

NEW DEGENERATION PHENOMENON FOR INFINITE-TYPE RIEMANN SURFACES

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ABSTRACT. Since the Teichmüller space of a surface R is a deformation space of complex structures defined on R , its Bers boundary describes the degeneration of complex structures in a certain sense. In this paper, by constructing a concrete example, we prove that if R is a Riemann surface of infinite type, then there exists a Riemann surface with a marking on the Bers boundary that is homeomorphic to the surface R . We also show that such points form an infinite-dimensional complex manifold on the Bers boundary.

1. INTRODUCTION

The Teichmüller space $\text{Teich}(\Gamma)$ of a Fuchsian group Γ is the space of quasiconformal deformations of Γ . In terms of the corresponding Riemann surface $R := \mathbb{H}/\Gamma$, the space $\text{Teich}(\Gamma)$ can be regarded as its deformation space; it is sometimes denoted by $\text{Teich}(R)$. Furthermore, $\text{Teich}(R)$ can be considered as a complex manifold, reflecting many properties of the Riemann surface R .

The motivation for studying the boundary of the Teichmüller space arises from the following reasoning: in complex analysis, the structure of the complex manifold $\text{Teich}(R)$ can be examined through the properties of proper holomorphic maps from the complex disk $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ to $\text{Teich}(R)$. Moreover, according to the Fatou–Riesz theorem, a holomorphic function from $\mathbb{D}$ to a complex manifold is characterized by its non-tangential limits at almost all points on the boundary $\partial\mathbb{D}$. Since the non-tangential limit points lie on the boundary of $\text{Teich}(R)$, understanding the boundary of $\text{Teich}(R)$ is crucial for analyzing its complex manifold structure. Furthermore, the boundary of $\text{Teich}(R)$ provides insights into the degeneration of complex structures on Riemann surfaces, thereby enhancing our understanding of these surfaces. Therefore, we study a boundary of the Teichmüller space.

Various boundaries of $\text{Teich}(R)$ have been considered. We focus on the Bers boundary $\partial_{\text{Bers}} \text{Teich}(\Gamma)$, which is obtained by embedding $\text{Teich}(R)$ into a bounded region of the complex Banach space of all holomorphic quadratic differentials. We will denote it $B(\Gamma)$ (cf. Subsection 2.1). The embedding is known as the Bers embedding $\mathcal{B} : \text{Teich}(\Gamma) \rightarrow B(\Gamma)$. The Teichmüller space which, when embedded in this manner, can be seen as the space of deformations of the Fuchsian group Γ within the representation space $\mathcal{R}(\Gamma)$, where $\mathcal{R}(\Gamma)$ is the space of the conjugacy classes of discrete and faithful homomorphisms from Γ to $PSL(2; \mathbb{C})$. When R is of finite type, it has been extensively studied by Bers [4], Maskit [12], and Abikoff [1].

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If R is compact, the Bers boundary consists of Kleinian groups corresponding to Riemann surfaces that degenerate topologically [4, 12]. When R has finitely many punctures, the same can be established by modifying the claim appropriately (more details will be given in Section 2.1). This implies that the degeneration of complex structures leads to that of the underlying topological structure. Some properties of $\text{Hol}(\mathbb{D}, \text{Teich}(R))$ have been investigated based on the properties of Kleinian groups lying on the Bers boundary. They help in understanding the complex manifold structure of $\text{Teich}(R)$ ([14, 15], [16, 17]).

This paper concerns the case where R is of infinite type. In this case, $\text{Teich}(R)$ becomes an infinite-dimensional complex manifold. We show that points of the Bers boundary are homeomorphic to the base surface R . Moreover, these points can be parametrized by an open subset of a certain Banach space. These results highlight differences between finite and infinite types. In this paper, we shall call the Kleinian group corresponding to such a Riemann surface a David-Fuchsian b-group. A detailed definition is described in Definition 2.1, Definition 2.3, and a construction is presented in Section 4.

The quasiconformal maps are characterized as the solution of the partial differential equation called the Beltrami equation $f_{\bar{z}} = \mu f_z$ considered for a function $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_{L^\infty} < 1$. Since the Teichmüller space is the set of all quasiconformal deformations of the Riemann surface, a deformation by the solution of the degenerate Beltrami equation is expected to lie on the Bers boundary. The degenerate Beltrami equation means that $\|\mu\|_{L^\infty} = 1$. David [5] has studied sufficient conditions on $\mu \in L^\infty(\mathbb{C})$ for the degenerate Beltrami equation to have a homeomorphic solution; its condition is called David's condition. We construct a David-Fuchsian b-group by constructing a deformation of the Fuchsian group Γ , which satisfies David's condition. To prove that the constructed David deformation of the Fuchsian group Γ belongs to the Bers boundary, it is necessary to construct a sequence in $\text{Teich}(\Gamma)$ that converges to the point in $B(\Gamma)$ corresponding the constructed David deformation. For that, we apply the estimate given by McMullen [13, Theorem 1.2]; we impose the condition that $\text{short}(R)$ is positive, where $\text{short}(R)$ is the infimum of lengths of essential simple closed geodesics of R . The first result is as follows.:

Theorem (Theorem 4.4). *Let R be a Riemann Surface of analytically infinite type with $\text{short}(R) > 0$ and infinitely many pairwise disjoint essential closed geodesics $\{\gamma_n^*\}_{n \geq 0}$ whose lengths are bounded. Then, there exists a David-Fuchsian b-group on the Bers boundary of $\text{Teich}(R)$.*

A natural question arising from Theorem 4.4 is whether a family of David-Fuchsian b-groups can be constructed, and if so, what degrees of freedom are available in the construction. In this regard, the following results have been obtained.

Theorem (Theorem 5.1). *Let R be a Riemann Surface of analytically infinite type with $\text{short}(R) > 0$ and infinitely many pairwise disjoint essential closed geodesics $\{\gamma_n^*\}_{n \geq 0}$ whose lengths are bounded. Then, there exists a unit ball D of a Banach space and a holomorphic map $F : D \rightarrow B(\Gamma)$, where $F(D)$ is contained in the Bers boundary $\partial_{\text{Bers}} \text{Teich}(R)$, and consists only of David-Fuchsian b-group, and the dimension of the tangent space of the image is infinite.*

In Section 2, we provide the necessary background about quasiconformal maps, Kleinian groups, and David maps. In Section 3, we compute the variation of the

injective radius when the cylinders are stretched to apply them to McMullen's inequality. In Section 4, we construct a Kleinian group using a David map, and we prove that the Kleinian group exists on $\partial_{\text{Bers}} \text{Teich}(\Gamma)$, that is, we prove that David-Fuchsian b-group exists. In Section 5, we consider how many David-Fuchsian b-groups exist on $\partial_{\text{Bers}} \text{Teich}(\Gamma)$. In conclusion, it is shown that there exists open sets D of a certain infinite-dimensional Banach space and a holomorphic map $F : D \rightarrow B(\Gamma)$ such that $F(D)$ is contained in the Bers boundary and consists of David-Fuchsian b-groups.

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2. PRELIMINARIES

2.1. Teichmüller theory and Kleinian groups. Let R be a Riemann surface whose universal covering surface is $\mathbb{D}$, and let it be represented by a Fuchsian group Γ acting on $\mathbb{D}$ as $R = \mathbb{D}/\Gamma$. A quasiconformal mapping $f : \mathbb{D} \rightarrow \mathbb{D}$ is said compatible with Γ if $\Gamma^f := f\Gamma f^{-1}$ remains a Fuchsian group. The isomorphism $\Gamma \rightarrow \Gamma^f$, defined by conjugation by $\Gamma \ni \gamma \mapsto f\gamma f^{-1} \in \Gamma^f$, is referred to as a quasiconformal deformation of Γ . Note that the deformation of Γ induces a quasiconformal deformation of the Riemann surface R . Indeed, the map f induces a quasiconformal map $\tilde{f} : \mathbb{D}/\Gamma \rightarrow \mathbb{D}/\Gamma^f$. Let $L^\infty(\Gamma)$ denote the complex Banach space of all bounded measurable *Beltrami differentials* for Γ supported on $\mathbb{H}$. Each element $\mu \in L^\infty(\Gamma)$ satisfies $\mu \circ \gamma \cdot \bar{\gamma}' = \mu \cdot \gamma'$ for every $\gamma \in \Gamma$. Its open unit ball $\text{Bel}(\Gamma)$ is the space of Beltrami coefficients for Γ . For each element $\mu \in \text{Bel}(\Gamma)$, the unique quasiconformal homeomorphism $w^\mu : \mathbb{D} \rightarrow \mathbb{D}$ with complex dilatation μ that fixes $-1, 1$ and i , is compatible for Γ . Two elements μ and ν in $\text{Bel}(\Gamma)$ are said to be *Teichmüller equivalent* if $w^\mu|_{\mathbb{S}^1} = w^\nu|_{\mathbb{S}^1}$ holds. The *Teichmüller space* $\text{Teich}(\Gamma)$ of Γ is the quotient space of $\text{Bel}(\Gamma)$ by the Teichmüller equivalence relation.

In the following, we review how to embed $\text{Teich}(\Gamma)$ into a bounded domain of the Banach space $B(\Gamma)$ by a well-known method. $B(\Gamma)$ is the complex Banach space of all L^∞ hyperbolic holomorphic quadratic differentials $\varphi : \mathbb{D}^* \rightarrow \hat{\mathbb{C}}$ for Γ , where $\mathbb{D}^*$ is the exterior of $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$. Namely, the point φ of $B(\Gamma)$ satisfies $\varphi \circ \gamma \cdot (\gamma')'^2 = \varphi$ for every $\gamma \in \Gamma$, and the L^∞ hyperbolic norm $\|\varphi\|_{B(\Gamma)} := \sup_{z \in \mathbb{D}^*} |\varphi(z)|(1 - |z|^2)^2$ is finite. Like $\text{Bel}(\Gamma)$, the space $B(\Gamma)$ is sometimes considered the space of quadratic differentials on the conjugate Riemann surface R^* through the covering map $\pi : \mathbb{D}^* \rightarrow R^*$. It is denoted as $B(R)$ in such a sense. For each $\mu \in \text{Bel}(\Gamma)$, define

$$\begin{cases} \mu & \text{on } \mathbb{D}, \\ 0 & \text{on } \mathbb{D}^*. \end{cases}$$

Then, there exists a unique quasiconformal mapping $f_\mu : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that

$$\text{bel}(f_\mu) \left(:= \frac{(f_\mu)_{\bar{z}}}{(f_\mu)_z} \right) = \begin{cases} \mu & \text{on } \mathbb{D} \\ 0 & \text{on } \mathbb{D}^* \end{cases}, \quad f_\mu(z) = z + o(1/z) \text{ (near } z = \infty).$$

The Schwarzian derivative of f_μ is defined on $\mathbb{D}^*$ as

$$S(f_\mu|_{\mathbb{D}^*}) = \left(\frac{f_\mu''}{f_\mu'} \right)' - \frac{1}{2} \left(\frac{f_\mu''}{f_\mu'} \right)^2.$$

The mapping $\mathcal{B} : \text{Teich}(\Gamma) \ni [\mu] \mapsto S(f_\mu|_{\mathbb{D}^*}) \in B(\Gamma)$ is well-defined and holomorphic. The map is also a biholomorphic map onto its image, called the *Bers embedding*. As a known fact, its image is a bounded domain in $B(\Gamma)$. Therefore, the boundary can be defined via the Bers embedding. This is called the *Bers boundary* $\partial_{\text{Bers}} \text{Teich}(\Gamma) := \partial \mathcal{B}(\text{Teich}(\Gamma))$. Next, we summarize the holonomy homomorphism defined at each point in $B(\Gamma)$. For each a point $\varphi \in B(\Gamma)$, then there exists a unique, locally univalent holomorphic map $W_\varphi : \mathbb{D}^* \rightarrow \hat{\mathbb{C}}$ such that

$$S(W_\varphi) = \varphi, \quad W_\varphi(z) = z + o(1/z), \text{ near } z = \infty.$$

Then, for every element $\gamma \in \Gamma$, there exists a unique Möbius transformation g such that

$$W_\varphi \circ \gamma = g \circ W_\varphi.$$

We denote the mapping $\gamma \mapsto g$ by $\chi_\varphi : \Gamma \rightarrow PSL(2; \mathbb{C})$, called a monodromy homomorphism. If W_φ is univalent, $\chi_\varphi(\Gamma)$ is also discrete. Bers proved that if φ is contained in $\text{Cl}(\text{Teich}(\Gamma)) := \text{Teich}(\Gamma) \cup \partial_{\text{Bers}} \text{Teich}(\Gamma)$, the function W_φ is univalent [4, Theorem 5]. Then, we classify φ according to the properties of $\chi_\varphi(\Gamma)$.

Definition 2.1 (Classification of elements in $\text{Cl}(\text{Teich}(\Gamma))$). Let a quadratic differential φ be in $\text{Cl}(\text{Teich}(\Gamma))$. The group $\chi_\varphi(\Gamma)$ or the differential φ is called

- a *quasiFuchsian group* if the function W_φ can be extended to a homeomorphic map on $\hat{\mathbb{C}}$,
- a *cusp* if there exists a hyperbolic element $\gamma \in \Gamma$ such that $\chi_\varphi(\gamma)$ is a parabolic element, and
- a *totally degenerate group* if $\chi_\varphi(\Gamma)$ is not a cusp and $W_\varphi(\mathbb{H}^*) \subset \hat{\mathbb{C}}$ is an open and dense subset.

Accola [2] has shown that when considering the Teichmüller space of a Fuchsian group Γ of the first kind, which is not necessarily finitely generated, the three conditions appearing in the above definition provide a classification of the elements of $\text{Cl}(\text{Teich}(\Gamma))$.

Bers [4] and Maskit [12] proved that if a finitely generated Fuchsian group Γ is of the first kind, quasiFuchsian b-groups do not lie on the Bers boundary. The following are known.

Theorem 2.2 ([4] and [12]). *Let R be a Riemann surface whose Fuchsian group Γ is finitely generated of the first kind. If a quadratic differential $\psi \in \partial_{\text{Bers}} \text{Teich}(S)$ is not a cusp, it is a totally degenerate group. Moreover, if a quadratic differential $\varphi \in \text{Cl}(\text{Teich}(\Gamma))$ is a quasiFuchsian group, then φ is in $\text{Teich}(\Gamma)$.*

In this paper, we prove that quasi-Fuchsian groups can lie on the Bers boundary when R is of infinite type. In other words, we show that Theorem 2.2 does not hold in the case of infinite type. For the sake of clarity, we define the following terms.

Definition 2.3. In this paper, we call the group $\chi_\varphi(\Gamma)$ or the differential φ a *topological quasiFuchsian b-group* if $\varphi \in \partial_{\text{Bers}} \text{Teich}(\Gamma)$ and φ is a quasiFuchsian group. In particular, when the function W_φ satisfies a David condition (see Definition 2.4), we call $\chi_\varphi(\Gamma)$ or the quadratic differentials φ a *David-Fuchsian b-group*.

Tanigawa [16, Proposition 4.4] and Yamamoto [19, Theorem 3.1] also discuss Kleinian groups that lie on the Bers boundary in the case where R is not of analytically finite type. The comparison and the relationships with our results are summarized in Section 6.

2.2. David map. This section summarizes the notion of David maps, characterized as solutions to the degenerate Beltrami equations. In particular, we discuss the results studied by David [5]. Since quasiconformal maps are characterized as solutions of Beltrami equations, David maps can be considered as an extension to quasiconformal maps.

Definition 2.4. Let μ be a measurable function on an open set $U \subset \mathbb{C}$ and $|\mu(z)| < 1$ for almost all $z \in U$ with respect to the Lebesgue measure. We say that μ satisfies a *David condition*, if there exist $\alpha > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that, for each $0 < \varepsilon < \varepsilon_0$,

$$(2.1) \quad |\{z \in U \mid |\mu(z)| > 1 - \varepsilon\}|_{sp} \leq C \exp\left(-\frac{\alpha}{\varepsilon}\right),$$

where $|\cdot|_{sp}$ is the spherical measure on $\hat{\mathbb{C}}$. Moreover, if $K_\mu(z) := \frac{1+|\mu|(z)}{1-|\mu|(z)}$ satisfies $\exp(K_\mu) \in L^p_{loc}$ for some $p \in (0, \infty)$, then μ is said to have *exponentially L^p_{loc} integrable distortion*.

Remark. (1) When the support of μ is contained by a bounded domain in the plane, then the Lebesgue measure can replace the spherical measure in (2.1).
 (2) Suppose that $\text{supp}(\mu) \subset \text{Cl}(\mathbb{D})$ and $K_\mu \in L^p(\mathbb{D})$, then μ satisfies the David condition by Chebychev's inequality,

$$\begin{aligned} |\{|\mu| > 1 - \varepsilon\}|_{\text{Leb}} &= \left| \left\{ K_\mu + 1 \geq \frac{2}{\varepsilon} \right\} \right|_{\text{Leb}} \\ &\leq \exp(-2/\varepsilon) \int_{\mathbb{D}} \exp(K_\mu(z) + 1) \, dx dy. \end{aligned}$$

Hence, the exponentially L^p_{loc} integrable distortion condition implies the David condition.

Theorem 2.5 (Generalized Riemann mapping theorem, [5]). *Let μ be a measurable function on $\mathbb{C}$ satisfies the David condition, then the Beltrami equation $f_{\bar{z}} = \mu f_z$ has a homeomorphic solution $f_\mu \in W^{1,1}_{loc}(\mathbb{C})$ which is unique up to postcomposition of a conformal map.*

Remark. Let Ω be an open subset in $\mathbb{R}^n$ for some $n \geq 2$. A map $f : \Omega \rightarrow \mathbb{R}^n$ is called *mapping of finite distortion* if $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$, $J_f \in L^1_{loc}(\Omega)$, and there exists a measurable function $K_f : \Omega \rightarrow [1, \infty)$ so that

$$|Df(x)|^n \leq K_f(x) J_f(x) \text{ a.e.,}$$

where J_f is the jacobian of f , Df is the matrix of partial derivative, and $|Df(x)|$ is the operator norm. A homeomorphic solution of the Beltrami equation with a coefficient μ which satisfies David condition, is an example of mappings of finite distortion. For more details, see [3, 8].

Definition 2.6. Let $\mu \in L^\infty(\mathbb{C})$ satisfy $|\mu(z)| < 1$ for almost everywhere $z \in \mathbb{C}$. A homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ is called *the principal solution* to the Beltrami equation with coefficient μ if the map f satisfies the following conditions:

- f satisfies the partial differential equation $f_{\bar{z}} = \mu f_z$ a.e.,
- there exists a discrete set $E \subset \mathbb{C}$ such that $f \in W_{\text{loc}}^{1,1}(\mathbb{C} \setminus E)$.
- around the point at infinity, f has the Laurent expansion $f(z) = z + \frac{a_1}{z} + \dots$.

Theorem 2.7 ([3, Theorem 20.4.7]). *Let μ be a measurable function on $\mathbb{C}$ with $|\mu| < 1$ a.e.. If there exist $p > 1$ and a measurable function K such that*

$$(2.2) \quad \exp(K) \in L^p(\mathbb{D}), \quad |\mu(z)| \leq \frac{K(z) - 1}{K(z) + 1} \chi_{\mathbb{D}},$$

where $\chi_{\mathbb{D}}$ is the characteristic function of $\mathbb{D}$, then the Beltrami equation $f_{\bar{z}} = \mu f_z$ admits a unique principal solution $f \in W_{\text{loc}}^{1,2}(\mathbb{C})$.

The following proposition can be obtained by rewriting the proof of the existence theorem for solutions of the degenerate Beltrami equation slightly [3, Theorem 20.4.7]. In [3, Proof of Theorem 20.4.7], it uses a specific sequence of approximations for μ . Still, any sequence of approximations that satisfies the appropriate conditions is sufficient.

Proposition 2.8. *Let μ be a measurable function, which satisfies the assumption of Theorem 2.7. Suppose that a sequence (μ_n) satisfies*

$$\|\mu_n\|_{L^\infty} < 1, \quad \mu_n(z) \rightarrow \mu(z) \text{ a.e.}, \quad |\mu_n| \leq |\mu|.$$

Then, the sequence of the principal solutions of μ_n has a subsequence that locally uniformly converges to the principal solution of μ .

Proof. For the Beltrami differential μ_n and μ , there exist BMO functions $(K_n)^*$ and K^* respectively, and

$$K_{\mu_n} \leq (K_n)^*, \quad K_\mu \leq K^*$$

holds. In particular, $|\mu_n| \leq |\mu|$, so we can see that $(K_n)^* \leq K^*$ is satisfied.

Next, let f_n be the principal solution for μ_n . Then, from the general theory of the Beltrami equation, we get

$$(2.3) \quad \|(f_n)_z - 1\|_{L^2(\mathbb{D})} = \|(f_n)_{\bar{z}}\|_{L^2(\mathbb{D})} \leq 4\|K^*\|_{L^2(\mathbb{D})},$$

(see [Proof of Theorem 20.4.7] in [AIM]). In particular, since μ satisfies the assumption of Theorem 2.7, there exists a constant $M > 0$ such that, for any $R > 0$,

$$\int_{\Delta(0;R)} |Df_n|^2 dx dy \leq 4\pi R^2 + M.$$

From modulus inequality ([3, Theorem 20.1.6]), for any $a, b \in \mathbb{D}$,

$$|f_n(a) - f_n(b)|^2 \leq \frac{\pi \int_{\Delta(0;R)} |\nabla f_n|^2 dx dy}{\log \left(e + \frac{1}{|a-b|} \right)},$$

the sequence (f_n) has a convergent subsequence. Moreover, from (2.3) and the weak sequential compactness of $W^{1,2}(\mathbb{C})$, by taking a subsequence, we may assume $f_n \rightarrow f$ locally uniformly and $Df_n \rightarrow Df$ weakly in L_{loc}^2 . Thus, we obtain

$$f_{\bar{z}} = \mu f_z \text{ a.e..}$$

Finally, it remains to be shown that f is the principal solution of μ . Let g_n be the inverse map of f_n , and then g_n has the following modulus of continuity: for

$a, b \in \mathbb{C}$,

$$|g_n(a) - g_n(b)| \leq 16\pi^2 \frac{1 + |a|^2 + |b|^2}{\log\left(e + \frac{1}{|a-b|}\right)} \|K_{\mu_n}\|_{L^1(\mathbb{D})},$$

which is in [3, Lemma 20.2.3]. Therefore, (g_n) has also a convergent subsequence. Since the limit function g is continuous and $g \circ f = \text{id}_{\mathbb{C}}$, the function f is a homeomorphism. Moreover, because f_n are conformal around $\infty \in \hat{\mathbb{C}}$, the function f is also conformal around $\infty \in \hat{\mathbb{C}}$. In particular, the Laurent expansion of f is

$$f(z) = z + \frac{a}{z} + \dots$$

□

3. DEFORMATION OF A FAMILY OF CYLINDERS

3.1. A result of McMullen.

Theorem 3.1 ([13, Theorem 1.2]). *Let R be a Riemann surface with $\text{short}(R) > 0$, where $\text{short}(R)$ denotes the infimum of the lengths of all essential simple closed geodesics on R . Let $[\mu]$ be a point in $\text{Teich}(R)$, and ν be a unit-norm Beltrami differential in the tangent space $T_{[\mu]} \text{Teich}(R)$ at $[\mu]$. If the support of ν is contained in the part of $f^\mu(R)$ of the injectivity radius less than $L < 1/2$, where the injectivity radius is measured using the hyperbolic metric of $f^\mu(R)$. Then, the image of ν under the derivative of Bers' embedding $\mathcal{B} : \text{Teich}(R) \rightarrow B(R)$ has norm at most $C(L \log 1/L)^2$, where the constant C depends only on $\text{short}(R)$.*

Remark. In the original paper [13], it seems that R is assumed to be of analytically finite type, but it is easy to see that his proof works without the assumption that R is to be of analytically finite type.

Applying this theorem, we estimate the distance in $B(\Gamma)$ between two points arising from a quasiconformal deformation in a thin region of R . Specifically, we show the following:

Theorem 3.2. *Let R be a Riemann surface with $\text{short}(R) > 0$. For $[\mu_0], [\mu_1] \in \text{Teich}(R)$, there exists a curve $\mu_t : [0, 1] \rightarrow \text{Bel}(R)$ such that*

$$\nu_t := \lim_{h \rightarrow 0} \frac{1}{h} \text{bel} \left(f^{\mu_{t+h}} \circ (f^{\mu_t})^{-1} \right)$$

has its support contained in the region of $f^{\mu_t}(R)$ where the injectivity radius satisfies $L_t < 1/2$. Moreover, if there exists a constant k such that $|\nu_t| \leq k < 1$ for all $t \in [0, 1]$, then

$$\|\mathcal{B}([\mu_0]) - \mathcal{B}([\mu_1])\|_{B(R)} \leq C_1 \left(L_0 \log \frac{1}{L_0} \right)^2,$$

where C_1 depends on $\text{short}(S)$, and $K = \frac{1+k}{1-k}$.

Remark. In the paper [13], there is a part where the main result is implicitly proven using the statement of Theorem 3.2.

Proposition 3.3. *Let R, S be hyperbolic Riemann surfaces, and $f : R \rightarrow S$ be a K -quasiconformal map. Then, there exists a constant $c := c(K) > 1$ such that, for all $p, q \in R$,*

$$\min \left\{ \frac{1}{c} d_R(p, q), \left(\frac{1}{c} d_R(p, q) \right)^K \right\} \leq d_S(f(p), f(q)) \leq c \max \{ d_R(p, q), d_R(p, q)^{1/K} \},$$

where d_R and d_S denote the hyperbolic distances on R and S , respectively.

Proof. Gehring–Osgood [6, Theorem 3] obtained how the quasi-hyperbolic distance of the hyperbolic domain in plane changes under K -quasiconformal mapping. In more detail, let D_1, D_2 be simply connected hyperbolic domains in the plane and $f : D_1 \rightarrow D_2$ be a K -quasiconformal map, then there exists a constant $c := c(K) > 1$ such that, for all $x, y \in D_1$,

$$d_{q,D_2}(f(x), f(y)) \leq c \max\{d_{q,D_1}(x, y), d_{q,D_1}(x, y)^{1/K}\},$$

where d_{q,D_j} are a quasi-hyperbolic distance for $j = 1, 2$. Note that the quasi-hyperbolic distance of the upper half plane $d_{q,\mathbb{H}}$ coincide with the hyperbolic distance $d_{\mathbb{H}}$ from the definition.

Moreover, let $\pi_R : \mathbb{H} \rightarrow R$ be an universal cover of R , then

$$(3.1) \quad \inf_{\substack{\tilde{p} \in \pi^{-1}(p) \\ \tilde{q} \in \pi^{-1}(q)}} d_{\mathbb{H}}(\tilde{p}, \tilde{q}) = d_S(p, q).$$

Finally, since $d_{\mathbb{H}}$ and $d_{\mathbb{H}}^{1/K}$ are continuous,

$$\max\{d_{\mathbb{H}}, d_{\mathbb{H}}^{1/K}\}$$

is also continuous. Therefore,

$$(3.2) \quad \inf_{\substack{\tilde{p} \in \pi^{-1}(p) \\ \tilde{q} \in \pi^{-1}(q)}} \max\{d_{\mathbb{H}}(\tilde{p}, \tilde{q}), d_{\mathbb{H}}(\tilde{p}, \tilde{q})^{1/K}\} = \max\{d_S(p, q), d_S(p, q)^{1/K}\}.$$

From (3.1) and (3.2), we obtain

$$(3.3) \quad d_S(f(p), f(q)) \leq c \max\{d_R(p, q), d_R(p, q)^{1/K}\}.$$

The inequality on the left-hand side of the claim can be obtained by applying (3.3) to the inverse map of f . □

Proposition 3.4. *Let R, S be hyperbolic Riemann surfaces, and $f : R \rightarrow S$ be a K -quasiconformal map. Then there exists a constant $c' := c'(K) > 0$ such that, if $r_R(p) < 1$,*

$$r_R(p) \leq c' r_S(f(p))^{1/K},$$

where $r_R(p)$ and $r_S(f(p))$ are the injectivity radii at p and $f(p)$ with respect to hyperbolic metrics R and S , respectively.

Proof. Let $r' < r_R(p)$. The set $\{q \in R \mid d_R(p, q) < r'\}$ is simply connected. Since f is a homeomorphism, $f(\{q \in R \mid d_R(p, q) < r'\})$ is also simply connected. Now, if $f(q) \in f(\{q \in R \mid d_R(p, q) < r'\})$, then

$$d_S(f(p), f(q)) \geq \min\left\{\frac{1}{c}d_R(p, q), \left(\frac{1}{c}d_R(p, q)\right)^K\right\} = \left(\frac{1}{c}d_R(p, q)\right)^K,$$

from Proposition 3.3 and $\frac{1}{c}d_R(p, q) < 1$. Thus, set $r := r'^K/c$, then $\{z \in S \mid d_S(f(p), z) < r\}$ is simply connected. Therefore,

$$\frac{1}{c}r' \leq r_S(f(p)).$$

Hence, set $c' := c^{1/K}$, we obtain $r' \leq c' r_S(f(p))^{1/K}$. □

Proof of Theorem 3.2. By using the curve μ_t , it follows from standard calculations, Theorem 3.1 and Proposition 3.4 through the following computation.

$$\begin{aligned} \|\mathcal{B}([\mu_0]) - \mathcal{B}([\mu_1])\|_{B(S)} &\leq \int_0^1 \left\| \frac{d}{dt} \mathcal{B}([\mu_t]) \right\| dt \leq \int_0^1 \|d_{[\mu_t]} \mathcal{B}\| \|\nu_t\| dt \\ &\leq kC \int_0^1 \left(L_t \log \frac{1}{L_t} \right)^2 dt \\ &\leq kC \left(L_0^{1/K} \log \frac{1}{L_0} \right)^2 \leq \frac{kC}{4} \left(L_0 \log \frac{1}{L_0} \right)^2, \end{aligned}$$

where C is the constant given in Theorem 3.1. So, we set $C_1 := \frac{kC}{4}$

□

3.2. Variation of injectivity radius in stretch deformation. We denote by $\text{St}(H) := \{\zeta \in \mathbb{C} \mid |\text{Im } \zeta| < H\}$, $C(H) := \text{St}(H)/\langle x \mapsto x + 2\pi \rangle$, where $H > 0$. This cylinder is biholomorphic to $\{z \in \mathbb{C} \mid 1/e^H < |z| < e^H\}$. In particular, the modulus is H/π . The simple closed curve $[0, 2\pi] \ni x \mapsto x$ in $C(H)$ is called the core curve. Let $H > 0$ and $0 < a < 1$. In the following, we consider a quasiconformal deformation of $\text{St}(H)$ in which the subdomain $\text{St}(aH)$ is stretched by a factor of two and $\text{St}(H) \setminus \text{St}(aH)$ is shifted vertically. This quasiconformal deformation induces a deformation of $C(H)$. That is, the quasiconformal map $\psi_H : C(H) \rightarrow C((1+a)H)$ and homotopy $\psi_{H,t} : C(H) \rightarrow C((1+at)H)$ we are dealing with are as follows:

$$(3.4) \quad \psi_H : x + iy \mapsto \begin{cases} x + iy + aHi & aH \leq y < H \\ x + 2yi & |y| \leq aH \\ x + iy - aHi & -H < y \leq -aH \end{cases},$$

and, for each $t \in [0, 1]$, we define

$$(3.5) \quad \psi_{H,t} : x + iy \mapsto \begin{cases} x + iy + atHi & aH \leq y < H \\ x + (1+t)yi & |y| \leq aH \\ x + iy - atHi & -H < y \leq -aH \end{cases}.$$

We denote by $\text{bel}(\psi_{H,t})$ the Beltrami differential $(\psi_{H,t})_{\bar{z}}/(\psi_{H,t})_z$. Then,

$$(3.6) \quad \text{bel}(\psi_{H,t})(x + iy) = \begin{cases} -\frac{t}{2+t} & |y| \leq aH \\ 0 & \text{the other.} \end{cases}$$

and

$$(3.7) \quad \lim_{h \rightarrow 0} \frac{1}{h} \text{bel}(\psi_{H,t+h} \circ \psi_{H,t}^{-1})(x + iy) = \begin{cases} -\frac{1}{2+2t} & |y| \leq (1+t)aH \\ 0 & \text{the other.} \end{cases}$$

Note that this Beltrami coefficient and infinitesimal Beltrami coefficient are independent of the height H of the cylinder and the constant a .

Denote by ρ^t the hyperbolic metric on $C((1+at)H)$ and by $r_{\rho^t}(p)$ the injectivity radius of $C((1+at)H)$ at $p \in C((1+at)H)$.

Proposition 3.5. *Taking $a = 1/2$, if $H > 2\pi^2$, we obtain*

$$(3.8) \quad \sup_{p \in C((1+t)aH)} r_{\rho^t}(p) \leq 1/2,$$

for all $t \in [0, 1]$. Hence, the path $[0, 1] \ni t \mapsto \text{bel}(\phi_{H,t})$ satisfies the assumption of Theorem 3.2. In particular,

$$(3.9) \quad \frac{\sup_{p \in C(3H/4)} r_{\rho^1}(p)}{\sup_{p \in C(H/2)} r_{\rho^0}(p)} \leq \frac{2\sqrt{2}}{3} (< 1).$$

Similarly, we denote the lengths l_0, l_1 of the shortest simple closed geodesics in $C(H)$ and $C(3H/2)$, respectively, then we get

$$(3.10) \quad \frac{l_1}{l_0} = \frac{2}{3}.$$

Proof. Note that the hyperbolic metric on $C((1+at)H)$ is

$$\frac{\pi}{2(1+at)H \cos\left(\frac{\pi}{2(1+at)H} \text{Im } z\right)} |dz|.$$

First, considering the closed curve $g : [-\pi, \pi] \ni x \mapsto x + (1+t)aHi$, we get

$$(3.11) \quad \sup_{p \in C((1+t)aH)} r_{\rho^t}(p) \leq \frac{1}{2} \text{length}_{\rho^t}(g) = \frac{\pi^2}{2(1+at)H \cos\left(\frac{a(1+t)}{2(1+at)}\pi\right)}.$$

Since we take $a = 1/2$,

$$\frac{\pi^2}{2(1+at) \cos\left(\frac{a(1+t)}{2(1+at)}\pi\right)} = \frac{\pi^2}{(2+t) \cos\left(\frac{1+t}{4+2t}\pi\right)} \leq \pi^2,$$

for all $t \in [0, 1]$. Therefore, if $H > 2\pi^2$, then

$$\sup_{p \in C((1+t)aH)} r_{\rho^t}(p) \leq \frac{1}{2}.$$

Next, we estimate $\sup_{p \in C(aH)} r_{\rho^0}(p)$. Since the geodesic through $0 \in C(aH)$ with respect to ρ^0 is $g_0 : [-\pi, \pi] \ni x \mapsto x \in C(aH)$ and its length is closed shortest geodesic in $C(aH)$, we get

$$(3.12) \quad \frac{\pi^2}{2H} = \frac{1}{2} \text{length}_{\rho^0}(g_0) \leq \sup_{p \in C(aH)} r_{\rho^0}(p).$$

Moreover, for the same reason as (3.11), the following inequality is obtained

$$\sup_{p \in C(3H/4)} r_{\rho^1}(p) \leq \frac{1}{2} \text{length}_{\rho^1}(g_1),$$

where $g_1 : [-\pi, \pi] \ni x \mapsto x + \frac{3}{4}Hi$. Therefore,

$$(3.13) \quad \frac{\sup_{p \in C(3H/4)} r_{\rho^1}(p)}{\sup_{p \in C(H/2)} r_{\rho^0}(p)} \leq \frac{2H}{\pi^2} \cdot \frac{\pi^2}{3H \cos\left(\frac{\pi}{4}\right)} = \frac{2\sqrt{2}}{3}.$$

Finally, the shortest closed geodesics of $C(H)$ and $C(3H/2)$ are $[-\pi, \pi] \ni x \mapsto x$, so their lengths are π^2/H and $2\pi^2/3H$ respectively. Then, the ratio is $2/3$. $\square$

From Theorem 3.2 and Proposition 3.5, we get

$$(3.14) \quad \|\mathcal{B}([\text{bel}(\psi_{H,0})]) - \mathcal{B}([\text{bel}(\psi_{H,1})])\|_{B(C(H))} \leq C_1 \left(L_0 \log \frac{1}{L_0} \right)^2.$$

Moreover, (3.9) and (3.10) ensure that the assumptions of Theorem 3.2 are satisfied even when the same stretching deformation (3.6) is applied repeatedly. See Subsection 4.1.

4. CONSTRUCTION OF DAVID-FUCHSIAN B-GROUPS

We would like to construct a topological quasi-Fuchsian b-group on the Bers boundary by repeatedly using the inequality (3.14) for a family of cylinder domains. To do so, we

- investigate whether a sequence $([\tau_n])_{n \in \mathbb{N}}$ given by a stretch deformation converges in the topology that defines Bers boundary in Subsection 4.1,
- construct the Beltrami differential μ so that it satisfies the David condition in Subsection 4.2,
- show that the Schwarzian derivative $S(f_\mu)$ of the constructed David map f_μ is not an interior point in Subsection 4.3, and
- see that it is the limit μ of the sequence of points that can be constructed by repeatedly applying the stretch deformation (slightly modifying the above sequence) in Subsection 4.4.

4.1. The deformation of a family of cylinders. We show that the same assertion as Proposition 3.5 holds for a union of pairwise disjoint cylinder regions. Assume that R has infinitely many essential closed geodesics $\{\gamma_n^*\}_{n \geq 0}$. On Riemann surfaces, the following theorem gives a way to pick out cylinders from the family of essential simple closed curves.

Theorem 4.1 (The collar Lemma, [9, Theorem 3.8.3]). *Let R be a Riemann surface. For an essential simple closed curve γ in R , we denote the geodesic which is homotopic to γ as γ^* . For a positive $\delta > 0$, we define the domain $A_\delta(\gamma) := \{x \in R \mid d(x, \gamma^*) < \delta\}$. Then, the following holds: Considering an (even infinite) set of essential simple closed curves $\{\gamma_1, \gamma_2, \dots\}$, if each geodesic is disjoint from the others, then the domains $A_{h(l_i)}(\gamma_i)$ are biholomorphic to cylinders and are mutually disjoint, where l_i is the length of the geodesic γ_i^* and $h(x) = \frac{1}{2} \log \frac{\cosh(x/2)+1}{\cosh(x/2)-1}$.*

Remark. The height of $A_{h(l_i)}(\gamma_i)$ is $h(l_i)$. The function h is monotonically decreasing such that $h(l) \rightarrow \infty$ as $l \rightarrow 0$.

Let $\{\text{Cyl}_n\}_{n \geq 1}$ be a family of cylinder regions whose core curves are homotopic to γ_n^* . Here, each Cyl_n is not necessarily one obtained by applying Theorem 4.1. For simplicity, we assume that each Cyl_n is conformal to a cylinder $C(H)$ whose height is $H > 0$. We also assume that the injectivity radius of $C(H)$ is less than $1/2$. For each $n \geq 0$, we fix a holomorphic embedding $\eta_n : C(H) \rightarrow \text{Cyl}_n$. See Figure 1. Denote by U the union of the cylinders Cyl_n .

The sequence of quasiconformal deformations of $\{R_j\}_{j \in \mathbb{N}}$ is defined inductively as follows: First, for each Cyl_n , consider the quasiconformal deformation $f_t : R \rightarrow R_t$ determined by the Beltrami coefficient τ_t obtained by

$$\tau_1 := \begin{cases} (\eta_n)_*(\text{bel}(\psi_H)) & \text{on } \text{Cyl}_n \\ 0 & \text{on } R \setminus U \end{cases}, \quad \tau_t := \begin{cases} (\eta_n)_*(\text{bel}(\psi_{H,t})) & \text{on } \text{Cyl}_n \\ 0 & \text{on } R \setminus U \end{cases}.$$

Also, let $\text{Cyl}_{(n,t)} := f_t(\text{Cyl}_n)$ for each $t \in [0, 1]$. From the definition, the following holds.

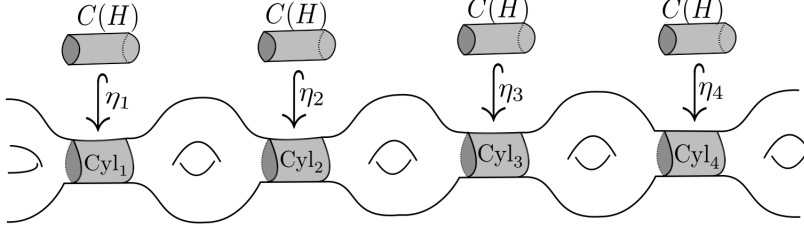


FIGURE 1

Claim 1. The map $\eta_n^{(t)} := f_t \circ \eta_n \circ \psi_{H,t}^{-1}$ is a holomorphic embedding from $\psi_{H,t}(C(H))$ to $\text{Cyl}_{(n,t)}$.

Proof. It is easy to see that $(\eta_n^{(t)})_{\bar{z}} = 0$ from the definitions. $\square$

The following can be proven.

Claim 2. The injectivity radius of each component in the support of

$$\lim_{h \rightarrow 0} \frac{1}{h} \text{bel}(f_{t+h} \circ f_t^{-1})$$

is less than $1/2$.

Proof. First,

$$\begin{aligned} \text{supp} \left(\lim_{h \rightarrow 0} \frac{1}{h} \text{bel}(f_{t+h} \circ f_t^{-1}) \right) &= \bigcup_{n \geq 0} \eta_n^{(t)} \left(\text{supp} \left(\lim_{h \rightarrow 0} \frac{1}{h} \text{bel}(\psi_{H,t+h} \circ \psi_{H,t}^{-1}) \right) \right) \\ &= \bigcup_{n \geq 0} \eta_n^{(t)}(\psi_{H,t}(C(H/2))). \end{aligned}$$

For each $x \in \eta_n^{(t)}(\psi_{H,t}(C(H/2)))$, from Schwarz–Pick’s lemma,

$$r_{\rho_{R_{\tau_t}}}(x) \leq r_{\rho^t}((\eta_n^{(t)})^{-1}(x)) \leq \sup_{p \in C((1+t)H/2)} r_{\rho^t}(p) \leq \frac{1}{2}.$$

The last inequality is from (3.8) in Proposition 3.5. $\square$

Next, we construct the j -th deformation. Consider a deformation of the standard cylinder $C(H)$ repeated j times. In detail, using the notation from Subsection 3.2, we take the quasiconformal map

$$(4.1) \quad \psi^{(j)} := \psi_{(3/2)^j H} \circ \cdots \circ \psi_{3H/2} \circ \psi_H.$$

In addition, we denote the injectivity radius in R_j of the support of $\text{bel}(\psi_{(3/2)^j H})$ as L'_j . From (3.9) in Proposition 3.5, for each $j \geq 0$, we get

$$(4.2) \quad L'_j \leq \frac{2\sqrt{2}}{3} L'_{j+1}.$$

Using the quasiconformal map $\psi^{(j)}$, we define

$$(4.3) \quad \tau_j := \begin{cases} (\eta_n)_*(\text{bel}(\psi^{(j)})) & \text{on } \text{Cyl}_n \ (n \geq j) \\ 0 & \text{on } R \setminus \bigcup_{n \geq j} \text{Cyl}_n \end{cases}.$$

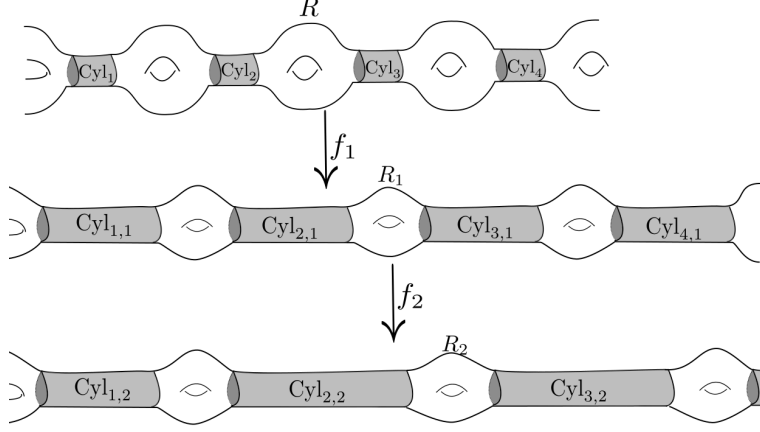


FIGURE 2

Note that we stop deforming Cyl_n for $j > n$. We write the quasiconformal map determined by τ_j as $f_j : R \rightarrow R_j$, the image of Cyl_n by f_j as $\text{Cyl}_{(n,j)}$, and the injectivity radius of the support of τ_j as L_j . See Figure 2. Moreover, for each $t \in [j-1, j]$, we define

$$(4.4) \quad \begin{aligned} \psi^{(t)} &:= \psi_{(3/2)^j H, t-(j-1)} \circ \psi_{(3/2)^{j-1} H} \circ \cdots \circ \psi_{3H/2} \circ \psi_H \\ &= \psi_{(3/2)^j H, t-(j-1)} \circ \psi^{(j-1)} \end{aligned}$$

and

$$(4.5) \quad \tau_t := \begin{cases} (\eta_n)_*(\text{bel}(\psi^{(t)})) & \text{on } \text{Cyl}_n \ (n \geq j) \\ 0 & \text{on } R \setminus \bigcup_{n \geq j} \text{Cyl}_n \end{cases}.$$

Similarly, we define R_t , f_t , and $\text{Cyl}_{(n,t)}$ for τ_t . At this time, the same thing as Claim 1 holds.

Claim 3. *Let $j \geq 0$. For each $0 \leq t \leq j$ and $n \geq j$, the map $\eta_n^{(t)} = f_t \circ \eta_n \circ (\psi^{(t)})^{-1}$ is a holomorphic embedding from $\psi^{(t)}(C(H))$ to $\text{Cyl}_{(n,t)}$.*

Proof. It is clear in the same way as Claim 1. □

Claim 4. *The injectivity radius L_{j-1} of each component of the support of*

$$\lim_{h \rightarrow 0} \frac{1}{h} \text{bel}(f_{t+h} \circ f_t^{-1})$$

is less than L'_{j-1} , where j is the natural number so that $j-1 \leq t \leq j$.

Proof. For each $t \in [0, \infty)$, we take $j \in \mathbb{N}$ with $j-1 \leq t \leq j$. Then, in the same way as Claim 4, we get

$$\begin{aligned} \text{supp} \left(\lim_{h \rightarrow 0} \frac{1}{h} \text{bel}(f_{t+h} \circ f_t^{-1}) \right) &= \bigcup_{n \geq 0} \eta_n^{(t)} \left(\text{supp} \left(\lim_{h \rightarrow 0} \frac{1}{h} \text{bel}(\psi_{(3/2)^j H, t+h} \circ \psi_{(3/2)^j H, t}^{-1}) \right) \right) \\ &= \bigcup_{n \geq 0} \eta_n^{(t)}(\psi_{(3/2)^j H, t}(C(H/2))), \end{aligned}$$

and, for each $x \in \eta_n^{(t)}(\psi_{(3/2)^j H, t}(C(H/2)))$,

$$r_{\rho_{R\tau_t}}(x) \leq r_{\rho^t}((\eta_n^{(t)})^{-1}(x)) \leq \sup_{p \in C\left(\frac{1+t}{2}\left(\frac{3}{2}\right)^{j-1}H\right)} r_{\rho^t}(p) \leq L'_{j-1},$$

from Schwarz–Pick’s lemma. Therefore $L_{j-1} \leq L'_{j-1}$

□

To summarize the above, we can see that the sequence $(\mathcal{B}([\tau_j]))_{j \geq 0}$ converges with respect to the norm of the Banach space $B(R)$. Indeed, for each $j \in \mathbb{N}$, we get

$$\|\mathcal{B}([\tau_j]) - \mathcal{B}([\tau_{j-1}])\|_{B(R)} \leq C_1 \left(L'_{j-1} \log \frac{1}{L'_{j-1}} \right)^2$$

from Theorem 3.2 and Claim 4. Since the function $s(x) = -x \log(x)$ is monotonically decreasing in the range $x > 1/e$, $L'_j \leq \frac{2\sqrt{2}}{3} L'_{j-1}$ and $L'_0 \leq 1/2$, we obtain

$$\begin{aligned} \sum_{j \geq 0} \|\mathcal{B}([\tau_j]) - \mathcal{B}([\tau_{j-1}])\|_{B(R)} &\leq C_1 \sum_{j \geq 0} \left(L'_{j-1} \log \frac{1}{L'_{j-1}} \right)^2 \\ (4.6) \quad &\leq C_1 \sum_{j \geq 0} \left(\frac{2\sqrt{2}}{3} \right)^{2j} \left(\log \left(\frac{3}{2\sqrt{2}} \right)^j \right)^2 \\ &\leq C_1 \left(\log \left(\frac{3}{2\sqrt{2}} \right) \right)^2 \sum_{j \geq 0} j^2 \left(\frac{2\sqrt{2}}{3} \right)^{2j} < \infty. \end{aligned}$$

Thus, the sequence $(\mathcal{B}([\tau_j]))_{j \in \mathbb{N}}$ converges. In the following subsection, we will discuss a modified version of this construction that provides the existence and uniqueness of solutions to the Beltrami equation corresponding to this limit, that is, the David condition in Definition 2.4.

4.2. Construction. As in Subsection 4.1, we assume the existence of infinitely many essential closed geodesics $\{\gamma_n^*\}_{n \geq 0}$. Let Γ be a Fuchsian model of R acting $\mathbb{D}$ with a universal covering map $\pi : \mathbb{D} \rightarrow R$ and ω be a fundamental domain of Γ in $\mathbb{D}$. For every n , we can find a doubly connected domain Cyl_n of R containing γ_n^* such that $\text{Cl}(\text{Cyl}_n)$ is compact in R and $n \neq m$ implies that $\text{Cl}(\text{Cyl}_n) \cap \text{Cl}(\text{Cyl}_m) = \emptyset$. For example, we can take a collar of γ_n^* . We denote by $T_m = \pi^{-1}(\text{Cyl}_m) \cap \omega$. Let $C(H)$ be a standard cylinder that is biholomorphic to Cyl_n , and let the holomorphic $\eta_n : C(H) \rightarrow \text{Cyl}_n$ be fixed.

Consider a positive sequence $(p_j)_{j \in \mathbb{N}}$ such that $\sum_{j \in \mathbb{N}} p_j < \infty$.

Lemma 4.2. *For any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|\bigcup_{m > N} \Gamma T_m|_{\text{Leb}} < \varepsilon$.*

Proof. Since $\mathbb{D} = \Gamma\omega \supset \Gamma(\bigcup_{m \in \mathbb{N}} T_m)$, one easily sees the Lemma. □

First, consider the following as the quasiconformal deformation of $C(H)$. In more detail, considering the quasiconformal deformation given by (4.4) in subsection 4.1, we define

$$\nu_j := \text{bel}(\psi^{(j)}),$$

for each $j \in \mathbb{N}$. Then, from Lemma 4.2, there exists a natural number $M(j)$ such that

$$|\Gamma T_{M(j)}|_{\text{Leb}} \cdot \exp(2K_j) < p_j,$$

where

$$K_j := \frac{1 + \|(\eta_{M(j)})_* \nu_j\|_{L^\infty}}{1 - \|(\eta_{M(j)})_* \nu_j\|_{L^\infty}}.$$

For simplicity, we select and renumber (γ_j^*) . In line with this, the corresponding cylinder $\text{Cyl}_{M(j)}$ and the region $T_{M(j)}$ should also have their subscripts replaced in the same way. This is defined so that the product of the maximum dilatation of the quasiconformal deformation and the area of the support in the universal covering is less than p_j when the deformation that repeats stretch deformation j times is defined on the j -th cylinder. Now, let μ be defined by

$$(4.7) \quad \begin{aligned} \tilde{\mu}(z) &:= \begin{cases} (\eta_j)_* \nu_j(\pi(z)) & \text{if } z \in T_j \\ 0 & \text{if } z \in R \setminus \bigcup_{j \in \mathbb{N}} \text{Cl}(T_j) \end{cases} \\ &= \sum_{j \in \mathbb{N}} (\eta_j)_* \nu_j(\pi(z)) \chi_{T_j}. \end{aligned}$$

Here χ_{T_j} is the characteristic function of T_j . The function $\tilde{\mu}$ has exponentially L^2 integrable distortion. That is, $\tilde{\mu}$ satisfies the assumptions of Theorem 2.7. Indeed, one can see from the following calculation:

$$(4.8) \quad \int_{\mathbb{D}} \exp(2K_{\tilde{\mu}}(z)) \, dx dy \leq e^2 \pi + \sum_{j \in \mathbb{N}} |\Gamma T_j|_{\text{Leb}} \cdot \exp(2K_j) \leq e^2 \pi + \sum_{j \in \mathbb{N}} p_j < \infty,$$

where

$$K_{\tilde{\mu}}(z) = \frac{1 + |\tilde{\mu}|}{1 - |\tilde{\mu}|}(z).$$

Therefore, if we extend $\tilde{\mu}$ to be identically zero in $\hat{\mathbb{C}} \setminus \text{Cl}(\mathbb{D})$, the Beltrami equation for $\tilde{\mu}$ has the principal solution $f_{\tilde{\mu}}$. Moreover, pushing μ of $\tilde{\mu}$ on R under the covering map $\mathbb{D} \rightarrow R$ induced the homeomorphism $f_{\mu} : R \rightarrow f_{\mu}(R)$. It is denoted by R_{μ} its image.

4.3. R_{μ} is not an interior point of $\text{Teich}(R)$. It follows immediately from the following theorem that it is not an interior point.

Theorem 4.3 ([18, Lemma 3.1]). *Let $f : R \rightarrow R$ be a K -quasiconformal map and γ be a closed curve on R . Denote by l the length of the hyperbolic geodesic in the free homotopy class of γ and by l' the length of the hyperbolic geodesic in the free homotopy class of $f(\gamma)$. Then*

$$l/K \leq l' \leq lK.$$

If R_{μ} is an interior point in $\text{Teich}(R)$, a K -quasiconformal map exists from R to R_{μ} . Then, from Theorem 4.3, it follows that the lengths of closed geodesics $f_{\mu}(\gamma_j)^*$ homotopic to $f_{\mu}(\gamma_j)$ in R_{μ} are greater than $\text{short}(R)/K (> 0)$. On the other hand, for each $n \in \mathbb{N}$, the length of hyperbolic geodesic of the shortest geodesic in $f_{\mu}(\text{Cyl}_j)$ is less than

$$\left(\frac{2}{3}\right)^{j-1} \frac{\pi^2}{H},$$

from (3.10) in Proposition 3.5 and Schwarz–Pick’s lemma. It means that

$$\lim_{j \rightarrow \infty} \text{length}_{\rho_{R_\mu}}(f_\mu(\gamma_j)^*) = 0.$$

It is a contradiction.

4.4. R_μ belongs to the Bers boundary. That R_μ belongs to the Bers boundary means that the Schwarzian derivative $S(f_{\tilde{\mu}})$ of the principal solution $f_{\tilde{\mu}}$ is on the Bers boundary. By the definition of the Bers boundary, it is sufficient to take a sequence (φ_n) in $\mathcal{B}(\text{Teich}(R))$ so that the sequence converges to φ_∞ in $B(R)$ and φ coincides with $S(f_{\tilde{\mu}})$. On the open set $U := \bigcup_{j \in \mathbb{N}} \text{Cyl}_j$, applying $\{\gamma_j^*\}$ to the discussion in Subsection 4.1, we can take the deformation of a family of cylinders $([\tau_j])_j$ in Subsection 4.1. The sequence converges in $\text{Teich}(R) \cup \partial_{\text{Bers}} \text{Teich}(R)$ from (4.6). Now, we denote $\mathcal{B}([\tau_j])$ by $\varphi_j \in B(R)$, and the limit of $\mathcal{B}([\tau_j])$ by φ_∞ . Since $S(W_{\varphi_j})$ is convergent to $S(W_{\varphi_\infty})$ with respect to the norm in $B(R)$, the sequence $(W_{\varphi_j})_j$ is local uniformly converges to W_{φ_∞} on the lower half plane $\mathbb{H}^*$.

Finally, we need to prove the sequence W_{φ_j} is local uniformly convergent to $f_{\tilde{\mu}}$. Indeed, from the uniqueness of the limit, we find $W_{\varphi_\infty} = f_{\tilde{\mu}}$. It is sufficient to prove that the sequence of $\pi^* \tau_j$ satisfies the assumptions of Proposition 2.8. For each j , the Beltrami coefficient τ_j induces the quasiconformal deformation in which Cyl_n is stretched j times for each $1 \leq n \leq j$ and Cyl_n is stretched n times for each $j < n$. From the definition, τ_j is coincide to $\tilde{\mu}$ in $\bigcup_{1 \leq n \leq j} \text{Cyl}_n$. Thus, the following statement holds: for each $z \in \pi^{-1}(U)$, there exists $j \in \mathbb{D}$ such that $z \in \Gamma(\bigcup_{1 \leq n \leq j} T_n)$, so $|\pi^* \tau_j(z) - \tilde{\mu}(z)| = 0$. Since $\text{supp}(\pi^* \tau_j), \text{supp}(\tilde{\mu}) \subset \pi^{-1}(U)$, the sequence $(\pi^* \tau_j)$ satisfies the assumptions of Proposition 2.8. Therefore, the sequence of the principal solutions of $\pi^* \tau_j$ is locally uniform convergent to $f_{\tilde{\mu}}$, from Proposition 2.8. That is, $f_{\tilde{\mu}}$ and W_{φ_∞} are the limit of the sequence (W_{φ_j}) . Hence $f_{\tilde{\mu}} = W_{\varphi_\infty}$. Therefore $S(f_{\tilde{\mu}}) = \varphi_\infty$. The following conclusions follow from the above:

Theorem 4.4. *Let R be a Riemann Surface of analytically infinite type with $\text{short}(R) > 0$ and infinitely many pairwise disjoint essential closed geodesics $\{\gamma_n^*\}_{n \geq 0}$ whose lengths are bounded. Then, there exists a David-Fuchsian b-group in the Bers boundary of $\text{Teich}(R)$.*

Proof. From the assumption that R has infinitely many pairwise disjoint essential closed geodesics $\{\gamma_n^*\}_{n \geq 0}$ whose lengths are bounded, there exists a quasiconformal mapping $f : R \rightarrow f(R)$ such that the set $\{f(\gamma_n^*)^*\}$ of geodesics homotopic to $f(\gamma_n^*)$ satisfies the conditions; their lengths are constant l and satisfy $h(l) > 2\sqrt{2}\pi^2$. In addition, $[f(R), f]$ is an element of $\text{Teich}(R)$. From the above, we can apply the discussion so far to $f(R)$ and the set $\{f(\gamma_n^*)^*\}$. □

Remark. The condition $\text{short}(R) > 0$ in Theorem 4.4 is essential to ensure the existence of a convergent sequence that approaches the point constructed on the Bers boundary. For instance, even if $\text{short}(R) = 0$, one may be able to construct a Beltrami differential $\mu \in L^\infty(\Gamma)$ satisfying the David condition, but proving that it lies on the Bers boundary appears to be nontrivial.

5. A HOLOMORPHIC FAMILY OF DAVID-FUCHSIAN B-GROUPS

5.1. Construct a holomorphic family of David-Fuchsian b-groups. The main purpose of this section is to prove the following:

Theorem 5.1. *Let R be a Riemann Surface of analytically infinite type with $\text{short}(R) > 0$ and infinitely many pairwise disjoint essential closed geodesics $\{\gamma_n^*\}_{n \geq 0}$ whose lengths are bounded. Then, there exists a unit ball D of a Banach space and a holomorphic map $F : D \rightarrow B(\Gamma)$, where $F(D)$ is contained in the Bers boundary $\partial_{\text{Bers}} \text{Teich}(R)$, and consists only of David-Fuchsian b-group, and the dimension of the tangent space of the image is infinite.*

Claim 5. *Let f be the principal solution to the Beltrami equation with coefficient μ which satisfies the assumption of Theorem 2.7. Then, for each $u \in W_{\text{loc}}^{1,2}(\mathbb{C})$, the composition $u \circ f^{-1}$ belongs to $W_{\text{loc}}^{1,1}(\mathbb{C})$.*

Proof. We will explain why the above argument is valid. First, from [8, Theorem 5.2], we get $f^{-1} \in W_{\text{loc}}^{1,1}(\mathbb{C})$ and there exists a measurable function $K_{f^{-1}} : \mathbb{C} \rightarrow [1, \infty)$ so that

$$|Df^{-1}|^2(z) \leq K_{f^{-1}}(z) J_{f^{-1}}(z) \text{ a.e.,}$$

where Df^{-1} is the matrix of partial derivative, $|Df^{-1}|$ is the operator norm and $J_{f^{-1}}$ is the Jacobian of f^{-1} . That is, f^{-1} is a mapping of finite distortion (see Remark of Theorem 2.5). Moreover, Gill [7, Theorem 1.1] has shown that we can assume $K \in L_{\text{loc}}^1(\mathbb{C})$. Finally, applying to [8, Theorem 5.13] with $p = 1, q = 2$, we get, for each $u \in W_{\text{loc}}^{1,2}(\mathbb{C})$, the composition $u \circ f^{-1}$ is in $W_{\text{loc}}^{1,1}(\mathbb{C})$. $\square$

Proof of Theorem 5.1. To prove Theorem 5.1, we use partial deformations on R . Applying the same arguments as in the proof of Theorem 4.4, we may assume that the lengths of the geodesics $\{\gamma_n^*\}_{n \geq 0}$ are constant. For each n , we consider the subcylinder $\widetilde{C}_n$ of the collar γ_n^* , whose center is γ_n^* and height is $h(l)/2$. Since the lengths are constant, their modulus are constant. Let R_2 be the union of the tubular neighborhoods $\{\widetilde{C}_n\}$ of $\{\gamma_n^*\}_{n \in \mathbb{N}}$. Applying $\{\gamma_n^*\}_{n \geq 0}$ to the discussion in Subsection 4.2, we take the Beltrami differential μ which induces a David-Fuchsian b-group. From the construction of μ , the support of μ is contained in R_2 . Then, the complex structure of $R_1 := R \setminus R_2$ can still be deformed. Moreover, the deformation space of R_1 is still an infinite-dimension space. By deforming the complex structure on R_1 , we construct a map such that Theorem 5.1 is satisfied.

$$(5.1) \quad P^\mu : \text{Bel}(R_1) \ni \tau \mapsto S(f_{(\mu+\tau)}) \in B(R).$$

Here, $\mu + \tau$ is precisely $\mu \chi_{R_2} + \tau \chi_{R_1}$. These Beltrami differentials satisfy the David condition. Indeed, let $\pi : \mathbb{D} \rightarrow R$ be an universal cover, and $\tilde{\mu}$ and $\tilde{\tau}$ be lifts of μ and τ , respectively. Set

$$K_{\tilde{\mu}+\tilde{\tau}}(z) := \frac{1 + |\tilde{\mu} + \tilde{\tau}|}{1 - |\tilde{\mu} + \tilde{\tau}|}.$$

We get

$$\begin{aligned}
& \int_{\mathbb{D}} \exp(2K_{\tilde{\mu}+\tilde{\tau}}(z)) \, dxdy \\
& \leq e^2\pi + \left(\int_{\text{supp}(\tilde{\mu})} + \int_{\text{supp}(\tilde{\tau})} \right) \exp(2K_{\tilde{\mu}+\tilde{\tau}}(z)) \, dxdy \\
& \leq e^2\pi + \int_{\text{supp}(\tilde{\mu})} \exp(2K_{\tilde{\mu}}(z)) \, dxdy + \int_{\text{supp}(\tilde{\tau})} \exp(2K_{\tilde{\tau}}(z)) \, dxdy < \infty.
\end{aligned}$$

However, since the supports of μ and τ are already included in R_2 and R_1 , respectively, the characteristic functions are omitted hereafter. The image of P^μ is almost evident that the image of this map is contained in the Bers boundary. Indeed, we consider the sequence $\{\tau_j + \tau\}$ with the sequence τ_j converging to μ in Subsection 4.2. From Proposition 2.8, we can prove $S(f_{\mu+\tau})$ lies on the Bers boundary. In addition, the Fréchet differentiability of P^μ at other points in $\text{Bel}(R_1)$ can be computed in the same manner as the derivative of the standard Bers embedding. Therefore, to complete the proof of Theorem 5.1, we shall prove that P^μ is Fréchet derivatable at $0 \in \text{Bel}(R_1)$ and the image of its tangent map is an infinite-dimensional subspace in $T_{S(f_\mu)}B(R)$. From the construction (4.7), the Beltrami differential μ satisfies the exponential integrability condition (4.8), so $f_{\mu+\tau}$ is in $W_{\text{loc}}^{1,2}(R)$ from Theorem 2.7. Therefore, the tangent map at μ on $\text{Bel}(R_1)$ is represented by

$$(5.2) \quad d_0 P^\mu[\nu](z) = -\frac{6}{\pi} (f_{\mu+0})_z^2(z) \int_R \frac{\nu(\zeta) \cdot (f_{\mu+0})_z^2(\zeta)}{(f_{\mu+0}(\zeta) - f_{\mu+0}(z))^4} \, d\xi d\eta, \quad (\zeta = \xi + i\eta)$$

in the same manner as the calculation for Bers embedding. For detailed calculations, see Claim 6 and Claim 7 given later. Briefly stated, since $f_{\mu+0}$ belongs to $W_{\text{loc}}^{1,2}(\mathbb{C})$, the coordinate transformation by the function $f_{\mu+0}$ can be done by multiplying the Jacobian of $f_{\mu+0}$ by the function under integration. In Subsection 5.2, we discuss the image of the tangent map of P^μ . □

Claim 6. *Let R be a Riemann surface, R_2 its sub-surface (but not necessarily connected), and $R_1 := R \setminus R_2$, and let μ be the Beltrami differential whose support is contained in R_2 , and the Beltrami equation of μ has the homeomorphic solution f_μ . Suppose the homeomorphic solution is an element of $W_{\text{loc}}^{1,2}(R)$. Then the map defined in (5.1)*

$$P^\mu : \text{Bel}(R_1) \ni \tau \mapsto S(f_{\mu+\tau}) \in B(R).$$

is differentiable, and its derivative at $0 \in \text{Bel}(R_1)$ is the formula (5.2):

$$d_0 P^\mu[\nu](z) = -\frac{6}{\pi} (f_{\mu+0})_z^2(z) \int_R \frac{\nu(\zeta) \cdot (f_{\mu+0})_z^2(\zeta)}{(f_{\mu+0}(\zeta) - f_{\mu+0}(z))^4} \, d\xi d\eta.$$

Proof. The following calculations are well known when μ is in the $\text{Bel}(\Gamma)$ (cf. [10, Theorem 6.11]).

We will use the notation f_μ instead of $f_{\mu+0}$ to simplify the notation. For the sake of proof, everything is lifted by the universal covering $\pi : \mathbb{D} \rightarrow R$ with Γ in the Fuchsian group model. The lifts of μ , R_1 and R_2 by the projection π are written as $\tilde{\mu}$, $\tilde{R}_1$ and $\tilde{R}_2$, respectively. Let $\tilde{\nu}$ be an element in $L^\infty(\Gamma)$ and its support be

contained in $\tilde{R}_1$. We will show that the limit

$$\lim_{t \rightarrow 0} \frac{1}{t} (S(f_{t\tilde{\nu}+\tilde{\mu}}) - S(f_{\tilde{\mu}}))$$

exists, where $f_{t\tilde{\nu}+\tilde{\mu}}$ and $f_{\tilde{\mu}}$ are the principal solution of the Beltrami equation of $t\tilde{\nu} + \tilde{\mu}$ and $\tilde{\mu}$, respectively. We define $\tilde{\mu}_t := \tilde{\mu} + t\tilde{\nu}$. Since $\text{supp}(\tilde{\mu}) \cap \text{supp}(\tilde{\nu}) = \emptyset$, $\tilde{\mu}_t$ satisfies the David condition, so we denote the principal homeomorphism solution of the Beltrami equation of $\tilde{\mu}_t$ by $\tilde{f}_t$. Moreover, set $g_t := f_t \circ f_{\tilde{\mu}}^{-1}$, we get $g_t \in W_{\text{loc}}^{1,1}(\mathbb{C})$ from Claim 5 and

$$\begin{aligned} \frac{(f_{\tilde{\mu}})_z}{(f_{\tilde{\mu}})_z} \frac{\tilde{\mu}_t - \tilde{\mu}}{1 - \tilde{\mu}\tilde{\mu}_t} \circ f_{\tilde{\mu}}^{-1} &= \frac{(f_{\tilde{\mu}})_z}{(f_{\tilde{\mu}})_z} \cdot \frac{t\tilde{\nu}}{1 - (t\tilde{\nu} \cdot \tilde{\mu} + |\tilde{\mu}|^2)} \circ f_{\tilde{\mu}}^{-1} \\ &= \frac{(f_{\tilde{\mu}})_z}{(f_{\tilde{\mu}})_z} \cdot t\tilde{\nu} \circ f_{\tilde{\mu}}^{-1}. \end{aligned}$$

So, the norm of the Beltrami differential of g_t is less than 1, for sufficiently small. Therefore, g_t is quasiconformal with respect to the complex structure of $f_{\tilde{\mu}}(\hat{\mathbb{C}})$. Applying g_t to the standard argument (cf. [10, Theorem 6.10]), we can show that the following limit exists:

$$\lim_{t \rightarrow 0} \frac{1}{t} S(g_t|_{f_{\tilde{\mu}}(\mathbb{D}^*)}).$$

In addition, we get the represent formula:

$$\lim_{t \rightarrow 0} \frac{1}{t} S(g_t|_{f_{\tilde{\mu}}(\mathbb{D}^*)})(z) = -\frac{6}{\pi} \int_{f_{\tilde{\mu}}(\mathbb{D})} \frac{L^\mu[\nu](w)}{(w - f_{\tilde{\mu}}(z))^4} dudv \quad (w = u + iv),$$

where

$$(5.3) \quad L^\mu[\nu] = \frac{(f_{\tilde{\mu}})_z}{(f_{\tilde{\mu}})_z} \cdot \tilde{\nu} \circ f_{\tilde{\mu}}^{-1}.$$

Note that $\text{supp}(\tilde{\mu}) \cap \text{supp}(\tilde{\nu}) = \emptyset$. Since $\text{bel}(g_t) = tL^\mu[\nu] + \delta(t)$ satisfies that $\|\delta(t)\|_{L^\infty} \rightarrow 0$ as $t \rightarrow 0$, we get

$$S(f_{t\tilde{\nu}+\tilde{\mu}}) = S(g_t|_{f_{\tilde{\mu}}(\mathbb{D}^*)})(f_{\tilde{\mu}})_z^2 + S(f_{\tilde{\mu}}).$$

Thus we obtain

$$\lim_{t \rightarrow 0} \frac{1}{t} (\beta(t\tilde{\nu} + \tilde{\mu}) - \beta(\tilde{\mu})) = -\frac{6}{\pi} \int_{f_{\tilde{\mu}}(\mathbb{D})} \frac{L^\mu[\nu](w)}{(w - f_{\tilde{\mu}}(z))^4} dudv \cdot (f_{\tilde{\mu}})_z^2$$

Finally, by substituting $f_{\tilde{\mu}}(\zeta)$ for w in the above integral, we obtain the representation formula (5.2). Note that from the assumption that $f_{\tilde{\mu}}$ is an element of $W_{\text{loc}}^{1,2}$, substitution by $f_{\tilde{\mu}}$ can be computed using the Jacobian of $f_{\tilde{\mu}}$, and that is

$$|(f_{\tilde{\mu}})_z|^2 - |(f_{\tilde{\mu}})_{\bar{z}}|^2.$$

□

5.2. Image of the tangent map d_0P^μ . The remaining tasks to prove are: first that d_0P^μ is a bounded linear operator, and second, that the image of d_0P^μ is infinite-dimensional. From the definition of the tangent space in $L^\infty(R)$,

$$T_0\text{Bel}(R_1) \cong L^\infty(R_1, R),$$

where

$$L^\infty(R_1, R) = \{\mu \in L^\infty(R) \mid \mu = \chi_{R_1}\mu\}.$$

Next, from (5.2), we get the following claim:

Claim 7. *The operator $d_0 P^\mu$ is bounded.*

Proof. Let $C \subset \hat{\mathbb{C}}$ be a Jordan curve and $\Omega \cup \Omega^* = \hat{\mathbb{C}} \setminus C$ be connected decomposition. In addition, if necessary, we apply a Möbius transformation to make $\infty \in C$. For each $\zeta \in \Omega^*$,

$$(5.4) \quad \int_{\Omega} \left| \frac{1}{z - \zeta} \right|^4 dx dy \leq 4^2 \pi \rho_{\Omega^*}^2(\zeta),$$

where ρ_{Ω^*} is the hyperbolic metric on Ω^* . Indeed, we obtain

$$(5.5) \quad \begin{aligned} \int_{\Omega} \left| \frac{1}{z - \zeta} \right|^4 dx dy &\leq \int_{\hat{\mathbb{C}} \setminus \Delta(0; \text{dist}(\zeta; C))} \frac{1}{|z|^4} dx dy \\ &\leq 2\pi \int_{\text{dist}(\zeta; C)}^{\infty} \frac{1}{r^3} dr \\ (5.6) \quad &\leq \pi (\text{dist}(\zeta; C))^{-2}. \end{aligned}$$

In addition, from Koebe's 1/4 Theorem, $\text{dist}(\zeta; C)^{-1} \geq 4\rho_{\Omega^*}$. Thus, we get (5.4). Therefore, for each $\nu \in d_0 \Lambda_\mu(T_0(\text{Bel}(R_1)))$,

$$\begin{aligned} \frac{\pi}{6} \|d_\mu \beta(\nu)\|_{T_{S(f_\mu(R))} B(R)} &= \sup_{z \in \mathbb{D}^*} \left| \int_{f_\mu(\mathbb{D})} \frac{L^\mu[\nu](w)}{(w - f_\mu(z))^4} dudv \cdot (f_\mu)_z^2(z) \cdot \rho_{\mathbb{D}^*}^{-2}(z) \right| \\ &= \sup_{\zeta \in f_\mu(\mathbb{D}^*)} \left| \int_{f_\mu(\mathbb{D})} \frac{L^\mu[\nu](w)}{(w - \zeta)^4} dudv \cdot \rho_{f_\mu(\mathbb{D}^*)}^{-2}(\zeta) \right| \\ &\leq 4^2 \pi \|L^\mu[\nu]\|_{L^\infty} = 4^2 \pi \|\nu\|_{L^\infty}. \end{aligned}$$

□

Finally, we only need to consider

$$(5.7) \quad d_0 P^\mu(T_0 \text{Bel}(R_1)) \cong \left(\frac{L^\infty(R_1, R)}{\ker(d_0 P^\mu) \cap T_0 \text{Bel}(R_1)} \right).$$

It is sufficient to prove that the right side of (5.7) is an infinite-dimensional closed subspace. From (5.2) and [10, Chapter 7.2.3], we obtain

$$\ker(d_0 P^\mu) = \left\{ \nu \in L^\infty(R) \left| \int_{f_\mu(R)} L^\mu(\nu) \cdot \varphi d\xi d\eta = 0 (\forall \varphi \in Q(f_\mu(R_1), f_\mu(R))) \right. \right\},$$

where

$$Q(f_\mu(R_1), f_\mu(R)) = \{ \varphi \in Q(f_\mu(R_1)) \mid \exists \Phi \in Q(f_\mu(R)) \text{ s.t. } \Phi|_{f_\mu(R_1)} = \varphi \}.$$

For our purpose, we use the well-known fact:

Theorem 5.2 ([10, Theorem 7.5 and Chapter 7.2.3]). *The mapping $L^\infty(f_\mu(R_1)) \rightarrow A^*(f_\mu(R_1))$ given by*

$$\Theta_\mu : Q(f_\mu(R_1)) \ni \varphi \mapsto \int_{f_\mu(R)} \mu \varphi d\xi d\eta \in \mathbb{C}$$

induces an isomorphism from $L^\infty(f_\mu(R_1))/N(f_\mu(R_1))$ onto $Q^(R_1)$, where*

$$N(R_1) := \left\{ \mu \in L^\infty(R_1) \left| \int_{f_\mu(R_1)} L^\mu(\mu) \cdot \varphi d\xi d\eta = 0 (\forall \varphi \in Q(f_\mu(R_1))) \right. \right\}.$$

We find $\ker(d_\mu\beta)$ is a closed subspace in $N(f_\mu(R_1))$ from the definition. Moreover, $Q(f_\mu(R_1), f_\mu(R))$ is also a subspace of $Q(f_\mu(R_1))$, but is not generally closed. Then, there exists the natural surjection ι^* from $Q^*(f_\mu(R_1))$ to $Q^*(f_\mu(R_1), f_\mu(R))$. Note that $Q^*(f_\mu(R_1), f_\mu(R))$ is a Banach space, and ι^* is a bounded operator. As above, from the universal property of the cokernel and the Hahn–Banach theorem, we get the following isomorphism in the sense of Banach spaces:

$$\begin{array}{ccc} \frac{\ker(d_0 P^\mu) \cap T_0 \text{Bel}(R_1)}{N(R_1)} \hookrightarrow L^\infty(R_1, R)/N(R_1) & \xrightarrow{\text{coker}} & \frac{L^\infty(R_1, R)}{\ker(d_0 P^\mu) \cap T_0 \text{Bel}(R_1)} \\ \Downarrow \Theta & \circlearrowright & \Downarrow \exists \tilde{\Theta} \\ Q^*(f_\mu(R_1)) & \xrightarrow{\iota^*} & Q^*(f_\mu(R_1), f_\mu(R)) \end{array}$$

Moreover, there exists a natural linear isomorphism:

$$(5.8) \quad \eta : Q(f_\mu(R)) \ni \Phi \mapsto \Phi|_{f_\mu(R_1)} \in Q(f_\mu(R_1), f_\mu(R))$$

from the identity theorem. So we obtain $\dim Q(f_\mu(R_1), f_\mu(R)) = \infty$. (Remark that η is not generally bounded. If $\mu = 0$, it becomes bounded. See Subsection 5.3) It means that $d_0 P^\mu(T_0 \text{Bel}(R_1))$ is an infinite dimensional closed space. This completes the proof of Theorem 5.1.

5.3. The boundedness of the operator η . Let R be a Riemann surface of analytically infinite type, and let R_1 be an open subset of R . In this Subsection, we consider the sufficient conditions on R and R_1 for the operator

$$\eta : Q(R) \ni \Phi \mapsto \Phi|_{R_1} \in Q(R_1, R)$$

to be bounded. The following is the results.

Theorem 5.3. *Suppose that R is of analytically infinite type with $\text{short}(R) > 0$ and infinitely many pairwise disjoint essential simple closed geodesics $\{\gamma_n^*\}$. Let R_1 be a complement of an union of tubular neighbourhoods $\tilde{C}_n$ of γ_n^* whose core curve coincides with γ_n^* . If there exist positive numbers m and M so that*

$$m < H(\tilde{C}_n)/H(C_n) < M,$$

where C_n is the collar neighbourhood γ_n^ , then there exists a constant C_q depending on R and R_1 such that,*

$$(\|\Phi\|_{L^1(R_1)} =) \int_R |\Phi| \leq C_q(R) \int_{R_1} |\Phi|,$$

for all integrable quadratic differentials Φ on R , that is the operator

$$\eta : Q(R) \ni \Phi \mapsto \Phi|_{R_1} \in Q(R_1, R),$$

is bounded.

The following lemma is a crucial complement to the proof of Theorem 5.3.

Lemma 5.4 (Puddings Lemma). *Let $\mathbb{A} = \{1 < |z| < R\}$ and $\tilde{\mathbb{A}} = \{r_1 < |z| < r_2\}$ with $1 < r_1 < r_2 < R$. Then, there exists a constant $C_a = C(r_1, r_2, R) > 0$ such that*

$$\|f\|_{L^1(\tilde{\mathbb{A}})} \leq C_a \|f\|_{L^1(\mathbb{A} \setminus \tilde{\mathbb{A}})},$$

for each a holomorphic function $f : \mathbb{A} \rightarrow \mathbb{C}$.

Proof. Let $1 < r_0 < r_1$ and $r_2 < r_3 < R$. Then, we obtain, for each $z = re^{i\theta}$ in $\mathbb{D}_{r_2} \setminus \text{Cl}(\mathbb{D}_{r_1})$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}_\alpha + \partial\mathbb{D}_\beta} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $r_0 < \beta < r_1, r_3 < \alpha < R$. Hence,

$$\begin{aligned} |f(z)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \alpha \frac{|f(\alpha e^{i\eta})|}{|\alpha e^{i\eta} - re^{i\theta}|} d\eta + \frac{1}{2\pi} \int_0^{2\pi} \beta \frac{|f(\beta e^{i\eta})|}{|\beta e^{i\eta} - re^{i\theta}|} d\eta, \\ &\leq \frac{1}{2\pi} \frac{1}{\alpha - r} \int_0^{2\pi} \alpha |f(\alpha e^{i\eta})| d\eta + \frac{1}{2\pi} \frac{1}{r - \beta} \int_0^{2\pi} \beta |f(\beta e^{i\eta})| d\eta. \end{aligned}$$

Next, both sides are multiplied by r and integrated.

$$\begin{aligned} \int_{r_1}^{r_2} |f(z)| r dr &\leq \frac{1}{2\pi} \int_{r_1}^{r_2} \frac{r}{\alpha - r} dr \int_0^{2\pi} \alpha |f(\alpha e^{i\eta})| d\eta + \frac{1}{2\pi} \int_{r_1}^{r_2} \frac{r}{r - \beta} dr \int_0^{2\pi} \beta |f(\beta e^{i\eta})| d\eta, \\ &\leq \frac{1}{2\pi} \int_{r_1}^{r_2} \frac{r}{r_3 - r} dr \int_0^{2\pi} \alpha |f(\alpha e^{i\eta})| d\eta + \frac{1}{2\pi} \int_{r_1}^{r_2} \frac{r}{r - r_0} dr \int_0^{2\pi} \beta |f(\beta e^{i\eta})| d\eta, \\ (5.9) \quad &\leq \frac{C(r_1, r_2, r_3)}{2\pi} \int_0^{2\pi} \alpha |f(\alpha e^{i\eta})| d\eta + \frac{C(r_0, r_1, r_2)}{2\pi} \int_0^{2\pi} \beta |f(\beta e^{i\eta})| d\eta, \end{aligned}$$

where

$$C(r_1, r_2, r_3) := \int_{r_1}^{r_2} \frac{r}{r_3 - r} dr, \quad C(r_0, r_1, r_2) := \int_{r_1}^{r_2} \frac{r}{r - r_0} dr.$$

Third, integrate (5.9) from 0 to 2π for θ .

$$(5.10) \quad 2\pi \|f\|_{L^1(\tilde{\mathbb{A}})} \leq C(r_1, r_2, r_3) \int_0^{2\pi} \alpha |f(\alpha e^{i\eta})| d\eta + C(r_0, r_1, r_2) \int_0^{2\pi} \beta |f(\beta e^{i\eta})| d\eta.$$

Finally, integrate (5.10) from r_3 to R with respect to α and from r_0 to r_1 with respect to β , we get

$$2\pi \|f\|_{L^1(\tilde{\mathbb{A}})} \leq (C(r_1, r_2, r_3)(r_1 - r_0) + C(r_0, r_1, r_2)(R - r_3)) \|f\|_{L^1(\mathbb{A} \setminus \tilde{\mathbb{A}})}.$$

To make a constant C that depends on $1, r_1, r_2$, and R , we can take r_0 and r_3 as, for example, $\frac{1+r_1}{2}$ and $\frac{r_2+R}{2}$, respectively. Further calculations lead to

$$\begin{aligned} &C(r_1, r_2, r_3)(r_1 - r_0) + C(r_0, r_1, r_2)(R - r_3) \\ &= -\frac{r_1 - 1}{2} \cdot \frac{r_2 + R}{2} \log \frac{r_3 - r_2}{r_3 - r_1} + \frac{R - r_2}{2} \cdot \frac{1 + r_1}{2} \log \frac{r_2 - r_0}{r_1 - r_0} \\ &\leq R^2 \log \frac{2r_2 - 1 - r_1}{2r_1 - 1 - r_1} \leq R^2 \log \frac{2r_2}{r_1 - 1}. \end{aligned}$$

Therefore, $C_a := \frac{1}{2\pi} R^2 \log \frac{2r_2}{r_1 - 1}$.

□

Remark. The constant C_a in the above Proposition 5.4 depends on the modulus of the annulus $\mathbb{A}$ and the annulus $\tilde{\mathbb{A}}$ and their relative positions. For example, if the core curves of $\mathbb{A}$ and $\tilde{\mathbb{A}}$ coincide, we know it depends on the modulus ratio.

Proof of Theorem 5.3. For each m , let Cyl_m be the collar of γ_m^* , then R_1 contains $\text{Cyl}_m \setminus \tilde{C}_m$. From Proposition 5.4, by using local coordinates if necessary, there exists a constant C_a such that

$$\|\Phi\|_{L^1(\tilde{C}_j)} \leq C_a \|\Phi\|_{L^1(C_j \setminus \tilde{C}_j)},$$

for each $j \in \mathbb{N}$. Therefore, we get

$$\begin{aligned} \int_R |\Phi| &= \left(\int_{R_1} + \int_{R_2} \right) |\Phi| = \int_{R_1} |\Phi| + \sum_j \int_{\tilde{C}_j} |\Phi|, \\ &\leq \int_{R_1} |\Phi| + C_a \sum_j \int_{\text{Cyl}_j \setminus \tilde{C}_j} |\Phi| = \int_{R_1} |\Phi| + C_a \int_{\bigcup \text{Cyl}_j \setminus \tilde{C}_j} |\Phi|, \\ &\leq \int_{R_1} |\Phi| + C_a \int_{R_1} |\Phi| = (C_a + 1) \int_{R_1} |\Phi|. \end{aligned}$$

Thus, we define C_q as $C_a + 1$. $\square$

6. COMPARISON WITH PREVIOUS STUDIES

Theorem 6.1 ([19, Theorem 3.1]). *Let $R \cong \mathbb{H}/\Gamma$ be a Riemann surface whose Fuchsian group is finitely generated Fuchsian group of the second kind. If $\psi \in \partial_{\text{Bers}} \text{Teich}(S)$ is not a cusp, then there exists a finitely generated quasi-Fuchsian group of the first kind G such that the limit set of G contains the limit set of $\chi_\psi(\Gamma)$.*

Theorem 6.1 is similar to Theorem 5.1, but the assumptions regarding the second kind and finite generation are different. In the case of the second kind, the free boundary part creates a gap in the limit set, and by connecting it appropriately, a first-kind finitely generated quasi-Fuchsian group is constructed.

Compare the following assertions about finite type Riemann surfaces and infinite generated Fuchs groups. In an earlier work by Tanigawa[16], the following assertion was considered:

Let R be a Riemann surface such that the Fuchsian group of R is infinitely generated of the first kind. Then, there exists a 1-dimensional analytic subset in $\partial_{\text{Bers}} \text{Teich}(R)$ consisting of Kleinian groups whose limit set is homeomorphic to S^1 .

The statement is similar to our Theorem 5.1. The results are similar because the limit set of David-Fuchsian b-group is homeomorphic to S^1 . However, in the present article, we have chosen a different way of proof to establish Theorem 5.1. The main reason is that a claim [16, Proposition 4.4] on the existence of non-tangential limit for holomorphic maps from $\mathbb{D}$ to $\text{Teich}(R)$, utilized in the earlier work, has a counterexample below. Therefore, the author must figure out how to prove the aforementioned statement now.

Eg. Let $f : \mathbb{D} \rightarrow \text{Teich}(\mathbb{D})$ be a holomorphic map into the universal Teichmüller space defined by $f(\lambda) = [\lambda]$. Here, $\lambda \in \mathbb{D}$ is identified with the function on $\mathbb{D}$ taking constant value λ . Then, f does not have a non-tangential limit at any point in $\partial\mathbb{D}$.

Proof. For each $\lambda \in \mathbb{D}$, the homeomorphism f_λ on $\hat{\mathbb{C}}$ corresponds to λ via the quasiconformal map $z \mapsto z + \lambda\bar{z}$ for $z \in \mathbb{D}$ and $z \mapsto z + \lambda/z$ for $z \in \mathbb{C} \setminus \text{Cl}(\mathbb{D})$. Hence its schwarzian derivative $S(f_\lambda)$ is

$$\frac{-6\lambda}{(z^2 - \lambda)^2}.$$

We get

$$\begin{aligned}
\|S(f_\lambda) - S(f_1)\|_{B(\mathbb{D})} &= 6 \sup_{z \in \hat{\mathbb{C}} \setminus \text{Cl}(\mathbb{D})} \left| \left(\frac{(\lambda - 1)(z^4 - \lambda)}{(z^2 - \lambda)^2(z^2 - 1)^2} \right) (|z|^2 - 1)^2 \right| \\
&= 6|\lambda - 1| \sup_{z \in \hat{\mathbb{C}} \setminus \text{Cl}(\mathbb{D})} \left| \frac{z^4 - \lambda}{(z^2 - \lambda)^2} \right| \left| \frac{(|z|^2 - 1)^2}{(z^2 - 1)^2} \right| \\
&\geq 6|\lambda - 1| \sup_{x > 1} \left| \frac{x^4 - \lambda}{(x^2 - \lambda)^2} \right|.
\end{aligned}$$

Moreover, $h_\lambda(x) := \frac{x^4 - \lambda}{(x^2 - \lambda)^2}$ ($\lambda \in [0, 1), x \in (1, \infty)$) is bounded below by $1/|1 - \lambda|$. Indeed, since

$$\begin{aligned}
h'_\lambda(x) &= \frac{4x^3(x^2 - \lambda)^2 - 4x(x^4 - \lambda)(x^2 - \lambda)}{(x^2 - \lambda)^4} \\
&= \frac{4\lambda x(1 - x^2)}{(x^2 - \lambda)^3},
\end{aligned}$$

h_λ has extrema at $x = 0, \pm 1$. From $h'(x) < 0$ ($\forall x > 1$), it follows that h_λ has maximum value at $x = 1$, hence

$$\|S(f_\lambda) - S(f_1)\|_{B(\mathbb{D})} \geq 6|\lambda - 1| \sup_{x > 1} |h_\lambda(x)| = 6|\lambda - 1|h_\lambda(1) = 6.$$

Finally, for each $e^{i\theta} \in \partial\mathbb{D}$, composing the rotation $z \mapsto e^{-i\theta}z$ with $f_{re^{i\theta}}$, we can prove that

$$\|S(f_{re^{i\theta}}) - S(f_{e^{i\theta}})\|_{B(\mathbb{D})} \geq 6$$

in the same manner. □

REFERENCES

- [1] W. ABIKOFF, On boundaries of Teichmüller spaces and on Kleinian groups III, *Acta Math.* 134 (1975), 211-234.
- [2] R. ACCOLA, Invariant domains for Kleinian groups, *Amer. J. Math.* 88 (1966), 329-336.
- [3] K. ASTALA, T. IWANIEC AND G. MARTIN, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Mathematical Series 48, Princeton University Press, Princeton, 2009.
- [4] L. BERS, On boundaries of Teichmüller spaces and on Kleinian groups I, *Ann. of Math.* 91 (1970), 570-600.
- [5] G. DAVID, Solutions de l'équation de Beltrami avec $|\mu| = 1$, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* 13 (1988), 25-70.
- [6] F. W. GEHRING AND B. OSGOOD, Uniform domains and the quasi-hyperbolic metric, *J. Anal. Math.* 36 (1979) 50-74.
- [7] J. T. GILL, Integrability of derivatives of inverses of maps of exponentially integrable distortion in the plane, *J. Math. Anal. Appl.* 352 (2009), 762-766.
- [8] S. HENCL AND P. KOSKELA, *Lectures on Mappings of Finite Distortion*, Lecture Notes in Mathematics 2096, Springer, Cham, 2014.
- [9] J. H. HUBBARD, *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics*, Vol. 1, Matrix Editions, Ithaca, 2006.
- [10] Y. IMAYOSHI AND M. TANIGUCHI, *An Introduction to Teichmüller Spaces*, Springer-Verlag, Tokyo, 1992.
- [11] S. LANG, *Differential Manifolds*, Springer-Verlag, Tokyo, 1985.
- [12] B. MASKIT, On boundaries of Teichmüller spaces and on Kleinian groups II, *Ann. of Math.* 91 (1970), 607-639.
- [13] C. MCMULLEN, Cusps are dense, *Ann. of Math.* 133 (1991), no. 1, 217-247.

- [14] H. SHIGA, On analytic and geometric properties of Teichmüller spaces, J. Math. Kyoto Univ. 24 (1984), 441-452.
- [15] H. SHIGA, Characterization of quasidisks and Teichmüller spaces, Tohoku Math. J. 37 (1985), 541-552.
- [16] H. TANIGAWA, Holomorphic families of geodesic discs in infinite dimensional Teichmüller spaces, Nagoya Math. J. 127 (1992), 117-128.
- [17] H. TANIGAWA, Holomorphic mappings into Teichmüller spaces, Proc. Amer. Math. Soc. 117 (1993), 71-78.
- [18] S. A. WOLPERT, The length spectra as moduli for compact Riemann surfaces, Ann. of Math. 109 (1979), 323-351.
- [19] H. YAMAMOTO, Boundaries of the Teichmüller spaces of finitely generated Fuchsian groups of the second kind, J. Math. Soc. Japan 35 (1983), no. 2, 307-321.

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