

Asymptotic behaviors of radially symmetric solutions of $\square u = |u|^p$ for super critical values p in high dimensions

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1. Introduction

We study asymptotic behaviors as $t \rightarrow \pm\infty$ of radially symmetric solutions of the nonlinear wave equation

$$(1.1) \quad u_{tt} - \Delta u = F(u) \quad \text{in } x \in \mathbb{R}^n, t \in \mathbb{R},$$

where $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$ with $p > 1$ and $n \geq 2$.

Let $p_0(n)$ be the positive root of the quadratic equation in p :

$$(1.2) \quad \Phi(n, p) \equiv \frac{n-1}{2}p^2 - \frac{n+1}{2}p - 1 = 0.$$

Note that $p_0(n)$ is strictly decreasing with respect to n and $p_0(4) = 2$. If $1 < p < p_0(n)$, it is known that the Cauchy problem for (1.1) with initial data prescribed on $t = 0$ does not admit global (in time) solutions, provided the initial data are chosen appropriately, even if they are sufficiently small. (See [6], [8] and [19]). The same is true for $p = p_0(n)$ if $n = 2$ or $n = 3$. (See [18]).

On the other hand, the case where $p > p_0(n)$ seems to be more complicated. When $2 \leq n \leq 4$, it is known that the problem admits a global solution for small initial data. (See [7], [8] and [24]). When $n \geq 5$, for $p \geq (n+3)/(n-1)$ a global weak solution of the problem obtained by [13] and [20]. (See also [3], [4], [11] and [12]). Recently, the case where p is between $p_0(n)$ and $(n+3)/(n-1)$ is treated by [5] and [14], independently.

Moreover, when $p > p_0(n)$ and either $n = 2$ or $n = 3$, it has been shown that the scattering operator for (1.1) exists on a dense set of a neighborhood of 0 in the energy space. (See [10],

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[17] and [23]). Namely, let $u_-(x, t)$ be the solution of the homogeneous wave equation

$$(1.3) \quad u_{tt} - \Delta u = 0 \quad \text{in } x \in \mathbb{R}^n, t \in \mathbb{R},$$

with small initial data

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \quad \text{for } x \in \mathbb{R}^n.$$

Then there exists a solution $u(x, t)$ of (1.1) such that $\|u(t) - u_-(t)\|_e \rightarrow 0$ as $t \rightarrow -\infty$, where

$$(1.4) \quad \|v(t)\|_e = \left\{ \int_{\mathbb{R}^n} (|\nabla v(x, t)|^2 + |v_t(x, t)|^2) dx \right\}^{1/2},$$

and there exists another solution $u_+(x, t)$ of (1.3) such that $\|u(t) - u_+(t)\|_e \rightarrow 0$ as $t \rightarrow \infty$. The analogous results have been obtained also for the high dimensional case, provided $p > p_1(n)$, where $p_1(n)$ is the largest root of the quadratic equation in p :

$$(n^2 - n)p^2 - (n^2 + 3n - 2)p + 2 = 0.$$

(See [13], [15], [16], and [20]). However here is a gap between $p_0(n)$ and $p_1(n)$. Indeed, since the left-hand-side of the above quadratic equation is rewritten as

$$2\{n\Phi(n, p) - 2(1 + \Phi(n, p))/p\},$$

it is easy to see that $p_0(n) < p_1(n)$.

The purpose of this note is to search the asymptotic behaviors of radially symmetric solutions of (1.1), which guarantee the existence of the scattering operator, for $p > p_0(n)$ in high dimensions $n \geq 5$.

2. Statements of main results

Throughout this section, we assume $n \geq 5$ (unless stated otherwise). First we shall consider the Cauchy problem for the homogeneous wave equation:

$$(2.1)_0 \quad u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0 \quad \text{in } \Omega,$$

$$(2.1)_1 \quad u(r, 0) = f(r), \quad u_t(r, 0) = g(r) \quad \text{for } r > 0,$$

where $\Omega = \{(r, t) \in \mathbb{R}^2; r > 0\}$ and $u(r, t)$ a real valued function. Then we have

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Theorem 1. Assume $f \in C^2([0, \infty))$ and $g \in C^1([0, \infty))$ satisfy

$$(2.2) \quad |f(r)|\langle r \rangle^{-1} + \sum_{j=0}^1 (|f^{(j+1)}| + |g^{(j)}(r)|) \leq \varepsilon \langle r \rangle^{-\kappa-(n+1)/2} \quad \text{for } r > 0,$$

where ε and κ are positive numbers and $\langle r \rangle = \sqrt{1+r^2}$. Here if n is even number, we further assume $\kappa < (n-1)/2$. Then (2.1) admits uniquely a weak solution $u(r, t) \in C^1(\Omega)$ such that for $(r, t) \in \Omega$ and $|\alpha| \leq 1$ we have

$$(2.3) \quad |D_{r,t}^\alpha u(r, t)| \leq C\varepsilon r^{1-m-|\alpha|} \langle r \rangle^{-1+|\alpha|} \Psi(r, |t|),$$

where we have set $m = [(n-2)/2]$ and

$$\Psi(r, t) = \langle r + |t| \rangle^{-\chi(n)} \langle r - t \rangle^{-\kappa}$$

with

$$\chi(n) = \begin{cases} 1/2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

and C is a constant depending only on m and κ .

Next we shall consider the nonlinear wave equation

$$(2.4) \quad u_{tt} - u_{rr} - \frac{n-1}{r} u_r = F(u) \quad \text{in } \Omega,$$

where $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$. Here we assume

$$(2.5) \quad p_0(n) < p < (n+3)/(n-1).$$

We shall introduce a function space X , in which we will look for solutions of (2.4), defined by

$$X = \{u(r, t) \in C^0(\Omega) : D_r u(r, t) \in C^0(\Omega), \|u\| < \infty\},$$

and

$$\|u\| = \sup_{(r,t) \in \Omega} \{(|u(r, t)|r^{m-1}\langle r \rangle + |D_r u(r, t)|r^m)\Psi^{-1}(r, |t|)\},$$

where Ψ is the same function as in (2.3). As for the parameter κ , we assume

$$(2.6) \quad \frac{1}{2} < \kappa \quad \text{and} \quad \frac{p+1}{p-1} - \frac{n+1}{2} < \kappa \leq q,$$

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where we have set

$$q = (1 + \Phi(n, p))/p = \frac{n-1}{2}p - \frac{n+1}{2}$$

with $\Phi(n, p)$ in (1.2). Note that there exist really numbers κ satisfying (2.6) for $p > p_0(n)$, because

$$\Phi(n, p) = (p-1)\left\{q - \left(\frac{p+1}{p-1} - \frac{n+1}{2}\right)\right\} > 0 \quad \text{for } p > p_0(n).$$

We are now in a position to state the main theorem in this note. Let $u_-(r, t)$ be the solution of (2.1) which is obtained in Theorem 1. Note that $u_- \in X$ and

$$(2.7) \quad \|u_-\| \leq C\varepsilon \quad \text{for any } \varepsilon > 0.$$

Then we have

Theorem 2. (Main theorem). *Assume conditions (2.2), (2.5) and (2.6) hold. Then there is positive constant ε_0 (depending only on p , n and κ) such that, if $0 < \varepsilon \leq \varepsilon_0$, there exists uniquely a weak solution $u(r, t)$ of the nonlinear wave equation (2.4) such that $u \in C^1(\Omega) \cap X$,*

$$(2.7) \quad \|u\| \leq 2\|u_-\|$$

and for $(r, t) \in \Omega$ and $|\alpha| \leq 1$ we have

$$(2.8)_- \quad |D_{r,t}^\alpha(u(r, t) - u_-(r, t))| \leq C\|u\|^p r^{1-m-|\alpha|} \langle r \rangle^{-1+|\alpha|} \Psi(r, t)$$

and

$$(2.9)_- \quad \|u(t) - u_-(t)\|_e \leq C\|u\|^p \langle t \rangle^{-\theta} \quad \text{if } t \leq 0,$$

where $\|\cdot\|$ is defined by (1.4) and we have set

$$\theta = \min\{q, \chi(n)p + p\kappa - 1\},$$

and C is a constant depending only on p , n and κ .

Moreover there exists uniquely a weak solution $u_+(r, t)$ of $(2.1)_0$ which belongs to $C^1(\Omega) \cap X$, such that for $(r, t) \in \Omega$ and $|\alpha| \leq 1$ we have

$$(2.8)_+ \quad |D_{r,t}^\alpha(u(r, t) - u_+(r, t))| \leq C\|u\|^p r^{1-m-|\alpha|} \langle r \rangle^{-1+|\alpha|} \Psi(r, -t)$$

and

$$(2.9)_+ \quad \|u(t) - u_+(t)\|_e \leq C\|u\|^p \langle t \rangle^{-\theta} \quad \text{if } t \geq 0.$$

Remarks. 1) If n is odd, in Theorems 1 and 2, one can replace $u \in C^1(\Omega)$ by $u \in C^2(\Omega)$. Moreover in (2.6) we can replace $\kappa > 1/2$ by $\kappa > 0$. In this case, we interpret $(2.9)_\pm$ as follows. When $\kappa > 1/2p$, $(2.9)_\pm$ is still valid. When $0 < \kappa \leq 1/2p$, it holds with $\theta = \kappa$. (See [9]).

2) For $n \geq 2$, consider the following Cauchy problem

$$(2.10) \quad \begin{cases} u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u) & \text{in } r > 0, t > 0, \\ u(r, 0) = 0, \quad u_t(r, 0) = g(r) & \text{for } r > 0. \end{cases}$$

It is known that, if $g(r) \geq Mr^{-\mu}$ for $r \geq 1$ with some positive constants M, μ and $\mu < (p+1)/(p-1)$, then (2.10) does not admit global solutions. (See [1], [2], [21] and [22]). Therefore condition (2.6) is partially necessary to obtain Theorem 2.

3) One can also show that the Cauchy problem for the nonlinear wave equation (2.4) admits a unique global solution, provided the hypotheses of Theorems 1 and 2 are fulfilled.

In the proof of Theorem 1, the following lemma plays a key role. Moreover Theorem 2 is obtained by considering the associated integral equation with the differential equation (2.4). So the lemma below is very essential in our work.

Lemma 3. Let $g \in C^0((0, \infty))$ and

$$g(r) = O(r^{-m-1}) \quad \text{as } r \downarrow 0.$$

For $r > 0$ and $t \geq 0$ we define a function $\Theta(g)$ as follows.

(1) n is odd : $n = 2m + 3$ ($m = 1, 2, \dots$).

$$\Theta(g)(r, t) = \int_{|t-r|}^{t+r} g(\lambda) K(\lambda, r, t) d\lambda,$$

where we have set

$$K(\lambda, r, t) = r^{2-n} \lambda^{2m+1} H_m(\lambda, r, t),$$

$$H_m(\lambda, r, t) = \left(\frac{\partial}{\partial \lambda} \frac{-1}{2\lambda} \right)^m (r^2 - (\lambda - t)^2)^{(n-3)/2}.$$

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(2) n is even : $n = 2m + 2$ ($m = 1, 2, \dots$).

$$\Theta(g)(r, t) = \int_{|t-r|}^{t+r} g(\lambda) K_1(\lambda, r, t) d\lambda + \int_0^{\max(t-r, 0)} g(\lambda) K_2(\lambda, r, t) d\lambda,$$

where we have set

$$K_1(\lambda, r, t) = r^{2-n} \lambda^{2m+1} \int_{\lambda}^{t+r} \frac{H_m(\rho, r, t)}{\sqrt{\rho^2 - \lambda^2}} d\rho,$$

$$K_2(\lambda, r, t) = r^{2-n} \lambda^{2m+1} \int_{t-r}^{t+r} \frac{H_m(\rho, r, t)}{\sqrt{\rho^2 - \lambda^2}} d\rho,$$

and

$$H_m(\rho, r, t) = \left(\frac{\partial}{\partial \rho} \frac{-1}{2\rho} \right)^m (r^2 - (\rho - t)^2)^{(n-3)/2}.$$

And we extend $\Theta(g)(r, t)$ as an odd function with respect to t . Then $\Theta(g) \in C^0(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

$$|\Theta(g)(r, t)| \leq C_B r^{-m} \quad \text{for } (r, t) \in B.$$

Moreover, if we set $u(x, t) = \Theta(g)(|x|, t)$, then $u(\cdot, t) \in C^0(\mathbb{R}; L_{loc}^2(\mathbb{R}^n))$ and u is a weak solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = 0, \quad u_t(x, 0) = c_n g(|x|) & \text{for } x \in \mathbb{R}^n \end{cases}$$

in the sense of distribution, where

$$c_n = \begin{cases} 2 \Gamma(\frac{n-1}{2}) & \text{if } n \text{ is odd,} \\ \sqrt{\pi} \Gamma(\frac{n-1}{2}) & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, if $g \in C^1((0, \infty))$ and for $j = 0, 1$

$$g^{(j)}(r) = O(r^{-m-j}) \quad \text{as } r \downarrow 0,$$

then $\Theta(g) \in C^1(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

$$|D_{r,t}^\alpha \Theta(g)(r, t)| \leq C_B r^{1-m-|\alpha|} \quad \text{for } (r, t) \in B \text{ and } |\alpha| \leq 1.$$

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