

The Solutions to The Radial Schrödinger Equation of The Hydrogen Atom by Means of N- Fractional Calculus Operator

Katsuyuki Nishimoto

Institute for Applied Mathematics, Descartes Press Co.
2-13-10 Kaguike, Koriyama 963-8833, Japan

Abstract

In this article, the solutions to the radial Schrödinger equation of the Hydrogen atom (in the Coulomb field)

$$\varphi_2 \cdot x^2 + \varphi_1 \cdot 2x + \varphi \cdot \{-(1/4)x^2 + vx - l(l+1)\} = 0$$

are discussed by means of N-fractional calculus operator.

A particular solution to the equation above is shown as follows for example.

$$\varphi = \varphi_{[1]} = x^l e^{x/2} (e^{-x} \cdot x^{v-(l+1)})_{l+v}$$

(fractional differintegrated form)

$$= (e^{i\pi})^{l+v} e^{-x/2} x^{v-1} {}_2F_0(-l-v, l+1-v; -1/x)$$

$$(|-1/x| < 1)$$

where

${}_pF_q(\cdots)$; Generalized Gauss hypergeometric functions.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i \operatorname{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i \operatorname{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C .)

Moreover, let $f = f(z)$ be a regular function in D ($z \in D$),

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}^-), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

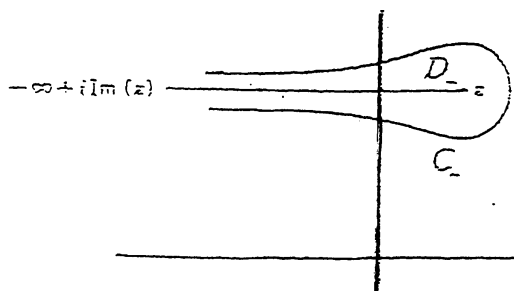


Fig. 1.

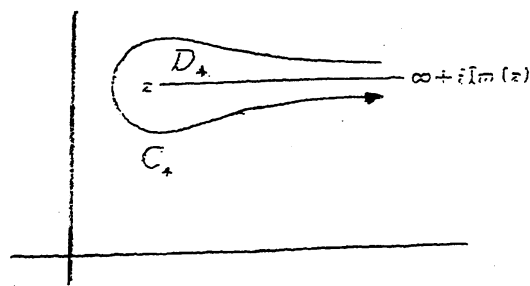


Fig. 2.

Notice that (1) is reduced to Goursat's integral for $\nu = n$ ($\in \mathbb{Z}^+$) and is reduced to the famous Cauchy's integral for $\nu = 0$. That is, (1) is an extension of Cauchy's integral and of Goursat's one, conversely Cauchy's and Goursat's ones are special cases of (1).

Moreover, notice that (1) is the representation which unifies the derivatives and integrations.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}^-) \quad [\text{Refer to (1)}] \quad (3)$$

$$\text{with} \quad N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that

$$f \in F = \{f | 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}, \text{ where } f = f(z) \text{ and } z \in \mathbb{C}. (\text{viz. } -\infty < \nu < \infty).$$

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. The "F.O.G. $\{N^\nu\}$ " is an "Action product group which has continuous index ν " for the set F . (F.O.G.; Fractional calculus operator group) [3]

Theorem C. Let be

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) **Lemma.** We have [1]

$$(i) \quad ((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii).

$$(i.v) \quad (u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

§ 1. Preliminary

(I) The theorem below is reported by the author already (cf. J.F.C.Vol. 27, May (2005),83 - 88.) [31]

Theorem D. *Let be*

$$P = P(\alpha, \beta, \gamma) := \frac{\sin \pi \alpha \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \alpha)} \quad (|P(\alpha, \beta, \gamma)| = M < \infty) \quad (1)$$

and

$$Q = Q(\alpha, \beta, \gamma) := P(\beta, \alpha, \gamma), \quad (|P(\beta, \alpha, \gamma)| = M < \infty) \quad (2)$$

When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^+$, we have

$$(i) \quad ((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = e^{-i\pi\gamma} P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha+\beta-\gamma}, \quad (3)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, \quad (1 + \alpha - \gamma) \notin \mathbb{Z}_0^-),$$

$$(ii) \quad ((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = e^{-i\pi\gamma} Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha+\beta-\gamma}, \quad (4)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, \quad (1 + \beta - \gamma) \notin \mathbb{Z}_0^-),$$

$$(iii) \quad ((z-c)^{\alpha+\beta})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha+\beta-\gamma}, \quad (5)$$

where

$$z-c \neq 0, \quad \left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right| < \infty.$$

Then the inequalities below are established from this theorem.

Corollary 1. *We have the inequalities*

$$(i) \quad ((z-c)^\alpha \cdot (z-c)^\beta)_\gamma \neq ((z-c)^\beta \cdot (z-c)^\alpha)_\gamma, \quad (6)$$

and

$$(ii) \quad ((z-c)^\alpha \cdot (z-c)^\beta)_\gamma \neq ((z-c)^{\alpha+\beta})_\gamma, \quad (7)$$

where

$$\alpha, \beta, \gamma \notin \mathbb{Z}_0^+, \quad \alpha \neq \beta, \quad z-c \neq 0.$$

Corollary 2.

$$(i) \quad \text{When } \alpha, \beta, \gamma \notin \mathbb{Z}_0^+, \text{ and } P(\alpha, \beta, \gamma) = Q(\beta, \alpha, \gamma) = 1, \quad (8)$$

we have

$$((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = ((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = ((z-c)^{\alpha+\beta})_\gamma, \quad (9)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, \quad (1 + \alpha - \gamma) \notin \mathbb{Z}_0^-, \quad (1 + \beta - \gamma) \notin \mathbb{Z}_0^-),$$

(ii) When $\gamma = m \in \mathbb{Z}_0^+$, we have ;

$$((z-c)^\alpha \cdot (z-c)^\beta)_m = ((z-c)^\beta \cdot (z-c)^\alpha)_m = ((z-c)^{\alpha+\beta})_m. \quad (10)$$

§ 2. The Solutions to The Radial Schrödinger Equation of The Hydrogen Atom

Theorem 1. *Let be $\varphi = \varphi(x) \in F$, then the homogeneous linear ordinary differential equation (Radial Schrödinger equation of the Hydrogen atom in the Coulomb field)*

$$L[\varphi; x; \nu, l] = \varphi_2 \cdot x^2 + \varphi_1 \cdot 2x + \varphi \cdot \{-(1/4)x^2 + \nu x - l(l+1)\} = 0 \quad (1)$$

$$(\varphi_\alpha = d^\alpha \varphi / dx^\alpha \text{ for } \alpha > 0, \varphi_0 = \varphi, x \neq 0)$$

has particular solutions of the forms (fractional differintegrated forms);

Group I;

$$1) \quad \varphi = x^l e^{x/2} (e^{-x} \cdot x^{\nu-(l+1)})_{l+\nu} \equiv \varphi_{[1]}, \quad (\text{denote}) \quad (2)$$

$$2) \quad \varphi = x^l e^{x/2} (x^{\nu-(l+1)} \cdot e^{-x})_{l+\nu} \equiv \varphi_{[2]} \quad (3)$$

$$3) \quad \varphi = x^l e^{-x/2} (e^x \cdot x^{-\nu-(l+1)})_{l-\nu} \equiv \varphi_{[3]} \quad (4)$$

$$4) \quad \varphi = x^l e^{-x/2} (x^{-\nu-(l+1)} \cdot e^x)_{l-\nu} \equiv \varphi_{[4]} \quad (5)$$

Group II;

$$1) \quad \varphi = x^{-(l+1)} e^{x/2} (e^{-x} \cdot x^{\nu+l})_{\nu-(l+1)} \equiv \varphi_{[5]} \quad (6)$$

$$2) \quad \varphi = x^{-(l+1)} e^{x/2} (x^{\nu+l} \cdot e^{-x})_{\nu-(l+1)} \equiv \varphi_{[6]} \quad (7)$$

$$3) \quad \varphi = x^{-(l+1)} e^{-x/2} (e^x \cdot x^{-\nu+l})_{-\nu-(l+1)} \equiv \varphi_{[7]} \quad (8)$$

$$4) \quad \varphi = x^{-(l+1)} e^{-x/2} (x^{-\nu+l} \cdot e^x)_{-\nu-(l+1)} \equiv \varphi_{[8]} \quad (9)$$

Proof of Group I.

Set

$$\varphi = x^\mu \phi, \quad (\phi = \phi(x)) \quad (10)$$

we have then

$$\varphi_1 = \mu x^{\mu-1} \phi + x^\mu \phi_1 \quad (11)$$

and

$$\varphi_2 = \mu(\mu-1)x^{\mu-2}\phi + 2\mu x^{\mu-1}\phi_1 + x^\mu \phi_2. \quad (12)$$

Hence, yields

$$\phi_2 \cdot x^{\mu+2} + \phi_1 \cdot x^{\mu+1}(2\mu+2) + \phi \cdot \{-(1/4)x^{\mu+2} + vx^{\mu+1} + x^{\mu}(\mu^2 + \mu - l^2 - l)\} = 0 \quad (13)$$

from (1), applying (10), (11) and (12).

Choose μ such that

$$\mu^2 + \mu - l^2 - l = 0 \quad (14)$$

hence

$$\mu = \begin{cases} l & (15) \\ -(l+1) & (16) \end{cases}$$

(I) Case of $\mu = l$;

In this case we have

$$\phi_2 \cdot x + \phi_1 \cdot (2l+2) + \phi \cdot \{v - (1/4)x\} = 0 \quad (17)$$

from (13). Next we set

$$\phi = e^{\lambda x} \psi \quad (\psi = \psi(x)) \quad (18)$$

we have then

$$\phi_1 = \lambda e^{\lambda x} \psi + e^{\lambda x} \psi_1 \quad (19)$$

and

$$\phi_2 = \lambda^2 e^{\lambda x} \psi + 2\lambda e^{\lambda x} \psi_1 + e^{\lambda x} \psi_2. \quad (20)$$

Therefore, we obtain

$$\psi_2 \cdot x + \psi_1 \cdot (2\lambda x + 2l+2) + \psi \cdot \{x(\lambda^2 - \frac{1}{4}) + 2\lambda l + 2\lambda + v\} = 0 \quad (21)$$

from (17), using (18), (19) and (20).

Choose

$$\lambda = \begin{cases} 1/2 & (22) \\ -1/2 & (23) \end{cases}$$

(i) Case of $\lambda = 1/2$;

In this case we have

$$\psi_2 \cdot x + \psi_1 \cdot (x + 2l+2) + \psi \cdot (l+1+v) = 0, \quad (24)$$

from (21).

Operate N^α (N-fractional calculus operator of order α) to the both sides of (24), yields

$$N^\alpha(\psi_2 \cdot x) + N^\alpha(\psi_1 \cdot (x + 2l + 2)) + N^\alpha(\psi \cdot (l + 1 + \nu)) = 0, \quad (\alpha \notin \mathbb{Z}_0^-) \quad (25)$$

hence

$$\psi_{2+\alpha} \cdot x + \psi_{1+\alpha} \cdot (x + \alpha + 2l + 2) + \psi_\alpha \cdot (\alpha + l + 1 + \nu) = 0 \quad (26)$$

from (25), by Lemma (iv), since

$$N^\alpha(\psi_2 \cdot x) = (\psi_2 \cdot x)_\alpha \quad (27)$$

for example.

Choose α such that

$$\alpha = -(l + 1 + \nu) \equiv p \quad (\text{denote}) \quad (28)$$

we have then

$$\psi_{2+p} \cdot x + \psi_{1+p} \cdot (x + p + 2l + 2) = 0 \quad (29)$$

from (26), applying (28).

Therefore, we obtain

$$u_1 \cdot x + u \cdot (x + p + 2l + 2) = 0 \quad (30)$$

from (29), setting

$$u = \psi_{1+p} \cdot \quad (u_{-(1+p)} = \psi) \quad (31)$$

A particular solution to this variable separable form equation (30) is given by

$$u = e^{-x} x^{-(p+2l+2)}. \quad (32)$$

We have then

$$\psi = u_{-(1+p)} = (e^{-x} \cdot x^{-(p+2l+2)})_{-(1+p)} \quad (33)$$

from (31), and hence

$$\phi = e^{x/2} (e^{-x} \cdot x^{-(p+2l+2)})_{-(1+p)} \quad (34)$$

from (18), applying (22) and (33).

Hence we obtain

$$\varphi = x^l e^{x/2} (e^{-x} \cdot x^{-(p+2l+2)})_{-(1+p)} \quad (35)$$

$$= x^l e^{x/2} (e^{-x} \cdot x^{\nu-(l+1)})_{l+\nu} \equiv \varphi_{[1]}, \quad (2)$$

from (10) using (15) and (34).

Next we obtain

$$\varphi = x^l e^{x/2} (x^{\nu-(l+1)} \cdot e^{-x})_{l+\nu} \equiv \varphi_{[2]} , \quad (3)$$

changing the order of

$$e^{-x} \quad \text{and} \quad x^{\nu-(l+1)} \quad \text{in the parenthesis} \quad (\cdot)_{l+\nu} \quad \text{in (2)}.$$

We have

$$\varphi_{[1]} = \varphi_{[2]} \quad \text{when} \quad (l+\nu) \in \mathbb{Z}_0^+ .$$

(ii) Case of $\lambda = -1/2$;

In this case we have

$$\psi_2 \cdot x + \psi_1 \cdot (-x + 2l + 2) + \psi \cdot (\nu - l - 1) = 0 , \quad (36)$$

from (21).

Therefore, in the same manner as above, we obtain

$$\varphi = x^l e^{-x/2} (e^x \cdot x^{-\nu-(l+1)})_{l-\nu} \equiv \varphi_{[3]} \quad (4)$$

from (36) and

$$\varphi = x^l e^{-x/2} (x^{-\nu-(l+1)} \cdot e^x)_{l-\nu} \equiv \varphi_{[4]} , \quad (5)$$

changing the order of

$$e^x \quad \text{and} \quad x^{-\nu-(l+1)} \quad \text{in the parenthesis} \quad (\cdot)_{l-\nu} \quad \text{in (4)}.$$

(II) Case of $\mu = -(l+1)$;

Set $-(l+1)$ instead of l in the solutions

$$\varphi_{[1]}, \quad \varphi_{[2]}, \quad \varphi_{[3]}, \quad \text{and} \quad \varphi_{[4]},$$

we have then the solutions

$$\varphi_{[5]}, \quad \varphi_{[6]}, \quad \varphi_{[7]}, \quad \text{and} \quad \varphi_{[8]},$$

respectively.

§ 3. The Familiar Forms of The Solutions in Section 2.

The familiar forms of the solutions that are obtained ones in section 2 are shown as follows.

Corollary 1. *We have*

Group I.

$$1) \quad \varphi_{[1]} = (e^{i\pi})^{l+\nu} e^{-x/2} x^{\nu-1} {}_2F_0(-l-\nu, l+1-\nu; -1/x) \quad (|-1/x| < 1), \quad (1)$$

$$2) \quad \varphi_{[2]} = (e^{-i\pi})^{l+\nu} e^{-x/2} x^{-(l+1)} \frac{\Gamma(2l+1)}{\Gamma(l+1-\nu)} {}_1F_1(-l-\nu; -2l; x), \quad (|x| < 1), \quad (2)$$

$$(|\Gamma(2l+1-k)/\Gamma(l+1-\nu)| < \infty)$$

$$3) \quad \varphi_{[3]} = e^{x/2} x^{-(1+\nu)} {}_2F_0(-l+\nu, l+1+\nu; 1/x) \quad (|1/x| < 1), \quad (3)$$

$$4) \quad \varphi_{[4]} = (e^{-i\pi})^{l-\nu} e^{x/2} x^{-(l+1)} \frac{\Gamma(2l+1)}{\Gamma(l+1+\nu)} {}_1F_1(\nu-l; -2l; -x), \quad (|-x| < 1), \quad (4)$$

$$(|\Gamma(2l+1-k)/\Gamma(\nu+l+1)| < \infty)$$

Group II.

$$1) \quad \varphi_{[5]} = (e^{i\pi})^{\nu-(l+1)} e^{-x/2} x^{\nu-1} {}_2F_0(l+1-\nu, -l-\nu; -1/x), \quad (|-1/x| < 1), \quad (5)$$

$$2) \quad \varphi_{[6]} = (e^{-i\pi})^{-(l+1)+\nu} e^{-x/2} x^l \frac{\Gamma(-2l-1)}{\Gamma(-l-\nu)} {}_1F_1(l+1-\nu; 2l+2; x), \quad (|x| < 1), \quad (6)$$

$$(|\Gamma(-2l-1-k)/\Gamma(-\nu-l)| < \infty)$$

$$3) \quad \varphi_{[7]} = e^{x/2} x^{-(1+\nu)} {}_2F_0(l+1+\nu, \nu-l; 1/x) \quad (|1/x| < 1), \quad (7)$$

$$4) \quad \varphi_{[8]} = (e^{i\pi})^{l+1+\nu} e^{x/2} x^l \frac{\Gamma(-2l-1)}{\Gamma(\nu-l)} {}_1F_1(l+1+\nu; 2l+2; -x), \quad (|-x| < 1), \quad (8)$$

$$(|\Gamma(-2l-1-k)/\Gamma(\nu-l)| < \infty)$$

Where ${}_pF_q(\dots)$ are the generalized Gauss hypergeometric functions.

Proof of Group I.

We have

$$1) \quad \varphi_{[1]} = x^l e^{x/2} (e^{-x} \cdot x^{\nu-(l+1)})_{l+\nu} \quad (\S 2. (2))$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l+\nu+1)}{k! \Gamma(l+\nu+1-k)} (e^{-x})_{l+\nu-k} (x^{\nu-(l+1)})_k \quad (\text{by Lemma (iv)}) \quad (9)$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l+\nu+1)}{k! \Gamma(l+\nu+1-k)} (e^{i\pi})^{l+\nu-k} e^{-x} \left\{ e^{-i\pi k} \frac{\Gamma(k+l+1-\nu)}{\Gamma(l+1-\nu)} x^{\nu-(l+1)-k} \right\} \quad (10)$$

$$= (e^{i\pi})^{l+\nu} e^{-x/2} x^{\nu-1} \sum_{k=0}^{\infty} \frac{[-l-\nu]_k [l+1-\nu]_k}{k!} (-x)^{-k} \quad (11)$$

$$= (e^{i\pi})^{l+\nu} e^{-x/2} x^{\nu-1} {}_2F_0(-l-\nu, l+1-\nu; -1/x), \quad (|-1/x| < 1), \quad (1)$$

since

$$\Gamma(\lambda - k) = (-1)^{-k} \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(k+1-\lambda)} = (-1)^{-k} \frac{\Gamma(\lambda)}{[1-\lambda]_k}, \quad (k \in \mathbb{Z}_0^+) \quad (12)$$

$$(e^{\lambda x})_{l+\nu-k} = \lambda^{l+\nu-k} e^{\lambda x}, \quad (13)$$

$$(x^{\nu-(l+1)})_k = e^{-i\pi k} \frac{\Gamma(k+l+1-\nu)}{\Gamma(l+1-\nu)} x^{\nu-(l+1)-k} \quad (14)$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \text{ with } [\lambda]_0 = 1.$$

(Notation of Pochhammer).

$$2) \quad \varphi_{[2]} = x^l e^{x/2} (x^{\nu-(l+1)} \cdot e^{-x})_{l+\nu} \quad (\S 2. (3))$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l+\nu+1)}{k! \Gamma(l+\nu+1-k)} (x^{\nu-(l+1)})_{l+\nu-k} (e^{-x})_k \quad (15)$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [-l-\nu]_k}{k!} \{ e^{-i\pi(l+\nu-k)} \frac{\Gamma(2l+1-k)}{\Gamma(l+1-\nu)} x^{-2l-1+k} \} (e^{i\pi})^k e^{-x} \quad (16)$$

$$\left(\left| \frac{\Gamma(2l+1-k)}{\Gamma(l+1-\nu)} \right| < \infty \right)$$

$$= x^{-l-1} (e^{-i\pi})^{l+\nu} e^{-x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [-l-\nu]_k (-1)^{-k} \Gamma(2l+1)}{k! \Gamma(l+1-\nu) [-2l]_k} x^k \quad (17)$$

$$= (e^{-i\pi})^{l+\nu} \frac{\Gamma(2l+1)}{\Gamma(l+1-\nu)} e^{-x/2} x^{-l-1} \sum_{k=0}^{\infty} \frac{[-l-\nu]_k}{k! [-2l]_k} x^k \quad (18)$$

$$= (e^{-i\pi})^{l+\nu} e^{-x/2} x^{-l-1} \frac{\Gamma(2l+1)}{\Gamma(l+1-\nu)} {}_1F_1(-l-\nu; -2l; x), \quad (|x| < 1), \quad (2)$$

$$3) \quad \varphi_{[3]} = x^l e^{-x/2} (e^x \cdot x^{-\nu-(l+1)})_{l-\nu} \quad (\S 2. (4))$$

$$= x^l e^{-x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l-\nu+1)}{k! \Gamma(l-\nu+1-k)} (e^x)_{l-\nu-k} (x^{-\nu-(l+1)})_k \quad (19)$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [\nu-l]_k}{k!} \{ e^{-i\pi k} \frac{\Gamma(\nu+l+1+k)}{\Gamma(\nu+l+1)} x^{-\nu-(l+1)-k} \} \quad (20)$$

$$= x^{-\nu-1} e^{x/2} \sum_{k=0}^{\infty} \frac{[\nu-l]_k [\nu+l+1]_k}{k!} x^{-k} \quad (21)$$

$$= x^{-\nu-1} e^{x/2} {}_2F_0(\nu-l, \nu+l+1; 1/x), \quad (|1/x| < 1). \quad (3)$$

$$4) \quad \varphi_{[4]} = x^l e^{-x/2} (x^{-\nu-(l+1)} \cdot e^{-x})_{l-\nu} \quad (\S 2. (5))$$

$$= x^l e^{-x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l-\nu+1)}{k! \Gamma(l-\nu+1-k)} (x^{-\nu-(l+1)})_{l-\nu-k} (e^{-x})_k \quad (22)$$

$$= x^l e^{-x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [\nu-l]_k}{k!} \{ e^{-i\pi(l-\nu-k)} \frac{\Gamma(2l+1-k)}{\Gamma(\nu+l+1)} x^{-2l-1+k} \} e^x \quad (23)$$

$$\left(\left| \frac{\Gamma(2l+1-k)}{\Gamma(\nu+l+1)} \right| < \infty \right)$$

$$= (e^{-i\pi})^{l-\nu} x^{-l-1} e^{x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [\nu-l]_k (-1)^{-k} \Gamma(2l+1)}{k! \Gamma(\nu+l+1) [-2l]_k} (-x)^k \quad (24)$$

$$= (e^{-i\pi})^{l-\nu} x^{-l-1} e^{x/2} \frac{\Gamma(2l+1)}{\Gamma(\nu+l+1)} {}_1F_1(\nu-l; -2l; -x), \quad (|-x| < 1). \quad (4)$$

Proof of Group II.

We can obtain the solutions in Group II from the ones $\varphi_{[1]} \sim \varphi_{[4]}$ in Group I, setting $-(l+1)$ instead of l , respectively. (cf. § 2.)

§ 4. Commentary

[I] In the quantum mechanics, it is requested that

$$l \leq (n-1) \in \mathbb{Z}_0^+; (l = 0, 1, 2, \dots, n-1), \quad (1)$$

n ; The principal quantum number, ($n = \nu$)

l ; Orbital angular momentum quantum number.

Moreover, for the relationship (1), we have

$$(i) \quad \varphi_{[1]} = \varphi_{[2]} = x^l e^{x/2} (x^{n-(l+1)} \cdot e^{-x})_{l+n} \quad ((l+n) \in \mathbb{Z}^+) \quad (2)$$

$$= (e^{-ix})^{l+n} e^{-x/2} x^{-l-1} \frac{\Gamma(2l+1)}{\Gamma(l+1-n)} {}_1F_1(-l-n; -2l; x), \quad (|x| < 1), \quad (3)$$

and

$$(ii) \quad \varphi_{[5]} = \varphi_{[6]} = x^{-(l+1)} e^{x/2} (x^{n+l} \cdot e^{-x})_{n-(l+1)} \quad ((n-l-1) \in \mathbb{Z}_0^+) \quad (4)$$

$$= (e^{-ix})^{n-l-1} x^l e^{-x/2} \frac{\Gamma(-2l-1)}{\Gamma(-n-l)} {}_1F_1(1+l-n; 2l+2; x), \quad (|x| < 1), \quad (5)$$

respectively.

We have the below classically.

$$\varphi_{[1]} = \varphi_{[2]} = x^l e^{x/2} D^{l+n} (x^{n-(l+1)} \cdot e^{-x}) \quad (D = d/dx) \quad (6)$$

and

$$\varphi_{[5]} = \varphi_{[6]} = x^{-(l+1)} e^{x/2} D^{n-(l+1)} (x^{n+l} \cdot e^{-x}), \quad (7)$$

from (2) and (4), respectively.

Next we have

$$(iii) \quad \varphi_{[3]} = \varphi_{[4]} = x^l e^{-x/2} (e^x \cdot x^{-n-(l+1)})_{l-n} \quad (8)$$

$$= x^l e^{-x/2} (e^x \cdot x^{-n-(l+1)})_{-m} \quad ((n-l) = m \in \mathbb{Z}^+) \quad (9)$$

$$= x^l e^{-x/2} \int \dots \int e^x x^{-(n+l+1)} (dx)^m \quad (10)$$

and

$$(iv) \quad \varphi_{[7]} = \varphi_{[8]} = x^{-(l+1)} e^{-x/2} (e^x \cdot x^{-n+l})_{-n-(l+1)} \quad (11)$$

$$= x^{-(l+1)} e^{-x/2} (e^x \cdot x^{-n+l})_{-m} \quad ((n+l+1) = m \in \mathbb{Z}^+) \quad (12)$$

$$= x^{-(l+1)} e^{-x/2} \int \dots \int e^x x^{-n+l} (dx)^m, \quad (13)$$

omitting the additional arbitrary constants of the integrations.

Note. We have (for $m, n \in \mathbb{Z}_0^+$)

$$L_n(x) = e^x D^n (e^{-x} x^n) \quad ; \text{ Laguerre function.}$$

$$L_n^m(x) = D_n^m L_n(x) \quad ; \text{ Associated Laguerre function.}$$

[II] We have

$$\varphi_2 \cdot x^2 + \varphi_1 \cdot 2x + \varphi \cdot \{-(1/4)x^2 + nx - n(n-1)\} = 0, \quad (14)$$

from § 2. (1), setting $v = n$ and $l = n-1$.

And when $n = l+1$ we have

$$\varphi = \varphi_{[5]} = \varphi_{[6]} = x^{-n} e^{x/2} (e^{-x} \cdot x^{2n-1})_0 = x^{n-1} e^{-x/2} \quad (15)$$

from (4). The function shown by (15) satisfies equation (14) clearly.

[III] The homogeneous Fukuhara equation is given by

$$\varphi_2 + \varphi_1 \cdot (a + b/x) + \varphi \cdot (p + q/x + r/x^2) = 0, \quad (x \neq 0) \quad (16)$$

that is,

$$\varphi_2 \cdot x^2 + \varphi_1 \cdot (ax^2 + bx) + \varphi \cdot (px^2 + qx + r) = 0. \quad (17)$$

Therefore, we have

$$\varphi_2 \cdot x^2 + \varphi_1 \cdot 2x + \varphi \cdot \{(-1/4)x^2 + vx - l(l+1)\} = 0 \quad (\text{§ 2. (1)})$$

from (17), setting

$$a = 0, b = 2, p = -1/4, q = v \text{ and } r = -l(l+1). \quad (18)$$

Namely, the radial Schrodinger equation of the Hydrogen atom is a special one of the homogeneous Fukuhara equation (16).

(cf. Fractional Calculus, Vol. 5, (1996) pp.136 - 142 ; by K. Nishimoto)

[IV]. Usually the solutions to the equation so-called special differential equations such as the above ones are obtained by means of Frobenius.

Then compare the our solving manner (Method of NFCO) with that of Frobenius and others.

[V] The equation § 2. (1) is shown in Chapter 5 in the volume of Ian N. Sneddon (Special Functions of Mathematical Physics and Chemistry (1961); Oliver and Boyd, Edingburgh.) in the form

$$\frac{d^2 R}{dx^2} + \frac{2}{x} \cdot \frac{dR}{dx} - \left\{ \frac{1}{4} - \frac{\nu}{x} + \frac{l(l+1)}{x^2} \right\} R = 0, \quad (x \neq 0) \quad (19)$$

and a solution is given by

$$R = R_{n,l}(x) = e^{-(1/2)x} x^l L_{n+l}^{2l+1}(x) \quad (n \geq l+1). \quad (20)$$

[VI] Indeed we have (see the Note above)

$$R = R_{n,l}(x) = e^{-(1/2)x} x^l D^{2l+1}(e^x D^{n+l}(e^{-x} x^{n+l})), \quad (21)$$

hence we obtain

$$R = R_{1,0}(x) = e^{-(1/2)x} D^1(e^x D^1(e^{-x} x)) = -e^{-(1/2)x}, \quad (22)$$

for $l=0$ and $n=1$.

And we obtain, when $l=0$ and $\nu=n=1$,

$$\varphi_{[1]} = \varphi_{[2]} = x^l e^{x/2} (e^{-x} \cdot x^{\nu-(l+1)})_{l+\nu} = e^{x/2} (e^{-x} \cdot 1)_1 \quad (23)$$

$$= -e^{-x/2}. \quad (24)$$

References

- [1] K. Nishimoto; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5 (1996), Descartes Press, Koriyama, Japan.
- [2] K. Nishimoto; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st. Century); Integrals and Differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.
- [3] K. Nishimoto; On Nishimoto's fractional calculus operator N^ν (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.
- [4] K. Nishimoto; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol. 6, Nov. (1994), 1 - 14.
- [5] K. Nishimoto; Ring and Field produced from The Set of N-Fractional Calculus Operator, J. Frac. Calc. Vol. 24, Nov. (2003), 29 - 36.
- [6] K. Nishimoto; An application of fractional calculus to the nonhomogeneous Gauss equations, J. Coll. Engng. Nihon Univ., B - 28 (1987), 1 - 8.
- [7] K. Nishimoto and S. L. Kalla; Application of Fractional Calculus to Ordinary Differential Equation of Fuchs Type, Rev. Tec. Ing. Univ. Zulia, Vol. 12, No. 1, (1989).
- [8] K. Nishimoto; Application of Fractional Calculus to Gauss Type Partial Differential Equations, J. Coll. Engng. Nihon Univ., B - 30 (1989), 81 - 87.
- [9] Shih - Tong Tu, S. - J. Jaw and Shy - Der Lin; An application of fractional calculus to Chebychev's equation, Chung Yuan J. Vol. XIX (1990), 1 - 4.
- [10] K. Nishimoto, H. M. Srivastava and Shih - Tong Tu; Application of Fractional Calculus in Solving Certain Class of Fuchsian Differential Equations, J. Coll. Engng. Nihon Univ., B - 32 (1991), 119 - 126.
- [11] K. Nishimoto; A Generalization of Gauss Equation by Fractional Calculus Method, J. Coll. Engng. Nihon Univ., B - 32 (1991), 79 - 87.
- [12] Shy - Der Lin, Shih - Tong Tu and K. Nishimoto; A generalization of Legendre's equation by fractional calculus method, J. Frac. Calc. Vol. 1, May (1992), 35 - 43.
- [13] N. S. Sohi, L. P. Singh and K. Nishimoto; A generalization of Jacobi's equation by fractional calculus method, J. Frac. Calc. Vol. 1, May (1992), 45 - 51.
- [14] K. Nishimoto; Solutions of Gauss equation in fractional calculus, J. Frac. Calc. Vol. 3, May (1993), 29 - 37.
- [15] K. Nishimoto; Solutions of homogeneous Gauss equations, which have a logarithmic function, in fractional calculus, J. Frac. Calc. Vol. 5, May (1994), 11 - 25.
- [16] K. Nishimoto; Application of N - transformation and N - fractional calculus method to nonhomogeneous Bessel equations (I), J. Frac. Calc. Vol. 8, Nov. (1995), 25 - 30.
- [17] K. Nishimoto; Operator N^ν method to nonhomogeneous Gauss and Bessel equations, J. Frac. Calc. Vol. 9, May (1996), 1 - 15.
- [18] K. Nishimoto and Susana S. de Romero; N - fractional calculus operator N^ν method to nonhomogeneous and homogeneous Whittaker equations (I), J. Frac. Calc. Vol. 9, May (1996), 17 - 22.
- [19] K. Nishimoto and Judith A. de Duran; N - fractional calculus operator N^ν method to nonhomogeneous Fukuhara equations (I), J. Frac. Calc. Vol. 9, May (1996), 23 - 31.
- [20] K. Nishimoto; N - fractional calculus operator N^ν method to nonhomogeneous Gauss equations, J. Frac. Calc. Vol. 10, Nov. (1996), 33 - 39.

- [21] K. Nishimoto ; Kummer's twenty - four functions and N- fractional calculus, *Nonlinear Analysis, Theory, Method & Applications*, Vol. 30, No.2, (1997), 1271 - 1282.
- [22] Shih- Tong Tu, Ding-Kuo Chyan and Wen- Chieh Luo ; Some solutions to the nonhomogeneous Jacobi equations via fractional calculus operator N^ν method, *J. Frac. Calc.* Vol.12, Nov. (1997), 51 - 60.
- [23] Shih- Tong Tu, Ding- Kuo Chyan and Erh- Tsung Chin ; Solutions of Gegenbauer and Chebysheff equations via operator N^ν method, *J. Frac. Calc.* Vol.12, Nov. (1997), 61 - 69.
- [24] K. Nishimoto ; N- method to Hermite equations, *J. Frac. Calc.* Vol. 13, May (1998), 21 - 27.
- [25] K. Nishimoto ; N- method to Weber equations, *J. Frac. Calc.* Vol. 14, Nov. (1998), 1 - 8.
- [26] K. Nishimoto ; N- method to generalized Laguerre equations, *J. Frac. Calc.* Vol. 14, Nov. (1998), 9 - 21.
- [27] Shy-Der Lin, Jaw-Chian Shyu, Katsuyuki Nishimoto and H. M. Srivastava ; Explicit Solutions of Some General Families of Ordinary and Partial Differential Equations Associated with the Bessel Equation by Means of Fractional Calculus, *J. Frac. Calc.* Vol. 25, May (2004), 33 - 45.
- [28] K. Nishimoto ; N- Fractional Calculus of Products of Some Power Functions, *J. Frac. Calc.* Vol. 27, May (2005), 83 - 88.
- [29] K. Nishimoto ; N- Fractional Calculus of Some Composite Functions, *J. Frac. Calc.* Vol. 29, May (2006), 35 - 44.
- [30] K. Nishimoto ; Solutions to Some Extended Hermite's equations by Means of N- Fractional Calculus, *J. Frac. Calc.* Vol. 29, May (2006), 45 - 56.
- [31] Shy-Der Lin, Yen-Shung Tsai, H. M. Srivastava and Pin-Yu Wang ; Fractional Calculus Derivations of the Explicit Solutions of Confluent Hypergeometric and Associated Laguerre Equations, *J. Frac. Calc.* Vol.29, May (2006), 75 - 86.
- [32] Tsuyako Miyakoda ; Solutions to An Extended Hermite's equation by Means of N- Fractional Calculus, *J. Frac. Calc.* Vol. 30, Nov. (2006), 23 - 32.
- [33] K. Nishimoto ; Solutions to Some Extended Weber's equations by Means of N- Fractional Calculus, *J. Frac. Calc.* Vol. 30, Nov. (2006), 1 - 11.
Associated Laguerre Equations, *J. Frac. Calc.* Vol.29, May (2006), 75 - 86.
- [34] K. Nishimoto ; N- Fractional Calculus of Some Composite Algebraic Functions, *J. Frac. Calc.* Vol.31, May (2007), 11 - 23.
- [35] K. Nishimoto ; N- Fractional Calculus of Some Logarithmic Functions, *J. Frac. Calc.* Vol.32, Nov. (2007), 17 - 28.
- [36] K. Nishimoto ; N-Fractional Calculus Operator Method to Chebyshev's Equation , *J. Frac. Calc.* Vol. 33, May (2008), 71 - 90.
- [37] K. Nishimoto ; N-Fractional Calculus Operator Method to Associated Laguerre's Equation (I), *J. Frac. Calc.* Vol. 35, May (2009), 33 - 45.
- [38] K. Nishimoto ; N-Fractional Calculus Operator Method to Associated Laguerre's Equation (II), *J. Frac. Calc.* Vol. 36, Nov. (2009), 1 - 13.
- [39] T. Miyakoda and K. Nishimoto ; N-Fractional Calculus Operator (NFCO) Method to An Extended Chebyshev's Equation , *J. Frac. Calc.* Vol. 36, Nov. (2009), 49 - 63.
- [40] K. Nishimoto ; Solutions to An Euler's Equation by Means of N- Fractional Calculus, *J. Frac. Caluc.* Vol.37, May (2010), 29 - 42.

- [41] K. Nishimoto ; N- Fractional Calculus Operator Method to Homogeneous Chebyshev's Equation, J. Frac. Calc. Vol. 38, Nov. (2010), 1 - 24.
- [42] K. Nishimoto ; N- Fractional Calculus Operator Method to The Nonhomogeneous Chebyshev's Equation, J. Frac. Calc. Vol. 38, Nov. (2010), 25 - 34.
- [43] K. Nishimoto ; N- Fractional Calculus of Certain Generalized Logarithmic Functions, J. Frac. Calc. Vol. 39, May (2011), 1 -14.
- [44] K. Nishimoto ; The Integral Contour of N-Fractional Calculus, Interval of Fractional Integral of Riemann -Liouville and That of Weyl, and N-Fractional Calculus of Some Functions, J. Frac. Calc. Vol. 39, May (2011), 43 -58
- [45] K. Nishimoto ; On The Set $\{N^\nu\}$, Where N^ν is The N- Fractional Calculus Operator of Order ν , J. Frac. Calc. Vol. 40, Nov. (2011), 49 - 59.
- [46] K. Nishimoto ; Solutions to Some Homogeneous Special Ordinary Differential Equation by Means of N- Fractional Calculus Operator, J. Frac. Calc. Vol. 41, May (2012), 1 - 23.
- [47] K. Nishimoto ; Solutions to The Homogeneous Bessel Equation by Means of N- Fractional Calculus Operator, J. Frac. Calc. Vol. 42, Nov. (2012), 1 - 19.
- [48] Saad Naji Al- Azawi ; Some Results in Fractional Calculus, LAPLAMBERT Academic Publishing GebH & Co. KG , Saarbrucken, Germany. Copyright 2011.
- [49] David Dummit and Richard M. Foote ; Abstract Algebra, Prentice Hall (1991).
- [50] K. B. Oldham and J. Spanier ; The Fractional Calculus, Academic Press (1974).
- [51] A. C. McBride ; Fractional Calculus and Integral Transforms of Generalized Functions, Research Notes, Vol. 31, (1979), Pitman.
- [52] S> G. Samko, A. A. Kilbas and O. I. Marichev ; Fractional Integrals and Derivatives, and Some Their Applications (1987), Nauka, USSR.
- [53] K. S. Miller and B. Ross ; An Introduction to The Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, (1993).
- [54] V. Kiryakova ; Generalized fractional calculus and applications, Pitman Reseach Notes, No.301, (1994), Longman.
- [55] Igor Podlubny ; Fractional Differential Equations (1999), Academic Press.
- [56] R. Hilfer (ed.) ; Applications of Fractional Calculus in Physics, (2000), World Scientific, Singapore, New Jersey, London, Hong Kong.
- [57] Anatoly A. Kilbas, Hari M. Srivastava and Juan J. Trujillo ; Theory and Applications of Fractional Differential Equations (2006), Elsevier, North- Holland, Mathematics Studies 204.
- [58] Ian N. Sneddon ; Special Functions of Mathematical Physics and Chemistry (1961), Oliver and Boyd, Edingburgh.
- [59] David Bohm ; Quantum Theory (1951), Prentice-Hall, New York.