

Bounds of minimal dilatation for pseudo-Anosovs and the magic 3-manifold

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1 Minimal dilatation of pseudo-Anosovs

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus g with n punctures, and let $\text{Mod}(\Sigma)$ be the mapping class group. Mapping classes $\phi \in \text{Mod}(\Sigma)$ are classified into 3 types, periodic, reducible, pseudo-Anosov. There exist two numerical invariants of pseudo-Anosov mapping classes. One is the entropy $\text{ent}(\phi)$ which is the logarithm of the dilatation $\lambda(\phi) > 1$. The other is the volume $\text{vol}(\phi)$ which is the hyperbolic volume of the mapping torus of ϕ

$$\mathbb{T}(\phi) = \Sigma \times [0, 1] / \sim,$$

where $\sim$ identifies $(x, 1)$ with $(f(x), 0)$ for any representative $f \in \phi$.

We denote by $\delta_{g,n}$, the minimal dilatation for pseudo-Anosov elements $\phi \in \text{Mod}(\Sigma_{g,n})$. We set $\delta_g = \delta_{g,0}$. A natural question arises.

Question 1.1. *What is the value of $\delta_{g,n}$? Find a pseudo-Anosov element of $\text{Mod}(\Sigma_{g,n})$ whose dilatation is equal to $\delta_{g,n}$.*

The above question is hard in general. For instance, in the case of closed surfaces, it is open to determine the values δ_g for $g \geq 3$. On the other hand, one understands the asymptotic behavior of the minimal entropy $\log \delta_g$. Penner proved that $\log \delta_g \asymp \frac{1}{g}$ [12]. The following question posed by McMullen.

Question 1.2 ([11]). *Does $\lim_{g \rightarrow \infty} g \log \delta_g$ exist? What is its value?*

Note that $\lim_{g \rightarrow \infty} g \log \delta_g$ exists if and only if $\lim_{g \rightarrow \infty} |\chi(\Sigma_g)| \log \delta_g$ exists, where $\chi(\Sigma)$ is the Euler characteristic of Σ .

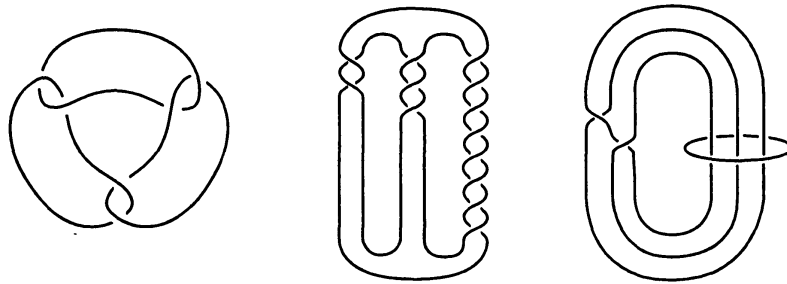


Figure 1: (left) 3 chain link $\mathcal{C}_3$. (center) $(-2, 3, 8)$ -pretzel link or Whitehead sister link. (right) link 6_2^2 . (N equals the exterior of $\mathcal{C}_3$. $N(\frac{-3}{2})$ is homeomorphic to the $(-2, 3, 8)$ -pretzel link exterior. $N(\frac{-1}{2})$ is homeomorphic to the 6_2^2 link exterior.)

Related questions on the minimal dilatation are ones for orientable pseudo-Anosovs. A pseudo-Anosov mapping class ϕ is said to be *orientable* if the invariant (un)stable foliation for a pseudo-Anosov homeomorphism $\Phi \in \phi$ is orientable. We denote by δ_g^+ , the minimal dilatation for orientable pseudo-Anosov elements of $\text{Mod}(\Sigma_g)$ for a closed surface Σ_g of genus g .

In this paper, we report our results in [7, 8] on the minimal dilatation by investigating the so called *magic manifold* N which is the exterior of the 3 chain link $\mathcal{C}_3$, see Figure 1(left). In Section 2, we describe a motivation for the study of pseudo-Anosovs which occur as the monodromies on fibers for Dehn fillings of N . In Section 3, we state our results.

We would like to note that this paper only contains some results in [7, 8] and does not contain their proofs. The readers who are interested in the details should consult (the introduction of) [7, 8].

2 Why is the magic manifold an intriguing example?

Gordon and Wu named the exterior of the link $\mathcal{C}_3$ the *magic manifold* N , see [3]. The reason why this manifold is called “magic” is that many important examples for the study of the exceptional Dehn fillings can be obtained from the Dehn fillings of a single manifold N . The magic

manifold is fibered and it has the smallest known volume among orientable hyperbolic 3-manifolds having 3 cusps. Many manifolds having at most 2 cusps with small volume are obtained from N by Dehn fillings, see [10].

2.1 Entropy versus volume

Both invariants entropy $\text{ent}(\phi)$ and volume $\text{vol}(\phi)$ know some complexity of pseudo-Anosovs ϕ . A natural question is how these are related.

Theorem 2.1 ([6]). *There exists a constant $B = B(\Sigma)$ depending only on the topology of Σ such that the inequality,*

$$B \text{vol}(\phi) \leq \text{ent}(\phi)$$

holds for any pseudo-Anosov ϕ on Σ . Furthermore, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \Sigma) > 1$ depending only on ε and the topology of Σ such that the inequality

$$\text{ent}(\phi) \leq C \text{vol}(\phi)$$

holds for any pseudo-Anosov ϕ on Σ whose mapping torus $\mathbb{T}(\phi)$ has no closed geodesics of length $< \varepsilon$.

The first part of Theorem 2.1 says that if the entropy is small, the volume can not be large.

For a non-negative integer c , we set

$$\begin{aligned} \lambda(\Sigma; c) &= \min\{\lambda(\phi) \mid \phi \in \text{Mod}(\Sigma), \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}, \\ \text{vol}(\Sigma; c) &= \min\{\text{vol}(\phi) \mid \phi \in \text{Mod}(\Sigma), \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}. \end{aligned}$$

A variation on the questions of the minimal dilatations is to determine $\lambda(\Sigma; c)$ and to find a mapping class realizing the minimum. In [6], the authors and S. Kojima obtain experimental results concerning the minimal dilatation. In the case the mapping class group $\text{Mod}(D_n)$ of an n -punctured disk D_n , they observe that for many pairs (n, c) , there exists a pseudo-Anosov element simultaneously reaching both minima $\lambda(D_n; c)$ and $\text{vol}(D_n; c)$. Experiments tell us that in case $c = 3$, the mapping tori reaching both minima are homeomorphic to N . Moreover when $c = 2$, it is observed that there exists a mapping class ϕ

realizing both $\lambda(D_n; 2)$ and $\text{vol}(D_n; 2)$ and its mapping torus $\mathbb{T}(\phi)$ is homeomorphic to a Dehn filling of N along one cusp. This is a motivation for us for focusing on N .

2.2 Small dilatation pseudo-Anosovs

After we have finished our papers [6, 7], we learned the small dilatation pseudo-Anosovs, introduced by Farb, Leininger and Margalit.

For any number $P > 1$, define the set of pseudo-Anosov homeomorphisms

$$\Psi_P = \{\text{pseudo-Anosov } \Phi : \Sigma \rightarrow \Sigma \mid \chi(\Sigma) < 0, |\chi(\Sigma)| \log \lambda(\Phi) \leq \log P\}.$$

Elements $\Phi \in \Psi_P$ are called *small dilatation pseudo-Anosov homeomorphisms*. If one takes P sufficiently large (e.g. $P \geq 2 + \sqrt{3}$), then Ψ_P contains a pseudo-Anosov homeomorphism $\Phi_g : \Sigma_g \rightarrow \Sigma_g$ for each $g \geq 2$. By a result in [5], Ψ_P also contains pseudo-Anosov homeomorphism $\Phi_n : D_n \rightarrow D_n$ for each $n \geq 3$. Let $\Sigma^\circ \subset \Sigma$ be the surface obtained by removing the singularities of the (un)stable foliation for Φ and $\Phi|_{\Sigma^\circ} : \Sigma^\circ \rightarrow \Sigma^\circ$ denotes the restriction. Observe that $\lambda(\Phi) = \lambda(\Phi|_{\Sigma^\circ})$. The set

$$\Psi_P^\circ = \{\Phi|_{\Sigma^\circ} : \Sigma^\circ \rightarrow \Sigma^\circ \mid (\Phi : \Sigma \rightarrow \Sigma) \in \Psi_P\}$$

is infinite. Let $\mathcal{T}(\Psi_P^\circ)$ be the set of homeomorphism classes of mapping tori by elements of Ψ_P° .

Theorem 2.2 ([2]). *The set $\mathcal{T}(\Psi_P^\circ)$ is finite. Namely, for each $P > 1$, there exist finite many complete, non compact hyperbolic 3-manifolds $M_1, M_2, \dots, M_r$ fibering over S^1 so that the following holds. Any pseudo-Anosov $\Phi \in \Psi_P$ occurs as the monodromy of a Dehn filling of one of the M_k . In particular, there exists a constant $V = V(P)$ such that $\text{vol}(\Phi) \leq V$ holds for any $\Phi \in \Psi_P$.*

It is not known that how large the set of manifolds $\{M_1, \dots, M_r\}$ is. By Theorem 2.2, one sees that the following set $\mathcal{V}$ is finite.

$$\mathcal{V} = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid n \geq 3, \Phi \text{ is pseudo-Anosov on } \Sigma = D_n, \lambda(\Phi) = \delta(D_n)\},$$

where $\delta(D_n)$ denotes the minimal dilatation for pseudo-Anosov elements of $\text{Mod}(D_n)$ on D_n .

In [7], we show that for each $n \geq 9$ (resp. $n = 3, 4, 5, 7, 8$), there exists a pseudo-Anosov homeomorphism $\Phi_n : D_n \rightarrow D_n$ with the smallest known entropy (resp. the smallest entropy) which occurs as the monodromy on an n -punctured disk fiber for the Dehn filling of N . A pseudo-Anosov homeomorphism $\Phi_6 : D_6 \rightarrow D_6$ with the smallest entropy occurs as the monodromy on a 6-punctured disk fiber for N . In particular, $N \in \mathcal{V}$. See also work of Venzke [13]. This result suggests that one may have a chance to find pseudo-Anosov homeomorphisms with small dilatation on other surfaces which arise as the monodromies on fibers for Dehn fillings of N . This is another motivation for us.

3 Results

Let us introduce the following polynomial

$$f_{(k,\ell)}(t) = t^{2k} - t^{k+\ell} - t^k - t^{k-\ell} + 1 \text{ for } k > 0, -k < \ell < k,$$

having a unique real root $\lambda_{(k,\ell)}$ greater than 1 [8]. For the rational number r , let $N(r)$ be the Dehn filling of N along the slope r .

Theorem 3.1. *Let $r \in \{-\frac{3}{2}, -\frac{1}{2}, 2\}$. For each $g \geq 3$, there exists a monodromy $\Phi_g = \Phi_g(r)$ on a closed fiber of genus g for a Dehn filling of $N(r)$, where the filling is on the boundary slope of a fiber of $N(r)$, such that*

$$\lim_{g \rightarrow \infty} g \log \lambda(\Phi_g) = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

In particular

$$\lim_{g \rightarrow \infty} \sup g \log \delta_g \leq \log\left(\frac{3+\sqrt{5}}{2}\right).$$

Remark 3.2. *Independently, Hironaka has obtained Theorem 3.1 in case $r = -\frac{1}{2}$ [4]. Independently, Aaber and Dunfield have obtained Theorem 3.1 in case $r = -\frac{3}{2}$ [1]. They have obtained similar results on the dilatation to those given in [8].*

By using monodromies on closed fibers coming from $N(-\frac{3}{2})$, we find an upper bound of δ_g for $g \equiv 0, 1, 5, 6, 7, 9 \pmod{10}$ and $g \geq 5$.

Theorem 3.3. (1) $\delta_g \leq \lambda_{(g+2,1)}$ if $g \equiv 0, 1, 5, 6 \pmod{10}$ and $g \geq 5$.

(2) $\delta_g \leq \lambda_{(g+2,2)}$ if $g \equiv 7, 9 \pmod{10}$ and $g \geq 7$.

For more details of an upper bound of δ_g for other g (e.g. $g \equiv 2, 4 \pmod{10}$), see [8]. The bound in Theorem 3.3 improves the one by Hironaka [4].

We turn to the study on the minimal dilatations δ_g^+ for orientable pseudo-Anosovs. The minima δ_g^+ were determined for $g = 2$ by Zhurov [14], for $3 \leq g \leq 5$ by Lanneau-Thiffeault [9], and for $g = 8$ by Lanneau-Thiffeault and Hironaka [9, 4]. Those values are given by $\delta_2^+ = \lambda_{(2,1)}$, $\delta_3^+ = \lambda_{(3,1)} = \lambda_{(4,3)} \approx 1.40127$, $\delta_4^+ = \lambda_{(4,1)} \approx 1.28064$, $\delta_5^+ = \lambda_{(6,1)} = \lambda_{(7,4)} \approx 1.17628$ and $\delta_8^+ = \lambda_{(8,1)} \approx 1.12876$.

We recall the lower bounds of δ_6^+ and δ_7^+ and the question on δ_g^+ for g even by Lanneau-Thiffeault.

Theorem 3.4 ([9]).

(1) $\delta_6^+ \geq \lambda_{(6,1)} \approx 1.17628$.

(2) $\delta_7^+ \geq \lambda_{(9,2)} \approx 1.11548$.

Question 3.5 ([9]). *For g even, is δ_g^+ equal to $\lambda_{(g,1)}$?*

We give an upper bound of δ_g^+ in case $g \equiv 1, 5, 7, 9 \pmod{10}$ using orientable pseudo-Anosov monodromies coming from $N(\frac{-3}{2})$.

Theorem 3.6. (1) $\delta_g^+ \leq \lambda_{(g+2,2)}$ if $g \equiv 7, 9 \pmod{10}$ and $g \geq 7$.

(2) $\delta_g^+ \leq \lambda_{(g+2,4)}$ if $g \equiv 1, 5 \pmod{10}$ and $g \geq 5$.

The bound in Theorem 3.6 improves the one by Hironaka [4]. Theorem 3.6(1) together with Theorem 3.4(2) gives:

Corollary 3.7. $\delta_7^+ = \lambda_{(9,2)}$.

Independently, Corollary 3.7 was established by Aaber and Dunfield [1].

The following tells us that the sequence $\{\delta_g^+\}_{g \geq 2}$ is not monotone decreasing.

Proposition 3.8. *If Question 3.5 is true, then $\delta_g^+ < \delta_{g+1}^+$ whenever $g \equiv 1, 5, 7, 9 \pmod{10}$ and $g \geq 7$. In particular the inequality $\delta_7^+ < \delta_8^+$ holds.*

Our pseudo-Anosov homeomorphisms providing the upper bound of δ_g in Theorem 3.3(1) are not orientable. This together with the inequality $\lambda_{(7,1)} < \lambda_{(6,1)} = \delta_5^+$ implies:

Corollary 3.9. $\delta_5 < \delta_5^+$.

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