

NON-TRIVIAL LIMIT LAWS IN TOPOLOGICAL GROUPS: HOW SIMPLE CAN THEY BE?

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ABSTRACT. A *limit law* is a map $f : (D, \leq) \rightarrow F(X)$ from a directed set $(D, \leq)$ to a free group $F(X)$ over some set X . A topological group G *satisfies* limit law f (we also say that f *holds* in G) provided that for every group homomorphism $\pi : F(X) \rightarrow G$ from $F(X)$ to G and each open set U containing the identity element e_G of G there exists some $d \in D$ such that $\pi(f(c)) \in U$ for all $c \geq d$. For a group G a limit law $f : (D, \leq) \rightarrow F(X)$ is called *G-algebraic* provided that there exists $d \in D$ such that $\pi(f(c)) = e_G$ whenever $c \geq d$ and $\pi : F(X) \rightarrow G$ is a group homomorphism. A limit law that is not *G-algebraic* is called *essentially G-topological*. Main result: If a Hausdorff group G satisfies some essentially *G-topological* limit law $f : (D, \leq) \rightarrow F(X)$ such that $(D, \leq)$ is either a linearly ordered set or a countable partially ordered set, then G also satisfies some essentially *G-topological* limit law $f' : (\mathbb{N}, \leq) \rightarrow F(X)$ having the usual set of integers $(\mathbb{N}, \leq)$ as its domain. It follows that if G is one of the three classical locally compact groups, $\mathbb{Z}$ (integers), $\mathbb{R}$ (reals) or $\mathbb{T}$ (unit circle), then every limit law with a linearly ordered domain that holds in G is *G-algebraic*.

As usual, the symbol $F(X)$ denotes the free group over a set X . If G is a group, then e_G denotes the identity element of G . The identity element of $F(X)$ will be simply denoted by e .

A *partially ordered set* (or shortly, *poset*) is a pair $(D, \leq)$ consisting of a set D together with a relation $\leq$ which is:

- (i) *reflexive*, i.e. $d \leq d$ for each $d \in D$, and
- (ii) *transitive*, i.e. $d_0 \leq d_1$ and $d_1 \leq d_2$ implies $d_0 \leq d_2$.

A partially ordered set $(D, \leq)$ is *directed* provided that for every pair $d_0, d_1 \in D$ of elements of D there exists $d \in D$ such that $d_0 \leq d$ and $d_1 \leq d$.

Limit laws were introduced in [4] and recently studied extensively in [1, 3]. A *limit law* is a map $f : (D, \leq) \rightarrow F(X)$ from a directed set $(D, \leq)$ to a free group $F(X)$ over some set X . We say that a limit law $f : (D, \leq) \rightarrow F(X)$ *holds* in a topological group G , or that G *satisfies law* f , provided that for every group homomorphism $\pi : F(X) \rightarrow G$ from $F(X)$ to G the directed set $\{\pi(f(d)) : d \in D\}$ converges to the identity element e_G of G ; that is, for every open set U containing e_G there exists $d \in D$ such that $\pi(f(c)) \in U$ for all $c \geq d$.

Let G be a group. A limit law $f : (D, \leq) \rightarrow F(X)$ will be called *G-algebraic* provided that there exists some $d \in D$ such that $\pi(f(c)) = e_G$ whenever $c \geq d$ and $\pi : F(X) \rightarrow G$ is a group homomorphism. If G is a topological group, then a *G-algebraic* limit law automatically holds in G for an obvious algebraic reason, thereby justifying its name.

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Note that the topology of the group G plays absolutely no role in “deciding” whether a G -algebraic law holds in G or not; everything is determined by the algebraic structure of G . Therefore, from a topological point of view, G -algebraic laws are trivial and not particularly interesting. A limit law that is not G -algebraic will be called *essentially G -topological*. Contrary to G -algebraic laws, the topology of the group G plays a crucial role in (really!) deciding whether an essentially G -topological law holds in G or not; this both explains the choice of our terminology and indicates that essentially G -topological laws are of special interest from the topological point of view.

It appears to be natural to adopt the (luck of) complexity of a partially ordered set $(D, \leq)$ as a measure of “simplicity” of a limit law $f : (D, \leq) \rightarrow F(X)$. The main purpose of this article is to demonstrate that the three classical locally compact groups, the group $\mathbb{Z}$ of integer numbers, the group $\mathbb{R}$ of real numbers and the unit circle group $\mathbb{T}$, do not satisfy any “simple” essentially G -topological law (see Corollary 14).

Lemma 1. *Let $f : (D, \leq) \rightarrow F(X)$ be a limit law that holds in a Hausdorff topological group G . If $(D, \leq)$ has a biggest element (in particular, if the poset $(D, \leq)$ is finite), then f is G -algebraic.*

Proof. Let a be a biggest element of $(D, \leq)$. First suppose that there exists a group homomorphism $\pi : F(X) \rightarrow G$ such that $\pi(f(a)) \neq e_G$. Then $U = G \setminus \{\pi(f(a))\}$ is an open neighbourhood of e_G by Hausdorffness of G . Since f holds in G , there exists $c \in D$ with $\pi(f(d)) \in U$ for all $d \geq c$. Since a is the biggest element of $(D, \leq)$, it follows that $\pi(f(a)) \in U = G \setminus \{\pi(f(a))\}$, a contradiction. Therefore $\pi(f(a)) = e_G$ for every group homomorphism $\pi : F(X) \rightarrow G$. Since a is the biggest element of $(D, \leq)$, we also have that $\pi(f(d)) = e_G$ whenever $d \in D$, $d \geq a$ and $\pi : F(X) \rightarrow G$ is a group homomorphism. This means that f is G -algebraic. $\square$

A subset C of a directed set $(D, \leq)$ is called *cofinal in $(D, \leq)$* if for every $d \in D$ there exists $c \in C$ with $d \leq c$.

Let f and g be limit laws. We will write $f \Leftarrow g$ provided that f holds in every topological group in which g holds.

Lemma 2. *If $f : (D, \leq) \rightarrow F(X)$ is a limit law and C is a cofinal subset of $(D, \leq)$, then the restriction $f|_C : C \rightarrow F(X)$ of f to C is a limit law and $f|_C \Leftarrow f$.*

Proof. Being a cofinal subset of a directed set $(D, \leq)$, the partially ordered set $(C, \leq)$ is also directed, and so $f|_C$ is a limit law. Let G be a topological group in which f holds. We are going to prove that $f|_C$ also holds in G . Indeed, let $\pi : F(X) \rightarrow G$ be a homomorphism from $F(X)$ to G . Let U be an open subset of G which contains the identity element e_G . Since f holds in G , there exists some $d \in D$ such that $\pi(f(c)) \in U$ for all $c \geq d$. Since C is cofinal in $(D, \leq)$, one can find $c_0 \in C$ with $c_0 \geq d$. Clearly, $\pi(f(c)) \in U$ for all $c \geq c_0$. $\square$

A *sequential law* is a limit law $f : (\mathbb{N}, \leq) \rightarrow F(X)$ with the set $(\mathbb{N}, \leq)$ of natural numbers as its directed set. A *countable law* is a limit law $f : (D, \leq) \rightarrow F(X)$ whose domain $(D, \leq)$ is a countable directed set.

In view of Lemma 1, the cardinality of the domain of an essentially G -topological law must be infinite, and thus countable laws are potentially the simplest possible essentially G -topological laws. This explains why our first theorem considers such laws.

Theorem 3. *Let G be a Hausdorff group. If G satisfies some essentially G -topological countable law, then it also satisfies some essentially G -topological sequential law.*

Proof. Let G be a Hausdorff group and let $f : (D, \leq) \rightarrow F(X)$ be an essentially G -topological countable limit law that holds in G . Let $D = \{d_n : n \in \mathbb{N}\}$ be an enumeration of D . According to Lemma 1 the poset $(D, \leq)$ does not have the biggest element. Using this fact, directedness of $(D, \leq)$ and the fact that f is essentially G -topological we can easily choose, by induction on n , an element $c_n \in D$ and a group homomorphism $\pi_n : F(X) \rightarrow G$ such that $d_n \leq c_n$, $c_{n-1} < c_n$ and $\pi_n(f(c_n)) \neq e_G$. By our construction, $C = \{c_n : n \in \mathbb{N}\}$ is cofinal in $(D, \leq)$ and therefore $f|_C \Leftarrow f$ by Lemma 2. Since f holds in G , so does $f|_C$. By our construction, $(C, \leq)$ is order isomorphic to $(\mathbb{N}, \leq)$ and $\pi_n(f(c)) \neq e_G$ for all $c \in C$. Thus $f|_C$ is an essentially G -topological sequential law. $\square$

Recall that a cardinal τ is called *singular* provided that there exists a cardinal $\kappa < \tau$ and a transfinite sequence $\{\tau_\beta : \beta < \kappa\}$ of cardinals such that $\sup\{\tau_\beta : \beta < \kappa\} = \tau$ and $\tau_\beta < \tau$ for each $\beta < \kappa$. A cardinal is *regular* if it is not singular.

If X is a set, G is a group and $\varphi : X \rightarrow G$ is a map, then $\widehat{\varphi} : F(X) \rightarrow G$ will denote the (unique) extension of φ over $F(X)$ that is a group homomorphism. If $y \in F(X)$ and $y \neq e$, then $\text{supp } y$ denotes the smallest subset Y of X such that y belongs to the subgroup of $F(X)$ generated by Y . Note that $\text{supp } y$ is always finite.

Our next lemma establishes an algebraic fact about free groups that is perhaps of some independent interest.

Lemma 4. *If X is a set and Z is a subset of $F(X)$ of uncountable regular cardinality, then there exist $Y \subseteq Z$, $y^* \in Y$ and a map $\varphi : X \rightarrow X$ such that $|Y| = |Z|$ and $\widehat{\varphi}(y) = y^*$ for all $y \in Y$.*

Proof. Without loss of generality we will assume that $z \neq e$ for each $z \in Z$. Note that $\{\text{supp } z : z \in Z\}$ is a family of non-empty finite subsets of X , so by the Δ -system Lemma (see, for example, [2, Ch. II, Theorem 1.6]) there exists a finite (possibly empty) set $T \subseteq X$ and $Z' \subseteq Z$ such that $|Z'| = |Z|$ and $\text{supp } z \cap \text{supp } z' = T$ whenever $z, z' \in Z'$ and $z \neq z'$. For each $n \in \mathbb{N} \setminus \{0\}$ define $Z'_n = \{z \in Z' : |\text{supp } z| = n\}$ and note that $Z' = \bigcup\{Z'_n : n \in \mathbb{N} \setminus \{0\}\}$. Since $|Z'| = |Z|$ is an uncountable regular cardinal, it follows that $|Z'_n| = |Z'|$ for some $n \in \mathbb{N} \setminus \{0\}$. Pick arbitrarily $z^* \in Z'_n$. For each $z \in Z'_n \setminus \{z^*\}$ choose a bijection $h_z : \text{supp } z \rightarrow \text{supp } z^*$ such that $h_z(t) = t$ for all $t \in T$, and let $\widehat{h}_z : F(\text{supp } z) \rightarrow F(\text{supp } z^*)$ be the natural homomorphic extension of h_z over $F(\text{supp } z)$. Since the set $F(\text{supp } z^*)$ is at most countable, and $|Z'_n \setminus \{z^*\}| = |Z'_n| = |Z'| = |Z|$ is an uncountable regular cardinal, there exist $g \in F(\text{supp } z^*)$ and $Y \subseteq Z'_n$ such that $|Y| = |Z'_n|$ and $\widehat{h}_y(y) = g$ for all $y \in Y$. Pick $y^* \in Y$ arbitrarily. For each $y \in Y$ define the map $f_y : \text{supp } y \rightarrow \text{supp } y^*$ by $f_y = h_{y^*}^{-1} \circ h_y$ and note that the restriction of f_y to T is the identity map of T . This allows us to define the map $\varphi : X \rightarrow X$ by $\varphi(x) = f_y(x)$ if $x \in \text{supp } y$ for some $y \in Y$ and $\varphi(x) = x$ if $x \in X \setminus \bigcup\{\text{supp } y : y \in Y\}$. Finally, by our construction

$$\widehat{\varphi}(y) = \widehat{f_y}(y) = \widehat{h_{y^*}}^{-1}(\widehat{h_y}(y)) = \widehat{h_{y^*}}^{-1}(g) = y^*$$

for each $y \in Y$. $\square$

Another potential candidate for a “simple” limit law is the law with a linearly ordered domain. Recall that a poset $(D, \leq)$ is *linearly ordered* provided that for every pair d, d' of elements of D either $d \leq d'$ or $d' \leq d$ holds. A *linearly ordered law* is a limit law $f : (D, \leq) \rightarrow F(X)$ whose domain $(D, \leq)$ is a linearly ordered set. Sequential laws are particular types of linearly ordered laws.

Theorem 5. *If a Hausdorff topological group G satisfies some essentially G -topological linearly ordered law, then G also satisfies some essentially G -topological sequential law.*

Proof. The proof of this theorem will be split into a sequence of claims.

Let G be a Hausdorff topological group and $f : (D, \leq) \rightarrow F(X)$ be an essentially G -topological linearly ordered law which holds in G . Let τ be the smallest cardinality of a cofinal subset of $(D, \leq)$. Choose a cofinal subset $E = \{d_\alpha : \alpha < \tau\}$ of $(D, \leq)$ of cardinality τ .

Claim 6. *If $C \subseteq D$ and $|C| < \tau$, then there exists $d \in D$ such that $c < d$ for all $c \in C$.*

Proof. Since τ is a minimal cardinality of a cofinal subset of $(D, \leq)$, the set C cannot be cofinal in $(D, \leq)$. Therefore there exists some $d \in D$ such that for all $c \in C$ the inequality $d \leq c$ does *not* hold. It is precisely here where we use the fact that $(D, \leq)$ is a linearly ordered set to conclude that $c < d$ for all $c \in C$. $\square$

By transfinite recursion we will choose points $\{c_\alpha : \alpha < \tau\} \subseteq D$ and a family $\{\pi_\alpha : \alpha < \tau\}$ of group homomorphisms from $F(X)$ to G in such way that, for every $\alpha < \tau$, one has $d_\alpha < c_\alpha$, $\pi_\alpha(f(c_\alpha)) \neq e_G$ and $c_\beta < c_\alpha$ for $\beta < \alpha$. Assume that $\alpha < \tau$ and that points $\{c_\beta : \beta < \alpha\} \subseteq D$ and group homomorphisms $\{\pi_\beta : \beta < \alpha\}$ from $F(X)$ to G have already been chosen. From Claim 6 it follows that there exists $d \in D$ such that $c_\beta < d$ for all $\beta < \alpha$. Since $(D, \leq)$ is directed, $d_\alpha \leq d'$ and $d \leq d'$ for some $d' \in D$. Now use the fact that f is essentially G -topological to pick $c_\alpha \in D$ and a group homomorphism $\pi_\alpha : F(X) \rightarrow G$ such that $d' \leq c_\alpha$ and $\pi_\alpha(f(c_\alpha)) \neq e_G$. Clearly c_α has all necessary properties.

Claim 7. *$\beta < \alpha < \tau$ implies $c_\beta < c_\alpha$.*

Proof. This was guaranteed as part of our inductive construction. $\square$

Claim 8. *$C = \{c_\alpha : \alpha < \tau\}$ is a cofinal subset of $(D, \leq)$.*

Proof. $E = \{d_\alpha : \alpha < \tau\}$ is cofinal in $(D, \leq)$ and $d_\alpha \leq c_\alpha$ for all $\alpha < \tau$ implies that C is also cofinal in $(D, \leq)$. $\square$

Claim 9. *If Γ is a cofinal subset of τ , then $\{c_\gamma : \gamma \in \Gamma\}$ is cofinal in $(D, \leq)$.*

Proof. Suppose that Γ is cofinal in τ . Let $d \in D$. From Claim 8 it follows that $d \leq c_\beta$ for some $\beta < \tau$. Cofinality of Γ in τ yields $\gamma \in \Gamma$ such that $\beta < \gamma$. Now $d \leq c_\beta \leq c_\gamma$ by Claim 7. $\square$

Claim 10. *τ is infinite.*

Proof. If τ is finite, then $(D, \leq)$ must have a biggest element a , and then f will be G -algebraic by Lemma 1. $\square$

Claim 11. *τ is a regular cardinal.*

Proof. Assume the contrary, i.e. that τ is singular. Then there exists a cardinal $\kappa < \tau$ and a transfinite sequence $\{\tau_\beta : \beta < \kappa\}$ of cardinals such that $\sup\{\tau_\beta : \beta < \kappa\} = \tau$ and $\tau_\beta < \tau$ for each $\beta < \kappa$. For each $\beta < \kappa$ applying $\tau_\beta < \tau$ and Claim 6 to the set $C_\beta = \{d_\alpha : \alpha < \tau_\beta\}$ one can find $b_\beta \in D$ such that $d_\alpha < b_\beta$ for $\alpha < \tau_\beta$. We now claim that the set $\{b_\beta : \beta < \kappa\}$ is cofinal in $(D, \leq)$, thereby contradicting minimality of τ . Indeed, let $d \in D$. Since E is cofinal in $(D, \leq)$, one has $d \leq d_\alpha$ for some $\alpha < \tau$. Since $\sup\{\tau_\beta : \beta < \kappa\} = \tau$, there exists $\beta < \kappa$ with $\alpha < \tau_\beta$. It remains only to note that $d \leq d_\alpha < b_\beta$. $\square$

Claim 12. τ is countable.

Proof. Assume the contrary. Then τ is an uncountable regular cardinal by Claims 10 and 11. We can now apply Lemma 4 to the set $Z = \{f(c_\alpha) : \alpha < \tau\}$ to find a subset $\Gamma \subseteq \tau$, an ordinal $\gamma^* \in \Gamma$ and a map $\varphi : X \rightarrow X$ such that $|\Gamma| = \tau$ and $\widehat{\varphi}(f(c_\gamma)) = f(c_{\gamma^*})$. Recall now that the group homomorphism $\pi_{\gamma^*} : F(X) \rightarrow G$ satisfies $g = \pi_{\gamma^*}(f(c_{\gamma^*})) \neq e_G$. Since G is Hausdorff, $U = G \setminus \{g\}$ is an open neighbourhood of e_G in G . Define a group homomorphism $\pi : F(X) \rightarrow G$ via $\pi = \pi_{\gamma^*} \circ \widehat{\varphi}$. Then $\pi(f(c_\gamma)) = \pi_{\gamma^*}(\widehat{\varphi}(f(c_\gamma))) = \pi_{\gamma^*}(f(c_{\gamma^*})) = g$ for $\gamma \in \Gamma$, and therefore

$$(1) \quad \pi(f(c_\gamma)) \notin U \text{ for each } \gamma \in \Gamma.$$

Since $|\Gamma| = \tau$, Γ is cofinal in τ , and so the set $\{c_\gamma : \gamma \in \Gamma\}$ is cofinal in $(D, \leq)$ by Claim 9. Cofinality of $\{c_\gamma : \gamma \in \Gamma\}$ in $(D, \leq)$ and (1) imply that f does not hold in G , a contradiction. $\square$

By Claim 8 $C = \{c_\alpha : \alpha < \tau = \omega\}$ is a cofinal subset of $(D, \leq)$, and so the restriction $h = f|_C$ of f to C is a limit law such that $h \Leftarrow f$ (see Lemma 2). Since f holds in G , so does h . From the choice of homomorphisms π_α it follows that h is essentially G -topological. Claim 7 implies that $(C, \leq)$ is order isomorphic to $(\mathbb{N}, \leq)$, i.e. that h is a sequential law. $\square$

From Theorems 3 and 5 we immediately get the following

Corollary 13. *For a Hausdorff group G the following conditions are equivalent:*

- (i) G satisfies some essentially G -topological linearly ordered law,
- (ii) G satisfies some essentially G -topological countable law,
- (iii) G satisfies some essentially G -topological sequential law.

If G is either the group $\mathbb{Z}$ of integer numbers, the group $\mathbb{R}$ of real numbers or the unit circle group $\mathbb{T}$, then each sequential law that holds in G is G -algebraic [1]. From this result and Corollary 13 we obtain

Corollary 14. *Let G be one of the groups $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{T}$. Then all countable or linearly ordered laws that hold in G are G -algebraic.*

If a locally compact Abelian group G satisfies some essentially G -topological sequential law, then G is totally disconnected [1]. From this and Corollary 13 we get our last

Corollary 15. *If a locally compact Abelian group G satisfies either some essentially G -topological linearly ordered law or some essentially G -topological countable law, then G is totally disconnected.*

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