady states in gene regulatory networks, *Journal of Theoretical Biology*, Vol. 236, No. 3, pp. 291–310 (2005)
 21. H.K. Khalil: Nonlinear Systems (Third Edition): *Prentice Hall, New Jersey* (2002)