

Graded algebras associated with indecomposable vector bundles over an elliptic curve

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§1. Introduction

Let X be an elliptic curve over an algebraically closed field k with $\text{char}(k) \neq 2$. Our object is to compute the graded algebra

$$\bigoplus_{i \geq 0} \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{L}^{\otimes i})$$

for a line bundle $\mathcal{L}$ and a vector bundle $\mathcal{E}$ over X defined as follows. Choose a point $P \in X$ and let $\mathcal{L} = \mathcal{L}(P)$ be the line bundle associated to the divisor P . Vector bundles over X were classified by Atiyah [1]. Among them we choose the following ones. For each positive integer n there exists uniquely an indecomposable vector bundle $\mathcal{E}_n$ of rank n which is a successive extension of the trivial bundle. That is,

$$\begin{aligned} \mathcal{O}_X &= \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \\ 0 \rightarrow \mathcal{E}_{n-1} &\hookrightarrow \mathcal{E}_n \rightarrow \mathcal{O}_X \rightarrow 0 \quad \text{exact, non split.} \end{aligned}$$

Now put

$$\Lambda(n) = \bigoplus_{i \geq 0} \Gamma(X, \text{End}(\mathcal{E}_n) \otimes \mathcal{L}^{\otimes i}) = \bigoplus_{i \geq 0} \text{Hom}(\mathcal{E}_n, \mathcal{E}_n \otimes \mathcal{L}^{\otimes i}).$$

We aim to give an explicit description of the algebra $\Lambda(n)$.

§2. Homogeneous coordinate ring

First of all, we look at the algebra

$$S = \bigoplus_{i \geq 0} \Gamma(X, \mathcal{L}^{\otimes i}).$$

We know the following presentation of S [2, p. 336].

$$\begin{aligned} \text{generators:} \quad & t \in S_1, \quad x \in S_2, \quad y \in S_3 \\ \text{relation:} \quad & y^2 = x(x - t^2)(x - \lambda t^2) \quad \text{with } \lambda \in k - \{0, 1\}. \end{aligned}$$

Also we have $S_0 = k$, $\dim S_i = i$ for $i > 0$ and a k -basis of S is given by $t^i x^j$, $t^i x^j y$ for $i, j \geq 0$. In addition, X is determined by λ as

$$X \cong \{x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)\} \subset \mathbb{P}^2$$

$$P \leftrightarrow (0 : 1 : 0)$$

We fix t, x, y, λ throughout.

§3. First properties of $\Lambda(n)$

We collect here some properties of $\Lambda(n)$ which are easily proved.

- The functor

$$\Gamma_* : \text{quasi-coherent } \mathcal{O}_X\text{-mod} \rightarrow \text{graded } S\text{-mod}$$

$$\mathcal{F} \mapsto \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes i})$$

is fully faithful, because $\mathcal{L}$ is ample. Hence we have an S -algebra isomorphism

$$\Lambda(n) \cong \text{End}_S(\Gamma_*(\mathcal{E}_n)).$$

We shall describe the S -module $\Gamma_*(\mathcal{E}_n)$ in §6.

- $\Lambda(n)$ is a maximal order in $\Lambda(n) \otimes_S \text{Frac}(S) \cong M_n(\text{Frac}(S))$.
- The degree 0 part $\Lambda(n)_0 = \text{End}(\mathcal{E}_n)$ is generated by a single endomorphism f defined by

$$f: \mathcal{E}_n \rightarrow \mathcal{E}_n / \mathcal{E}_1 \cong \mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n.$$

We have $f^n = 0$ and $\dim \Lambda(n)_0 = n$. We shall construct f explicitly in §7.

- The degree i part $\Lambda(n)_i$ has dimension $n^2 i$ for $i > 0$.

§4. Λ as an R -algebra

Write $\Lambda = \Lambda(n)$. Put $R = k[t, x]$, a polynomial subalgebra of S . Then $S = R \oplus Ry$. Λ is an R -free module of rank $2n^2$. We shall give an R -basis of Λ .

There exist $g \in \Lambda_1$, $h \in \Lambda_2$, $l \in \Lambda_3$ such that the following diagrams commute.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{g} & \mathcal{E} \otimes \mathcal{L} \\ \uparrow & & \downarrow \\ \mathcal{O} & \xrightarrow[t]{} & \mathcal{L} \end{array}$$

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{h} & \mathcal{E} \otimes \mathcal{L}^{\otimes 2} \\
\uparrow & & \downarrow \\
\mathcal{O} & \xrightarrow{s} & \mathcal{L}^{\otimes 2} \\
\mathcal{E} & \xrightarrow{l} & \mathcal{E} \otimes \mathcal{L}^{\otimes 3} \\
\uparrow & & \downarrow \\
\mathcal{O} & \xrightarrow{y} & \mathcal{L}^{\otimes 3}
\end{array}$$

Here the left vertical arrows are the inclusion map and the right ones are induced by the surjection $\mathcal{E} \rightarrow \mathcal{O}$. An explicit form of g will be given in §7. Then the following monomials form an R -basis of Λ .

$$f^i, f^i g f^j, f^i h f^j, f^i l \quad 0 \leq i \leq n-1, \quad 0 \leq j \leq n-2.$$

The quotient $\bar{\Lambda} = \Lambda / R_+ \Lambda = \Lambda / (t, x) \Lambda$ is a symmetric graded k -algebra of dimension $2n^2$. We have the following isomorphisms of bimodules over $\bar{\Lambda}_0 = \Lambda_0$.

$$\bar{\Lambda}_1 \cong \bar{\Lambda}_2 \cong \text{Ker}(\bar{\Lambda}_0 \otimes \bar{\Lambda}_0 \xrightarrow{\text{mult}} \bar{\Lambda}_0)$$

$$\bar{\Lambda}_3 \cong \bar{\Lambda}_0$$

$$\bar{\Lambda}_i = 0 \quad i > 3.$$

§5. Λ as a k -algebra

Let $n > 2$. Regard Λ as a left $\Lambda_0 \otimes \Lambda_0$ -module by $(a \otimes b) \cdot c = acb$.

PROPOSITION. $\Lambda_+ = \Lambda_1 \oplus \Lambda_2 \oplus \dots$ is a free $\Lambda_0 \otimes \Lambda_0$ -module with basis

$$(gf^{n-1})^i g, \quad (gf^{n-1})^i (gf^{n-2})^j gf^{n-3} g \quad \text{for } i, j \geq 0.$$

THEOREM. The k -algebra Λ is generated by f and g . The relations between them are generated by the following ones.

Case n is even: $f^n = 0$ and $n-2$ quadratic relations of the form

$$gf^k g = A_k \cdot gf^{n-3} g + B_k \cdot gf^{n-1} g$$

$$\text{with } A_k, B_k \in \Lambda_0 \otimes \Lambda_0 \text{ for } 0 \leq k \leq n-2, k \neq n-3$$

Case n is odd: $f^n = 0$ and $n-2$ quadratic relations as above and one cubic relation of the form

$$gf^{n-3} gf^{n-3} g = C \cdot gf^{n-2} gf^{n-3} g + D \cdot gf^{n-1} gf^{n-3} g + E \cdot gf^{n-1} gf^{n-1} g$$

$$\text{with } C, D, E \in \Lambda_0 \otimes \Lambda_0.$$

§6. S -module $\Gamma_*(\mathcal{E}_n)$

Put $v = x - (\lambda + 1)t^2$, $u = (x - t^2)(x - \lambda t^2)$. Define a graded S -module M as follows. M is R -free with basis $\alpha, \beta_i, \gamma_i$ for $i > 0$ with $\deg \alpha = 0$, $\deg \beta_i = 1$, $\deg \gamma_i = 2$. The action of y on M is given by

$$\begin{aligned} y\alpha &= x\beta_1 + t\gamma_1 \\ y\beta_i &= -\lambda t^3 O_i \beta_{i-1} - tx\beta_{i+1} + v\gamma_{i-1} - t^2 \gamma_{i+1} \\ y\gamma_i &= x^2 \beta_{i+1} + \lambda t^3 E_i \gamma_{i-1} + tx\gamma_{i+1} \end{aligned}$$

where $\beta_0 = -t\alpha$, $\gamma_0 = x\alpha$ and $O_i = 1$ for an odd i , $O_i = 0$ for an even i , $E_i = 1 - O_i$.

For $n \geq 1$ define a graded S -submodule $M(n)$ of M to be the free R -submodule generated by $\alpha, \beta_i, \gamma_i$ for $1 \leq i \leq n-1$ and $x\beta_n + t\gamma_n$.

PROPOSITION. $\Gamma_*(\mathcal{E}_n) \cong M(n)$ as graded S -modules.

So we may identify $\Lambda(n) = \text{End}_S(M(n))$.

Though the S -module M is not free, the $S[\frac{1}{y}]$ -module $M[\frac{1}{y}] = S[\frac{1}{y}] \otimes_S M$ is free with basis α_i , $i \geq 0$, given by

$$\begin{aligned} \alpha_i &= \frac{1}{x} \gamma_i & i: \text{ odd} \\ &= -\frac{1}{u} (\lambda t^3 \beta_i - v \gamma_i) & i: \text{ even.} \end{aligned}$$

§7. Generators

Let us construct $f, g \in \Lambda$ as endomorphisms of the S -module $M(n)$. Define an $S[\frac{1}{y}]$ -linear map $f: M[\frac{1}{y}] \rightarrow M[\frac{1}{y}]$ by

$$\begin{aligned} f(\alpha_i) &= \alpha_{i-1} - \frac{\lambda t^3 y}{ux} \alpha_{i-2} + \frac{((\lambda + 1)v + \lambda t^2)x}{u} \alpha_{i-3} \\ &\quad - \frac{\lambda ty}{u} \alpha_{i-4} + \frac{\lambda vx}{u} \alpha_{i-5} & \text{if } i \text{ is even} \\ f(\alpha_i) &= \alpha_{i-1} + \frac{\lambda t^3 y}{ux} \alpha_{i-2} \\ &\quad + \frac{(\lambda + 1)x - \lambda t^2}{x} \alpha_{i-3} + \frac{\lambda ty}{u} \alpha_{i-4} & \text{if } i \text{ is odd} \end{aligned}$$

where we understand $\alpha_i = 0$ for $i < 0$. Then

$$\begin{aligned}
 f(\alpha) &= 0 \\
 f(\beta_i) &= \beta_{i-1} + (\lambda + 1)\beta_{i-3} && i: \text{ even} \\
 &= \beta_{i-1} + (\lambda + 1)\beta_{i-3} + \lambda\beta_{i-5} && i: \text{ odd} \\
 f(\gamma_i) &= \gamma_{i-1} + (\lambda + 1)\gamma_{i-3} + \lambda\gamma_{i-5} - \lambda t\beta_{i-3} && i: \text{ even} \\
 &= \gamma_{i-1} + (\lambda + 1)\gamma_{i-3} + \lambda t\beta_{i-3} && i: \text{ odd}
 \end{aligned}$$

So M and $M(n)$ are stable under f . We denote also by f the restrictions of f to M and $M(n)$. Thus $f \in \Lambda(n)_0$ for all n .

Secondly, define an $S[\frac{1}{y}]$ -linear map $g: M[\frac{1}{y}] \rightarrow M(n)[\frac{1}{y}]$ as follows. When n is even,

$$\begin{aligned}
 g(\alpha_0) &= t\alpha_{n-1} - \frac{y}{x}\alpha_{n-2} \\
 g(\alpha_1) &= \frac{y}{x}\alpha_{n-1} + \frac{t((\lambda + 1)x - \lambda t^2)}{x}\alpha_{n-2} + \frac{\lambda t^2 y}{u}\alpha_{n-3} \\
 g(\alpha_2) &= -\frac{\lambda t^2 y}{u}\alpha_{n-2} + \frac{\lambda t v x}{u}\alpha_{n-3} \\
 g(\alpha_i) &= 0 \quad \text{for } i > 2,
 \end{aligned}$$

and when n is odd,

$$\begin{aligned}
 g(\alpha_0) &= t\alpha_{n-1} - \frac{vy}{u}\alpha_{n-2} \\
 g(\alpha_1) &= \frac{y}{x}\alpha_{n-1} + (\lambda + 1)t\alpha_{n-2} \\
 g(\alpha_2) &= -\frac{\lambda t^2 y}{u}\alpha_{n-2} + \sum_{i \geq 3, \text{ odd}} \lambda(-\lambda - 1)^{(i-3)/2} (t\alpha_{n-i} - \frac{vy}{u}\alpha_{n-i-1}) \\
 g(\alpha_i) &= 0 \quad \text{for } i > 2.
 \end{aligned}$$

Then it turns out that g maps M into $M(n)$. Its restriction $M(n) \rightarrow M(n)$ is denoted by g again. g increases degree by 1, so $g \in \Lambda_1$. These f, g are the desired generators.

§8. Explicit equations in case n even

When n is even, we can give explicit defining equations for Λ , using additional generators.

We define $e \in \Lambda_0$ and $g_+ \in \Lambda_1$ by

$$\begin{aligned} e(\alpha_i) &= \alpha_{i-2} \quad \text{for all } i \\ g_+(\alpha_0) &= t\alpha_{n-2} - \frac{vy}{u}\alpha_{n-3} \\ g_+(\alpha_1) &= t\alpha_{n-1} + (\lambda+1)t\alpha_{n-3} \\ g_+(\alpha_2) &= \frac{vy}{u}\alpha_{n-1} + (\lambda+1)t\alpha_{n-2} \\ g_+(\alpha_i) &= 0 \quad \text{for } i > 2. \end{aligned}$$

THEOREM. If n is even and $n > 2$, the k -algebra Λ has the following presentation. The generators are f, e, g, g_+ . The relations are

$$\begin{aligned} e^{\frac{n}{2}} &= 0 \\ f^2 &= (1 + (\lambda+1)e)(1 + \lambda e)(1 + e)e \\ fg(1 + (\lambda+1)e) &+ (1 + (\lambda+1)e)gf \\ &= g_+ + (\lambda+1)eg_+ + (\lambda+1)g_+e + \lambda e^2g_+ + ((\lambda+1)^2 + \lambda)eg_+e + \lambda g_+e^2 \\ &\quad + \lambda(\lambda+1)e^2g_+e + \lambda(\lambda+1)eg_+e^2 \\ ge^{\frac{n-4}{2}}g &= \lambda g_+e^{\frac{n-2}{2}}g_+ \\ g_+e^{\frac{n-4}{2}}g_+ &= (\lambda+1)g_+e^{\frac{n-2}{2}}g_+ \\ ge^jg &= ge^jg_+ = 0 \quad \text{for } 0 \leq j \leq \frac{n-6}{2}. \end{aligned}$$

Finally we give another presentation of Λ in line with the theorem of §5. Put

$$c = e \otimes 1, d = 1 \otimes e, p = f \otimes 1, q = 1 \otimes f \in \Lambda_0 \otimes \Lambda_0$$

and

$$\begin{aligned} \alpha &= (1 + (\lambda+1)c)(1 + (\lambda+1)d) - \lambda^2c^2d^2 \\ \gamma &= (\lambda+1)(1 + \lambda c)(1 + c)(1 + \lambda d)(1 + d) \\ &\quad + \lambda d(1 + \lambda c)(1 + c) + \lambda c(1 + \lambda d)(1 + d) \\ \beta &= (1 + \lambda cd)\alpha - (\lambda+1)cd\gamma \\ &= 1 + (\lambda+1)(c + d) + \lambda cd - (\lambda+1)^3(c^2d + cd^2) \\ &\quad - ((\lambda+1)^4 + \lambda(\lambda+1)^2 + \lambda^2)c^2d^2 - \lambda(\lambda+1)^2(c^3d + cd^3) \\ &\quad - \lambda(\lambda+1)((\lambda+1)^2 + \lambda)(c^3d^2 + c^2d^3) - \lambda^2((\lambda+1)^2 + \lambda)c^3d^3. \end{aligned}$$

Then $\alpha, \beta, \gamma \in \Lambda_0 \otimes \Lambda_0$ and β is invertible.

THEOREM. If n is even and $n > 2$, the k -algebra Λ has the following presentation. The generators are f, e, g . The relations are

$$e^{\frac{n}{2}} = 0$$

$$f^2 = (1 + (\lambda + 1)e)(1 + \lambda e)(1 + e)e$$

$$ge^{\frac{n-2}{2}}g = (\square_1 p + \square_2 q)ge^{\frac{n-4}{2}}fg + (\square_3 p + \square_4 q)ge^{\frac{n-2}{2}}fg$$

$$\square_1 = -\frac{1}{\beta}(1 + \lambda d)(1 + d)(1 + (\lambda + 1)d + \lambda cd)$$

$$\square_3 = \frac{1}{\beta}(1 + \lambda d)(1 + d)[(\lambda + 1)(1 + (\lambda + 1)d) + (\lambda + 1 + \frac{\lambda c}{(1 + \lambda c)(1 + c)})(1 + (\lambda + 1)d + \lambda cd)]$$

$$\square_1 \leftrightarrow \square_2, \quad \square_3 \leftrightarrow \square_4 \quad \text{by interchange } c \leftrightarrow d$$

$$ge^{\frac{n-4}{2}}g = (\square_1 p + \square_2 q)ge^{\frac{n-4}{2}}fg + (\square_3 p + \square_4 q)ge^{\frac{n-2}{2}}fg$$

$$\square_1 = -\frac{1}{\beta}d(1 + (\lambda + 1)d)(1 + (\lambda + 1)c + \lambda cd)$$

$$\square_3 = \frac{1}{\beta}(1 + (\lambda + 1)d)[(\lambda + 1)d(1 + (\lambda + 1)c + \lambda cd) + \frac{1 + (\lambda + 1)c}{(1 + \lambda c)(1 + c)}(1 + (\lambda + 1)d + \lambda cd)]$$

$$\square_1 \leftrightarrow \square_2, \quad \square_3 \leftrightarrow \square_4 \quad \text{by interchange } c \leftrightarrow d$$

$$ge^{\frac{n-k}{2}}g = 0 \quad \text{for } k > 4, \text{ even}$$

$$ge^{\frac{n-6}{2}}fg = (\square_1 + \square_2 pq)ge^{\frac{n-4}{2}}fg + (\square_3 + \square_4 pq)ge^{\frac{n-2}{2}}fg$$

$$\square_1 = \frac{1}{\beta}((\lambda + 1)\beta - \lambda \gamma cd)$$

$$\square_2 = -\frac{1}{\beta}\lambda(1 + \lambda cd)$$

$$\square_3 = \frac{1}{\beta}[\lambda(1 + \lambda cd)(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) - (\lambda + 1)^2\beta + \lambda(\lambda + 1)\gamma cd]$$

$$\square_4 = \frac{1}{\beta}\left(\frac{\lambda \gamma}{(1 + \lambda c)(1 + c)(1 + \lambda d)(1 + d)} + \lambda(\lambda + 1)(1 + \lambda cd)\right)$$

$$ge^{\frac{n-8}{2}}fg = (\square_1 + \square_2 pq)ge^{\frac{n-4}{2}}fg + (\square_3 + \square_4 pq)ge^{\frac{n-2}{2}}fg$$

$$\begin{aligned}
\Box_1 &= \frac{1}{\beta}(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) \\
&\quad \times (1 - (\lambda + 1)^2 cd - \lambda(\lambda + 1)(c^2 d + cd^2) - \lambda^2 c^2 d^2) \\
\Box_2 &= -\frac{1}{\beta}\lambda(\lambda + 1)cd \\
\Box_3 &= -\frac{1}{\beta}(\lambda + 1)(1 + (\lambda + 1)c)(1 + (\lambda + 1)d) \\
&\quad \times (1 - ((\lambda + 1)^2 + \lambda)cd - \lambda(\lambda + 1)(c^2 d + cd^2) - \lambda^2 c^2 d^2) \\
\Box_4 &= \frac{1}{\beta}\left(\frac{\lambda\alpha}{(1 + \lambda c)(1 + c)(1 + \lambda d)(1 + d)} + \lambda(\lambda + 1)^2 cd\right)
\end{aligned}$$

$$ge^{\frac{n-k}{2}}fg = 0 \quad \text{for } k > 8, \text{ even.}$$

References

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- [3] D. Tambara, Graded algebras of vector bundle maps over an elliptic curve, preprint.