

Yukawa Institute for Theoretical Physics
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Symmetric Orbifold CFT: Calculations and Applications

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Abstract

This thesis uses symmetric product orbifold CFTs as a controllable laboratory for $\text{AdS}_3/\text{CFT}_2$, with the aim of sharpening what it means for the theory to exhibit “holographic” behavior. In this setting, “holographic” is used in two senses: the usual notion tied to a holographic CFT admitting a semiclassical gravity description, and a tensionless viewpoint in which a bulk dual may exist but is organized in a manifestly stringy way. Guided by this two-layer usage, the thesis proceeds in two parts. We first examine how standard semiclassical benchmarks behave at the orbifold point, and we then develop a complementary framework aimed at the tensionless viewpoint and test it against bulk expectations.

The first part develops controlled probes of semiclassical benchmarks at the orbifold point. We set up an analytic framework for single-interval entanglement on the torus and obtain closed-form expressions for Rényi entropies at integer $n > 1$ in a Dirac-fermion symmetric-orbifold toy model, while clarifying the open issues surrounding the $n \rightarrow 1$ continuation needed for the entanglement entropy. We also study late-time spectral correlations through the progressively time-averaged spectral form factor and find that a smooth random-matrix-like ramp does not emerge at the orbifold point. Instead, coherent contributions from the light sector obstruct the ramp, strongly supporting the standard picture that twist-two marginal deformations must lift and reorganize a substantial part of the light spectrum before chaotic diagnostics compatible with a semiclassical gravity regime can become visible.

The second part is constructive and targets the tensionless viewpoint through open-string physics. In stringy AdS_3 backgrounds, D-branes introduce open-string sectors and new classes of observables, yet symmetric orbifolds have lacked a satisfactory CFT framework that makes such degrees of freedom concrete. We formulate an open-sector framework using interfaces that connect Sym^{N_-} and Sym^{N_+} by gluing P strands, identify the resulting “open twist” vertices, and develop diagrammatic rules for interface correlators. After lifting the construction to the grand-canonical ensemble natural for pure NS-NS AdS_3 holography, we find quantitative agreement of torus partition functions with the unit-flux ($k = 1$) string-theory side, and qualitative agreement of correlators at the level of genus expansion and covering-surface structure.

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Part I.

Foundations

1. Introduction and Summary

1.1. Overview

The goal of this thesis is to use symmetric product orbifold CFTs as a controllable laboratory for $\text{AdS}_3/\text{CFT}_2$, with the aim of sharpening what it means for the theory to exhibit “holographic” behavior. Before going into details, let us briefly summarize why symmetric orbifold CFTs are a particularly useful class of CFTs to study:

- First, in the most traditional (D1-D5) realization of $\text{AdS}_3/\text{CFT}_2$ in string theory, the bulk description admits a semiclassical gravity regime with appropriate matter content, while the CFT_2 side is realized as a deformation of a *specific* symmetric orbifold CFT.
- Second, symmetric orbifolds share several universal features with semiclassical AdS_3 gravity. A prominent example is a BTZ-like high-temperature regime controlled by Cardy growth, which arises *quite generally* in symmetric orbifolds¹.
- Third, *certain* symmetric orbifold CFTs are believed to be dual to string theory on AdS_3 in a tensionless, manifestly stringy regime.

The details of these viewpoints are reviewed in Section 2.1 and Section 6.1. For the present overview, the key point is that symmetric orbifold CFTs provide a controllable laboratory that is closely related to $\text{AdS}_3/\text{CFT}_2$ holography.

One may already notice that the AdS_3 side can appear either semiclassical or intrinsically stringy, depending on the regime and the duality frame. Symmetric orbifold CFTs sit at a point where this “two-faced” character becomes especially transparent. Accordingly, in this thesis we use “holographic” in two complementary senses. First, there is the semiclassical notion associated with an effective bulk gravity description, where one expects geometric entanglement, universal thermodynamics, and random-matrix-like signatures of chaos. Second, there is a more stringy notion appropriate to AdS_3 string theory, where a bulk dual is not organized purely in terms of semiclassical geometry, and where extended degrees of freedom and sector structure play an essential role.

¹This feature is insensitive to the choice of seed, provided one takes the large- N limit.

Despite this rich structure, symmetric orbifolds remain highly tractable². In this sense they can serve as a controllable laboratory for holography. Moreover, the two perspectives above are not independent. For example, when an observable departs from semiclassical expectations at the orbifold point, it is natural to interpret the deviation as a clue about how the stringy organization of degrees of freedom manifests itself. Conversely, since “BTZ-like” features are expected rather universally across broad classes of symmetric orbifolds, this universality can sharpen the interpretation of BTZ-like phenomena even in strongly stringy regimes.

The thesis follows this logic through two complementary strands. The first strand exploits analytic and numerical control at the orbifold point to probe semiclassical benchmarks. We study entanglement-type observables in thermal states and investigate late-time spectral statistics through the spectral form factor. The second strand is constructive and focuses on open-string physics in stringy AdS_3 backgrounds. Motivated by the fact that symmetric orbifolds have lacked a satisfactory CFT framework for open-string-like degrees of freedom, we develop an interface-based open sector and compare it with bulk expectations, including quantitative tests at the level of torus amplitudes and structural tests for correlation functions through their genus expansion.

Scope and conventions. We focus primarily on $\text{AdS}_3/\text{CFT}_2$ examples related to bottom-up approaches to holographic CFT_2 , and tensionless (or pure NS-NS) string backgrounds. We use standard CFT conventions for complex coordinates and thermal identification.

Related publications. Parts of this thesis are based on, or overlap with, the following publications and preprints:

- Chapter 3 is based in part on [1]:
[1] T. Takayanagi and T. Tsuda. “Free fermion cyclic/symmetric orbifold CFTs and entanglement entropy”. In: *JHEP* 12 (2022), p. 004.
- Chapter 4 overlaps with [2]:
[2] P. Caputa, B. Chen, T. Takayanagi, and T. Tsuda. “Thermal pseudo-entropy”. In: *JHEP* 01 (2025), p. 003.
- Chapters 5 and 6 are based on [3] and are complemented by [4]:

²For example, when the seed theory is a rational CFT, the corresponding symmetric orbifold is again rational.

- [3] S. Harris, Y. Hikida, V. Schomerus, and T. Tsuda. “Holographic Interfaces in Symmetric Product Orbifolds”. In: *JHEP* 12 (2025), p. 114.
- [4] S. Harris, V. Schomerus, and T. Tsuda. “Interface Correlators in Symmetric Product Orbifolds”. Preprint on arXiv in Dec. 2025.

1.2. Summary of results

This thesis consists of three main parts. The first two are primarily diagnostic: they probe how far semiclassical-holographic expectations can be pushed at the symmetric orbifold point and identify obstructions that prevent a straightforward semiclassical interpretation. The third part is constructive: it formulates an open-sector framework via interfaces and relates it to tensionless holography i.e. strings on AdS_3 with unit NS-NS flux.

1.2.1. Thermal states and entanglement (Chapter 3)

In Chapter 3, we first develop a technical reduction for torus observables in symmetric product orbifolds. The basic idea is that a broad class of quantities on the torus can be reorganized and expressed in terms of corresponding computations in cyclic orbifolds. We implement this reduction by introducing a new class of Hecke-like operators that package the sum over twisted sectors in a way adapted to torus geometry, and we explain how the relevant thermal winding data are identified on the symmetric-orbifold side and then extracted from cyclic-orbifold building blocks. While the reduction is formulated at the level of torus observables, it provides a flexible framework that is expected to be useful beyond the specific application considered in this chapter.

We then apply this torus reduction to entanglement-type observables in thermal states. Concretely, focusing on a single interval on the torus, we implement the replica method in a symmetric-orbifold toy model with a Dirac-fermion seed CFT and compute Rényi entropies for integer $n > 1$, obtaining closed-form expressions. These results provide a controlled analytic starting point for studying the entanglement structure of thermal states at the orbifold point and for identifying where deviations from semiclassical-holographic behavior first appear. We also discuss the status of the $n \rightarrow 1$ limit and outline future directions toward extracting the entanglement entropy and making quantitative comparisons with holographic expectations.

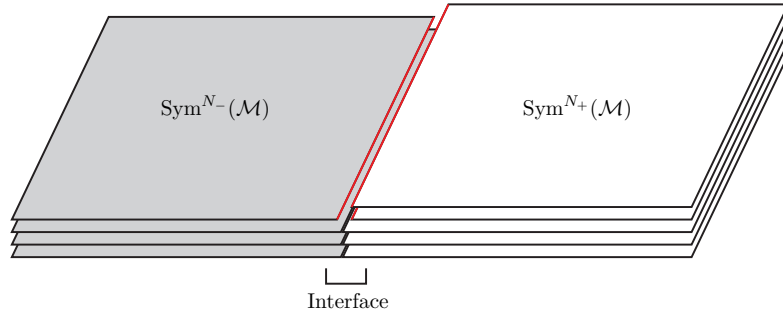


Figure 1.1.: Schematic picture of an interface connecting Sym^{N-} and Sym^{N+} , where a subset of the seed copies are glued across the interface.

1.2.2. Spectral statistics (Chapter 4)

Chapter 4 uses the spectral form factor (SFF) to probe late-time spectral correlations in symmetric product orbifolds, treating the undeformed orbifold point as a baseline for assessing semiclassical-holographic expectations. To obtain a stable late-time signal in 2d CFTs, we apply a progressive time average following [5]. We work on $\text{Sym}^N \mathcal{C}(R)$ with a compact-boson seed at several radii. Numerically, the averaged SFF shows a dip and subsequent growth, but the would-be ramp is replaced by a robust step pattern: the step locations are largely insensitive to β and, for sufficiently large N , to N itself, while remaining strongly seed-dependent and absent at $N = 1$. This points to persistent light-sector coherence at the orbifold point and suggests that recovering a smoother, more universal ramp likely requires the spectral reorganization expected from twist-two marginal deformations.

1.2.3. Interfaces and open-string-like DoF (Chapters 5 and 6)

A defining feature of stringy AdS backgrounds is the possibility of introducing branes, which in turn give rise to open-string degrees of freedom and new sectors of observables. In tensionless holography and related stringy regimes, understanding such brane-induced sectors from the CFT side is particularly important. Yet, in the context of symmetric orbifold CFTs, a satisfactory formulation of open-string-like degrees of freedom has been lacking.

In Chapter 5, we introduce interfaces in symmetric product orbifolds as a natural generalization of boundary states and topological defects. Concretely, an interface is an object that connects $\text{Sym}^{N-}(\mathcal{M})$ and $\text{Sym}^{N+}(\mathcal{M})$ by gluing P strands across the interface, as illustrated in Fig. 1.1. From the torus partition function in the presence of such interfaces, we identify new operator insertions that change the interface data, which we refer to as “open twist” vertices.

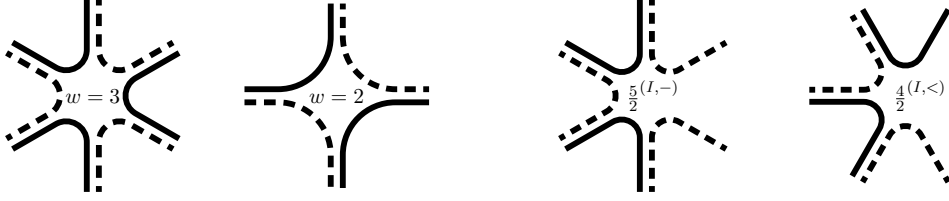


Figure 1.2.: Vertex diagrams for twist insertions. The two diagrams on the left represent the standard twist vertices in the symmetric orbifold, while the two diagrams on the right represent the corresponding open twist vertices associated with interfaces.

We then generalize the Feynman-like diagrammatics of symmetric orbifolds developed in [6] to the interface setting. In particular, we show that correlators with open twist insertions admit a natural diagrammatic representation that mirrors a bulk-inspired extension of the usual picture. See Fig. 1.2. Using these rules, we compute a class of interface correlation functions and analyze their large- N behavior. Requiring compatibility with a string genus expansion singles out a specific scaling regime for the interface data, namely $P \sim N$ with $N_{\pm} - P \sim \sqrt{N}$, and within this regime, we compare the resulting scaling and structure with expectations from the stringy bulk picture.

Chapter 6 reviews key aspects of pure NS-NS AdS_3 holography and uses them to test the bulk interpretation of the interface sector. In this pure NS-NS setting, the symmetric orbifold is naturally treated in a grand-canonical ensemble, and we lift the interface construction accordingly. We find quantitative agreement for the torus partition function with the unit flux $k = 1$ string-theory side.

For correlation functions, we compare the genus expansion and the associated covering-surface interpretation with the worldsheet picture and find qualitative agreement at this structural level. Taken together, these results support the view that the interface and open twist sector provides a concrete and computable CFT realization of open-string-like degrees of freedom in tensionless $\text{AdS}_3/\text{CFT}_2$.

1.3. Organization of the thesis

This thesis is organized as follows.

Chapter 2 reviews background on $\text{AdS}_3/\text{CFT}_2$ and symmetric orbifold CFTs, focusing on three motivations that guide the thesis: the old-fashioned D1-D5 perspective, a bottom-up approach to holographic CFTs, and the tensionless (pure NS-NS) viewpoint as context for later chapters. We also review tools used throughout, including boundary states, topological defects, diagrammatic methods for computing correlators, and the grand-canonical uplift.

Chapter 3 studies thermal states and entanglement observables in symmetric orbifolds. We begin by reviewing a reduction of torus observables in symmetric orbifolds to corresponding computations in cyclic orbifolds, which we view as a general framework expected to be useful beyond entanglement. We then implement the replica method in a symmetric-orbifold toy model with a Dirac-fermion seed CFT and obtain closed-form expressions for Rényi entropies at integer $n > 1$. We also discuss the status of the $n \rightarrow 1$ limit and future directions toward extracting the entanglement entropy.

Chapter 4 investigates spectral statistics in symmetric orbifolds using the spectral form factor. We discuss the obstruction to random-matrix-like behavior at the orbifold point and introduce a time-averaged coarse-graining motivated by twisted-sector averaging. We then present numerical results showing that this averaging alone does not restore the expected chaotic features, and we interpret the outcome as an indication of the persistence of intrinsically stringy structure.

Chapter 5 introduces interfaces in symmetric product orbifolds as a natural generalization of boundary states and topological defects. From the torus partition function in the presence of interfaces, we identify “open twists” as interface-changing operators and develop a diagrammatic framework for computing interface correlation functions, which generalizes the building blocks used in the closed sector. We then study their large- N behavior and compare it with expectations from string-theoretic considerations.

Chapter 6 reviews key aspects of pure NS-NS AdS_3 holography and uses them to test the bulk interpretation of the interface sector. We lift the interface construction to the grand-canonical ensemble required in this tensionless setting and match the resulting torus partition function quantitatively to the string-theory side. For correlation functions, we compare the genus expansion and the associated covering-surface interpretation with the worldsheet picture, finding qualitative agreement at this structural level.

We conclude in Chapter 7 with a summary of results and a discussion of open problems and future directions.

2. Symmetric Product Orbifold CFT

In this chapter, we give an overall framework for symmetric product orbifold CFTs, hereafter simply symmetric orbifolds.

Before we begin the main text, we first give an intuitive working definition; precise statements and computations will appear in later sections. Start with an arbitrary 2-dimensional CFT $\mathcal{M}$ on a Riemann surface X , with fields $\phi(x)$ for $x \in X$. We will refer to $\mathcal{M}$ as the *seed theory*. For a positive integer N , the symmetric orbifold is obtained by taking the tensor product $\mathcal{M}^{\otimes N}$ and orbifolding by the symmetric group S_N , namely $\text{Sym}^N(\mathcal{M}) = \mathcal{M}^{\otimes N}/S_N$. The central charge scales additively, $c = N c_{\mathcal{M}}$, and if the seed $\mathcal{M}$ is solvable, then in principle $\text{Sym}^N(\mathcal{M})$ remains exactly tractable.

To see how twisted sectors arise, introduce a g -twist operator $\sigma_g(x_0)$ with $g \in S_N$. Regard g as a permutation of the index set $\{1, \dots, N\}$. In the presence of σ_g , fields in the i -th copy of $\mathcal{M}$ pick up the monodromy of g when x circles x_0 :

$$\phi^i(e^{2\pi i}(x - x_0) + x_0) = \phi^{g(i)}(x).$$

Equivalently, the twist $\sigma_g(x_0)$ permutes copies along the sheet indices specified by g . A single ℓ -cycle $g = (i_1 i_2 \dots i_\ell)$ imposes

$$\phi^{i_k}(e^{2\pi i}(x - x_0) + x_0) = \phi^{i_{k+1}}(x) \quad (k = 1, \dots, \ell; i_{\ell+1} \equiv i_1).$$

Gauge-invariant twists are obtained by summing over the conjugacy class $[g]$:

$$\sigma_{[g]} = \sum_{h \in [g] \subset S_N} \sigma_h,$$

so that primary twist operators are labeled by the cycle structure of S_N , e.g., $[g] = \prod_{\ell \geq 1} (\ell)^{m_\ell}$ with $\sum_{\ell} \ell m_\ell = N$. This yields the familiar organization into untwisted and twisted sectors.

2.1. $\text{AdS}_3/\text{CFT}_2$ correspondence and symmetric orbifold

In this section, we explain why the symmetric orbifold is important from three perspectives. First, we review its role in the D1-D5 system and its deformations and moduli. We recall that the symmetric orbifold of $c = 6$ SCFT with target $\mathbb{T}^4$ or $K3$ has long been identified as a cousin of the low-energy effective theory of the D1-D5 brane system. We distinguish the exactly solvable orbifold point from marginal blow-up deformations that move along the moduli space and modify short-distance physics while preserving supersymmetry and the central charge.

Second, we place the symmetric orbifold CFT within bottom-up attempts to construct two-dimensional holographic CFTs, briefly stating criteria for a 2d CFT to admit a semiclassical AdS_3 gravity dual and assessing to what extent those criteria are satisfied.

Finally, we outline its correspondence with tensionless strings. For type II superstrings in the F1-NS5 background, and more generally for strings on $\text{AdS}_3 \times X$ supported by pure NSNS flux, we sketch the exact relation to the grand-canonical ensemble of the symmetric orbifold.

2.1.1. D1-D5 system and symmetric orbifold

Review of decoupling limit in D1-D5 system

Here we review the construction of the $\text{AdS}_3/\text{CFT}_2$ correspondence following [7]. See detail of this setup in e.g. [8, 9, 10, 11, 12, 13]. Consider Type IIB superstring theory. Among the directions 0-9, compactify 6-9 directions on a four-dimensional manifold $\mathcal{M}$, taken to be either the four-torus $\mathbb{T}^4$ or a $K3$ surface. Place Q_5 D5-branes and Q_1 D1-branes as in Table 2.1.

Table 2.1.: D1-D5 brane configuration that gives rise to the $\text{AdS}_3/\text{CFT}_2$ correspondence. A $\circ$ indicates the directions along which the brane extends.

direction	0	1	2	3	4	5	6	7	8	9
D5	$\circ$	$\times$	$\times$	$\times$	$\times$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
D1	$\circ$	$\times$	$\times$	$\times$	$\times$	$\circ$	$\times$	$\times$	$\times$	$\times$

The worldvolume theory flows at low energy to a 1+1-dimensional superconformal field theory; we will refer to it as the D1-D5 CFT. To focus on low-energy physics, take the decoupling limit that sends the string scale to zero while keeping dimensionless

quantities fixed:

$$\alpha' \rightarrow 0, \quad \frac{r}{\alpha'} = (\text{fixed}), \quad v \equiv \frac{V_4}{(2\pi)^4 \alpha'^2} = (\text{fixed}), \quad g_6 \equiv \frac{g_s}{\sqrt{v}} = (\text{fixed}), \quad (2.1)$$

where r measures transverse separations between branes and V_4 is the volume of $\mathcal{M}$. In this limit, interactions between the brane worldvolume theory and the bulk decouple.

On the string theory side, the D1-D5 solution can be written using $\sum_{i=1,\dots,4} x^i x^i = r^2$ and $\sum_{i=1,\dots,4} dx^i dx^i = dr^2 + r^2 d\Omega_3^2$ as

$$ds^2 = f_1^{-1/2} f_5^{-1/2} (dt^2 + dx_5^2) + f_1^{1/2} f_5^{1/2} (dr^2 + r^2 d\Omega_3^2) + f_1^{1/2} f_5^{-1/2} ds_M^2, \quad (2.2a)$$

with harmonic functions

$$f_1 = \left(1 + \frac{g_s \alpha' Q_1}{v r^2} \right), \quad (2.2b)$$

$$f_5 = \left(1 + \frac{g_s \alpha' Q_5}{r^2} \right). \quad (2.2c)$$

In the near-horizon region $r \rightarrow 0$ under the above limit (2.1), the constant terms in f_1, f_5 can be neglected and the metric becomes

$$ds^2 = \frac{r^2}{l^2} (dt^2 + dx_5^2) + \frac{l^2}{r^2} dr^2 + l^2 d\Omega_3^2 + \sqrt{\frac{Q_1}{Q_5}} \frac{ds_M^2}{\sqrt{v}}, \quad (2.3)$$

which is $\text{AdS}_3 \times S^3$ with the internal space M . The AdS radius (which also equals the S^3 radius) is

$$l^2 = g_6 \alpha' \sqrt{Q_1 Q_5}. \quad (2.4)$$

The symmetry of the 0-5 directions is $SO(2,2) \times SO(4)$: $SO(4)$ rotates the S^3 spanned by 1-4, and $SO(2,2)$ is the AdS isometry. In CFT terms, $SO(2,2) \simeq SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is the global conformal symmetry, while $SO(4) \simeq SU(2) \times SU(2)$ is the R-symmetry associated with supersymmetry.

Let us consider when the classical supergravity approximation on the metric (2.3) is valid. It requires both small curvature and small effective string coupling in the near-horizon region[13]:

$$(\text{curvature}) \quad \frac{l^2}{\alpha'} = \sqrt{g_6^2 Q_1 Q_5} \gg 1, \quad (2.5a)$$

$$(\text{loops}) \quad g_{\text{eff}} \sim g_6 \sqrt{\frac{Q_1}{Q_5}} \ll 1. \quad (2.5b)$$

Focusing on asymptotic symmetries that preserve asymptotically AdS boundary conditions, one finds a Virasoro algebra with central charge $3l/2G_3$ [14]. This matches the local conformal symmetry of the dual CFT, so the central charge computed from AdS equals that of the CFT. In the present case, one obtains

$$c = 6Q_1Q_5 \quad (\mathcal{M} = \mathbb{T}^4), \quad c = 6(Q_1Q_5 + 1) \quad (\mathcal{M} = K3). \quad (2.6)$$

Conformal manifold of D1-D5 CFT

To analyze the IR CFT in detail, we first examine the classical vacua of the 2d $\mathcal{N} = (4, 4)$ $U(Q_1)$ gauge theory living along the 05 directions. Its field content is

- vector multiplet with adjoint real scalars $\vec{X} = (X^1, X^2, X^3, X^4) \in (\mathbb{R}^{Q_1 \times Q_1})^4$,
- adjoint hypermultiplet with complex scalars $B_1, B_2 \in \mathbb{C}^{Q_1 \times Q_1}$,
- Q_5 -flavor fundamental hypers with complex scalars $I \in \mathbb{C}^{Q_1 \times Q_5}$, $J \in \mathbb{C}^{Q_5 \times Q_1}$,

and adjoint scalars can be decomposed into four real scalars as

$$B_1 = \frac{X^6 + iX^7}{\sqrt{2}}, \quad B_2 = \frac{X^8 + iX^9}{\sqrt{2}}, \quad (X^6, X^7, X^8, X^9 \in \mathbb{R}^{Q_1 \times Q_1}),$$

where X^{1-4} parametrize the common transverse directions and X^{6-9} parametrize the internal M -directions. Let the FI triplet that parametrizes B -field on $\mathcal{M}$ be $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ with $\zeta_{\mathbb{R}} \in \mathbb{R}$ and $\zeta_{\mathbb{C}} \in \mathbb{C}$. Up to an overall gauge coupling, the scalar potential is a sum of squares,

$$\begin{aligned} V = \text{Tr}_{Q_1} \Big\{ & \left(\mu_{\mathbb{R}} - \zeta_{\mathbb{R}} \mathbf{1}_k \right)^2 + \left| \mu_{\mathbb{C}} - \zeta_{\mathbb{C}} \mathbf{1}_k \right|^2 \\ & + \sum_{i < j} [X^i, X^j][X^i, X^j]^\dagger \\ & + \sum_{i=1}^4 \left([X^i, B_1][X^i, B_1]^\dagger + [X^i, B_2][X^i, B_2]^\dagger \right) \\ & + \sum_{i=1}^4 \left((X^i I)(X^i I)^\dagger + (J X^i)^\dagger (J X^i) \right) \Big\}, \end{aligned} \quad (2.7a)$$

where

$$\mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \quad \mu_{\mathbb{C}} = [B_1, B_2] + IJ. \quad (2.7b)$$

Vacua are obtained by setting each square to zero:

$$\begin{aligned}\mu_{\mathbb{R}} &= \zeta_{\mathbb{R}} \mathbf{1}_k, & \mu_{\mathbb{C}} &= \zeta_{\mathbb{C}} \mathbf{1}_k, \\ [X^i, X^j] &= 0, & [X^i, B_1] &= [X^j, B_2] = 0, \\ X^i I &= 0, & J X^i &= 0 \quad (i, j = 1, \dots, 4).\end{aligned}\tag{2.8}$$

The vacuum structure splits into three types: Higgs, Coulomb, and mixed branches. In the brane picture, the Higgs branch corresponds to D1s sitting on top of the D5s, whereas in the Coulomb branch the D1s separate from the D5s along the 1234 directions. We now summarize each branch.

Higgs branch. This branch corresponds to generic FI parameters ζ . In particular, the adjoint and fundamental hypermultiplet scalars can acquire vacuum expectation values, while the vector-multiplet scalars must vanish:

$$\vec{X} = 0, \quad (B_1, B_2; I, J) \text{ solve } \mu = \zeta, \quad \text{modulo } U(Q_1).\tag{2.9}$$

The solution set of $\mu = \zeta$ modulo $U(Q_1)$ is precisely the hyperkähler-quotient realization of the ADHM construction for $U(Q_5)$ instantons of charge Q_1 on $\mathcal{M}$. Accordingly, the IR theory on the Higgs branch flows to a $(4, 4)$ non-linear sigma model whose target is this instanton moduli space. For $\zeta = 0$ the target space degenerates to the symmetric product orbifold of the internal space, $\text{Sym}^{Q_1 Q_5}(\mathcal{M})$, while turning on a self-dual B -field on $\mathcal{M}$ (equivalently $\zeta \neq 0$) resolves the small-instanton singularities and yields the smooth hyperkähler manifold $\text{Hilb}^{Q_1 Q_5}(\mathcal{M})$ [15, 16]. In the symmetric orbifold side, this introduction of B -field is realized as twist sector marginal deformation; see [17, 18, 19, 20].

Coulomb branch. In this branch, the gauge group $U(Q_1)$ is generically broken by non-zero vacuum expectation values of the adjoint scalars $\vec{X}$. The commuting components of $\vec{X}$ can be simultaneously diagonalized, and the scalar potential vanishes only if the FI parameters are set to zero:

$$\vec{X} \neq 0, \quad B_{1,2} = I = J = 0, \quad \zeta_{\mathbb{R}} = \zeta_{\mathbb{C}} = 0.\tag{2.10}$$

Generically one finds $U(Q_1) \rightarrow U(1)^{Q_1}$, and the eigenvalues of $\vec{X}$ parametrize the locations of the D1-branes in the common transverse directions. In the IR, the Coulomb-branch fixed point is a *singular CFT* [21] that includes an $\mathcal{N} = (4, 4)$ linear-dilaton sector with a continuum spectrum; at large $|\vec{X}|$ the eigenvalue modes decouple, while the linear-dilaton throat appears in the $|\vec{X}| \rightarrow 0$ region [22, 23].

Mixed branch. Split $Q_1 = Q_{1,H} + Q_{1,C}$ and take block-diagonal VEVs: an ADHM Higgs block of size $Q_{1,H}$ and a Coulomb block of size $Q_{1,C}$. Concretely,

$$\vec{X} = \begin{pmatrix} 0 & 0 \\ 0 & \text{diag}(\dots) \end{pmatrix}, \quad (B_1, B_2; I, J) = \begin{pmatrix} \text{ADHM with charge } Q_{1,H} & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.11)$$

Physically this describes peeling off small instantons while retaining an instanton sector of lower charge. Locally the vacuum moduli factorizes into the product of a Higgs factor (the instanton moduli space at charge $Q_{1,H}$) and Coulomb directions (the eigenvalue space of $\vec{X}$ for $Q_{1,C}$ branes) [21].

Note that precise IR fixed point at the intersection $\zeta \sim 0$ and $|\vec{X}| \sim 0$ remains an open problem.

2.1.2. Necessary conditions for CFTs with semi-classical gravity dual

Necessary conditions are known for a CFT to be dual to a weakly coupled gravitational theory in AdS (possibly including higher-spin modes). They read as follows [24]:

1. The theory has a large number of degrees of freedom; in 2D this means a large central charge.
2. The low-energy spectrum is sparse.
3. Higher-point correlators factorize into products of two-point functions (large- N factorization).

These are commonly referred to as large central charge, sparse spectrum, and large- N factorization, respectively. While stated in general dimensions, we will henceforth specialize to the $\text{AdS}_3/\text{CFT}_2$ case.

The first condition corresponds, via $c = 3l/(2G)$, to small Newton's constant G in AdS, i.e. to the validity of the classical gravity approximation.

The third condition corresponds to weak coupling in AdS, where Wick's theorem applies.

For the second condition, a concrete upper bound on the spectrum i.e. the HKS bound is known [25]. See related work for correlators in [26]. We briefly review its origin. On the AdS side with torus boundary conditions, the dominant saddle points are thermal AdS and the BTZ black hole. As one raises the temperature of thermal AdS, a phase transition occurs to the BTZ black hole. Therefore, a phase transition is expected on the CFT side as one varies the temperature. In two-dimensional CFT the modular parameter τ of the torus plays the role of inverse temperature. Modular

invariance severely constrains the density of states, and as a result the low-energy spectrum obeys an upper bound. This is the HKS bound: the density of states $\rho(h)$ at conformal weight (h, h) satisfies

$$\rho(h) \lesssim e^{2\pi h}, \quad (h < \frac{c}{12}). \quad (2.12)$$

Moreover, when the bulk theory is Einstein gravity at low energies (i.e. without light higher-spin fields; we refer to such CFTs as *holographic CFTs* in this thesis), the theory exhibits chaotic behavior and there are even stronger constraints on the low-energy spectrum.

From the bottom-up perspective, for any seed $\mathcal{C}$, the symmetric orbifold $\text{Sym}^N \mathcal{C}$ can be arranged to satisfy several of the necessary holographic conditions by taking N large. For the first condition (large central charge), the central charge is N times that of the seed, so this is automatic at large N . For the second condition (sparse spectrum), it has been shown that symmetric orbifold CFTs satisfy the HKS bound at large N , independent of the seed [27, 25]. Finally, for the third condition (large- N factorization), this has been verified for all operators [6, 28, 29, 30].

However, properties such as chaos [31] and the stronger low-energy spectral constraints required of holographic CFTs are *not* satisfied. Thus a symmetric orbifold CFT, as is, is not holographic. To obtain a holographic CFT, one is led to turn on marginal deformations of $\text{Sym}^N \mathcal{C}$ for special choices of seeds, as in the D1-D5 case. See recent developments of symmetric orbifolds of supersymmetric minimal models and their marginal deformations in [32, 33, 34, 35, 36, 37, 38, 39, 40, 41].

2.1.3. AdS_3 string with pure NSNS flux

In very recently, AdS_3 string theory with pure NSNS flux background is proved to be dual to symmetric orbifold with some appropriate marginal deformations. In the string theory side, the worldsheet CFT is level k $\text{SL}(2, \mathbb{C})$ WZW model times X that total central charge is 26. In particular, type IIB superstring on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ with unit NSNS flux is completely dual to the grand-canonical ensemble of symmetric orbifold $\text{Sym}^N(\mathbb{T}^4 \text{ SCFT})$. We will see details in Chap. 6.

2.2. Preliminaries of Symmetric Orbifold CFT

We denote the base Riemann surface where CFT lives as X .

2.2.1. Definition of the symmetric product orbifold $\text{Sym}^N \mathcal{C}$

We start from a seed CFT $\mathcal{C}$ with central charge c and a positive integer N . The *symmetric product orbifold* is defined by gauging the permutation symmetry:

$$\text{Sym}^N \mathcal{C} := \mathcal{C}^{\otimes N} / S_N, \quad c_{\text{total}} = Nc. \quad (2.13)$$

This is the standard S_N orbifold construction specialized to permutation symmetry.

Hilbert space and twisted sectors. Let $\text{Conj}(S_N)$ denote the set of conjugacy classes of S_N (equivalently, partitions of N). The orbifold Hilbert space is the direct sum over twisted sectors labeled by conjugacy classes, followed by the S_N -invariant projection:

$$\mathcal{H}_{\text{orb}} = \bigoplus_{[\sigma] \in \text{Conj}(S_N)} (\mathcal{H}_{\mathcal{C}^{\otimes N}})^{\sigma} \Big|_{S_N\text{-inv.}}. \quad (2.14)$$

A conjugacy class is specified by a cycle type $\prod_{n \geq 1} n^{m_n}$ with $\sum_{n \geq 1} n m_n = N$. An n -cycle twist identifies n copies into a single “long string” of length n , leading to fractional modes by $1/n$ and a ground-state energy shift

$$h_{\text{tw}(n)} = \frac{h_0}{n} + \frac{c}{24} \left(n - \frac{1}{n} \right), \quad \bar{h}_{\text{tw}(n)} = \frac{\bar{h}_0}{n} + \frac{c}{24} \left(n - \frac{1}{n} \right), \quad (2.15)$$

where $(h_0, \bar{h}_0)$ is the seed ground weight in the corresponding sector.

Operator content. For a fixed cycle type $\prod_n n^{m_n}$ the sector factorizes into the tensor product of the m_n Fock spaces of length- n long strings, followed by the S_N -invariant projection. This “long-string” picture underlies the combinatorics entering the partition function and will be used below to derive the generating-function form leading to DMVV [42, 43].

Let us clarify the action of S_N on $\mathcal{C}^{\otimes N}$. Label fields living in $\mathcal{C}$ by an index I , writing them as ϕ_I . Then a field $\Phi_{I_1, \dots, I_N}$ in $\mathcal{C}^{\otimes N}$ is the tensor product

$$\Phi_{I_1, \dots, I_N} = \bigotimes_{i=1}^N \phi_{I_i}, \quad (2.16)$$

i.e. ϕ_{I_i} lives in the i -th factor of $\mathcal{C}$. As in Definition

$$g : \Phi_{I_1, \dots, I_N} \rightarrow \Phi_{I_{g^{-1}(1)}, \dots, I_{g^{-1}(N)}}, \quad \Phi_{I_{g^{-1}(1)}, \dots, I_{g^{-1}(N)}} = \bigotimes_{i=1}^N \phi_{I_{g^{-1}(i)}}, \quad (2.17a)$$

namely the field ϕ_{I_i} originally in the i -th copy moves to the $g(i)$ -th copy under the action of g .

2.2.2. Sphere partition function

The symmetric orbifold can be viewed as a discrete gauging of the permutation group S_N . From this point of view, the orbifold partition function on a manifold X is obtained by summing over flat S_N bundles on X with an overall factor $1/N!$ coming from the gauging [44]:

$$Z_{\text{Sym}^N \mathcal{C}}^X = \frac{1}{N!} \sum_{\text{flat } S_N \text{ bundles on } X} Z_{\mathcal{C}^{\otimes N}}^X(\text{holonomies on } X). \quad (2.18)$$

On the sphere $X = S^2$, the only flat S_N bundle is the trivial one. Let the sphere partition of the seed theory $\mathcal{C}$ be $z = Z_{\mathcal{C}}^{X=S^2}$. If we focus on the contribution of the untwisted vacuum sector, the sphere partition function of the symmetric product orbifold at fixed N therefore behaves as

$$Z_{\text{Sym}^N \mathcal{C}}^{S^2} = \frac{1}{N!} Z_{\mathcal{C}^{\otimes N}}^{S^2} = \frac{1}{N!} z^N, \quad (2.19)$$

where in the last equality we used $Z_{\mathcal{C}^{\otimes N}}^{S^2} = z^N$ for the tensor product theory. Thus the characteristic combinatorial factor $1/N!$ of the symmetric product orbifold already appears at the level of the sphere partition function and can be traced directly to the discrete gauging of the permutation group S_N .

Inner products and the sphere path integral. Accordingly, when we identify the inner product of any states $|a\rangle, |b\rangle$ with a sphere path integral with insertions a, b , the *physical* inner product must carry this $1/N!$ factor:

$$\langle a|b\rangle_{\text{phys}} = \frac{1}{N!} \langle a|b\rangle, \quad (2.20)$$

where $\langle a|b\rangle$ denotes the algebraic inner product computed in the symmetric product theory without the $1/N!$ prefactor. In particular, taking $a = b = 0$ gives the vacuum overlap¹,

$$\langle 0|0\rangle_{\text{phys}} = \frac{1}{N!} \langle 0|0\rangle = \frac{1}{N!} \left(Z_{\mathcal{C}}^{S^2} \right)^N = \frac{1}{N!} z^N, \quad (2.21)$$

in agreement with the sphere partition function (2.19).

¹More precisely, the vacuum state of the symmetric orbifold $|0\rangle$ is $|0\rangle_{\text{seed}}^{\otimes N}$.

2.2.3. Torus partition function

Following the standard orbifold trace formula [45, 46, 47, 48], the torus partition function of the symmetric product orbifold $\text{Sym}^N \mathcal{C}$ is

$$Z_{\text{Sym}^N \mathcal{C}}^{\mathbb{T}^2}(\tau) = \frac{1}{|S_N|} \sum_{\substack{g, h \in S_N \\ gh=hg}} Z_{\mathcal{C}^{\otimes N}}^{\mathbb{T}^2} \begin{bmatrix} g \\ h \end{bmatrix}(\tau), \quad (2.22)$$

where $Z \begin{bmatrix} g \\ h \end{bmatrix}$ denotes the trace on $\mathcal{C}^{\otimes N}$ with g -twist along the spatial cycle and h -insertion along the temporal cycle.

Any commuting pair (g, h) can be simultaneously brought to a canonical block form labeled by integers $a, d \geq 1$ and $b \in \{0, \dots, d-1\}$ with $n = ad$: in each block, g is a product of a cycles of length d , while h permutes these d cycles and translates by b inside each cycle. Geometrically this corresponds to an n -sheeted cover of the torus with $\tau \mapsto \frac{a\tau+b}{d}$. Therefore a single (a, d, b) -block contributes

$$Z_{\mathcal{C}}^{\mathbb{T}^2} \left(\frac{a\tau + b}{d} \right) \quad (2.23)$$

to the trace [43].

Counting inequivalent assemblies of such blocks with total size $\sum_{a,d,b} ad m_{a,d,b} = N$ and dividing by automorphisms uses the cycle-index theorem for S_N [43]. Passing to the p -generating function in N yields the exponentiated form

$$\sum_{N \geq 0} p^N Z_{\text{Sym}^N \mathcal{C}}^{\mathbb{T}^2}(\tau) = \prod_{a,d \geq 1} \exp \left[p^{ad} \frac{1}{ad} \sum_{b=0}^{d-1} Z_{\mathcal{C}}^{\mathbb{T}^2} \left(\frac{a\tau + b}{d} \right) \right]. \quad (2.24)$$

Hecke operator and torus partition function.

Let us first define Jacobi-like functions. Let $\phi(\tau, z)$ be a function on $\mathbb{H} \times \mathbb{C}$, which satisfies the following conditions:

$$\phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k \exp \left(\frac{2\pi i m c z^2}{c\tau + d} \right) \phi(\tau, z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad (2.25a)$$

$$\phi(\tau, z + \lambda\tau + \mu) = \exp \left(-2\pi i m (\lambda^2 \tau + 2\lambda z) \right) \phi(\tau, z), \quad \lambda, \mu \in \mathbb{Z}, \quad (2.25b)$$

for some *weight* $k \in \mathbb{Z}$ and *index* $m \in \mathbb{Z}_{\geq 0}$. If $k = m = 0$, ϕ is independent of z and becomes a modular invariant function.

Definition of Hecke operator. For a positive integer n , the n -th Hecke operator T_n acts on Jacobi-like function ϕ by

$$(T_n \phi)(\tau, z) := \frac{1}{n} \sum_{\substack{a, d \in \mathbb{Z}_{>0} \\ ad=n}} \sum_{b=0}^{d-1} \phi\left(\frac{a\tau + b}{d}, az\right). \quad (2.26)$$

When ϕ has index $m = 0$, say $\phi(\tau, z) = Z(\tau)$, this reduces to

$$(T_n Z)(\tau) := \frac{1}{n} \sum_{\substack{a, d \in \mathbb{Z}_{>0} \\ ad=n}} \sum_{b=0}^{d-1} Z\left(\frac{a\tau + b}{d}\right). \quad (2.27)$$

Note that the Hecke operator maps Jacobi-like functions with weight and index (k, m) to another Jacobi-like functions with weight and index (k, m) .

Torus partition function with Hecke operators. Grouping terms in (2.24) at fixed $n = ad$, one arrives at the Hecke-operator form of generating function

$$\sum_{N \geq 0} p^N Z_{\text{Sym}^N \mathcal{C}}^{\mathbb{T}^2}(\tau) = \exp\left(\sum_{n \geq 1} p^n (T_n Z_{\mathcal{C}}^{\mathbb{T}^2})(\tau)\right). \quad (2.28)$$

Let $(m_k)_{k \in \mathbb{Z}_{>0}}$ be a non-negative integer sequence that satisfies $\sum_{k > 0} k m_k = N$. The torus partition function of N -th symmetric orbifold can be written as a sum over such sequence as

$$Z_{\text{Sym}^N \mathcal{C}}^{\mathbb{T}^2}(\tau) = \sum_{\substack{m_k \in \mathbb{Z}_{\geq 0} \\ \text{for } k \in \mathbb{Z}_{>0} \\ \text{s.t. } \sum_k k m_k = N}} \prod_{k \geq 1} \frac{1}{m_k!} \left((T_k Z_{\mathcal{C}}^{\mathbb{T}^2})(\tau) \right)^{m_k}. \quad (2.29)$$

DMVV formula and the spectrum of symmetric orbifolds.

Let us rewrite the formula (2.24) to obtain the famous DMVV formula. Suppose the modular-invariant torus partition function of the seed CFT $\mathcal{C}$ is

$$Z_{\mathcal{C}}^{\mathbb{T}^2}(\tau) = \sum_{m, \bar{m} \in I} d(m, \bar{m}) q^m \bar{q}^{\bar{m}}. \quad (2.30)$$

R.H.S. of (2.24) can be rewritten as

$$\prod_{a, d \geq 1} \prod_{m, \bar{m} \in I} \left(\exp \left[\sum_{b=0}^{d-1} \frac{1}{ad} p^{ad} q^{\frac{a}{d}(m - \frac{c}{24})} \bar{q}^{\frac{a}{d}(\bar{m} - \frac{c}{24})} e^{2\pi i \frac{b}{d}(m - \bar{m})} \right] \right)^{d(m, \bar{m})}.$$

By using the following identities

$$\exp\left(\sum_{a=1}^{\infty} \frac{x^a}{a}\right) = \frac{1}{1-x} \quad \text{and} \quad \frac{1}{d} \sum_{b=0}^{d-1} e^{2\pi i \frac{b}{d}(m-\bar{m})} = \begin{cases} 1 & \text{if } m - \bar{m} \equiv 0 \pmod{d} \\ 0 & \text{else} \end{cases}, \quad (2.31)$$

we obtain

$$\sum_{N \geq 0} p^N Z_{\text{Sym}^N \mathcal{C}}^{\mathbb{T}^2}(\tau) = \prod_{w \geq 1} \prod_{\substack{(h, \bar{h}) \in I \\ h - \bar{h} \equiv 0 \pmod{w}}} \left(\frac{1}{1 - p^w q^{\frac{m}{w}} \bar{q}^{\frac{\bar{m}}{w}}} \right)^{d(h, \bar{h})}. \quad (2.32)$$

One can read off the spectrum of symmetric orbifolds from it.

2.2.4. Elliptic Genera

When the seed CFT $\mathcal{M}$ is an $\mathcal{N} = 2$ SCFT, the elliptic genus of $\mathcal{M}$ and that of the symmetric orbifold are related. If the elliptic genus $\chi_{\mathcal{M}}(\tau, z)$ of $\mathcal{M}$ is Jacobi-like function that has a fourier expansion

$$\chi_{\mathcal{M}}(\tau, z) = \sum_{n \geq 0, l \in \mathbb{Z}} c(n, l) q^n y^l, \quad (2.33)$$

then the generating function of elliptic genera $\chi_{\text{Sym}^N \mathcal{M}}$ of $\text{Sym}^N \mathcal{M}$ is known to be

$$\begin{aligned} \sum_{N \geq 0} p^N \chi_{\text{Sym}^N \mathcal{M}}(\tau, z) &= \prod_{n > 0} \prod_{m \geq 0} \prod_{l \in \mathbb{Z}} \frac{1}{(1 - p^n q^m y^l)^{c(nm, l)}} \\ &= \exp \left(\sum_{k \geq 1} p^k (T_k \chi_{\mathcal{M}})(\tau, z) \right). \end{aligned} \quad (2.34)$$

See [42] for a proof and further details.

2.3. Boundary states in symmetric orbifold

In the symmetric orbifold, one can consider the set of Cardy boundary states [49, 50] related to the character of S_N . In this section we review the construction of boundary states in symmetric product orbifolds following [49].

The boundary states preserving the extended symmetry algebra of the symmetric product orbifold are labeled by the data

$$(\vec{n}, \vec{r}; \vec{a}) = \left\{ (n_i, r_i; a_i) \right\}_{i=1}^{n_b}$$

with $\sum_i n_i = N$, n_b for the number of seed Cardy boundary states, a_i for the label of seed Cardy boundary states and r_i for the label of irreducible representations of S_{n_i} . Boundary states take the form

$$|\vec{n}, \vec{r}; \vec{a}\rangle = \sum_{h \in S_N} \prod_{i=1}^{n_b} \left(\sum_{g_i \in S_{n_i}} \frac{\chi_{r_i}(g_i)}{n_i!} |a_i\rangle_{hg_i h^{-1}} \right).$$

Note that we do *not* include any overall factor $1/\sqrt{N!}$ appeared in the original paper [49]. Let us start by explaining it.

2.3.1. Normalization of boundary states and Cardy condition

Before constructing the set of boundary states, let us recall that we normalize the inner product as in (2.20). Let $|a\rangle$ and $|b\rangle$ be the Cardy boundary states of the symmetric orbifold. The annulus amplitude in the closed string channel is obtained from the physical inner product,

$$Z_{ab}^{(\text{closed})}(\tilde{\tau}) = \langle a | \tilde{q}^{L_0 + \bar{L}_0 - \frac{c}{12}} | b \rangle_{\text{phys}} = \frac{1}{N!} \langle a | \tilde{q}^{L_0 + \bar{L}_0 - \frac{c}{12}} | b \rangle, \quad (2.35)$$

while the open string channel is given by

$$Z_{ab}^{(\text{open})}(\tau) = \sum_i n_{ab}^i \chi_i(\tau), \quad n_{ab}^i \in \mathbb{Z}_{\geq 0}. \quad (2.36)$$

Using (2.35), we see that the factor $1/N!$ appears as a universal overall multiplier of all closed-channel amplitudes. Therefore it does not affect the Cardy relation between closed and open channels, and the normalization (2.35) is fully compatible with the standard Cardy consistency condition, leaving the integer open-string multiplicities unchanged.

We also define the corresponding boundary entropy in the symmetric orbifold from the physical inner product with the orbifold vacuum,

$$g_a = \langle 0 | a \rangle_{\text{phys}}. \quad (2.37)$$

In terms of the algebraic overlap this becomes

$$g_a = \frac{1}{N!} \langle 0 | a \rangle, \quad \log g_a = \log \langle 0 | a \rangle - \log N!, \quad (2.38)$$

so that all boundary entropies are shifted by the same universal constant $-\log N!$.

2.3.2. Boundary states in twisted sectors

We now explain how to construct boundary states in the twisted sectors following [49], and how they are assembled into the building blocks $|a\rangle_g$ that appear in the full Cardy boundary states.

Seed Cardy states and twisted Cardy states.

We start from the seed theory $\mathcal{C}$ and assume that its conformal boundary conditions are known. Let $|a\rangle$ be a Cardy boundary state of the seed theory. As usual, it can be expanded in terms of Ishibashi states $|h\rangle\rangle$ labeled by Virasoro primaries with $h = \bar{h}$:

$$|a\rangle = \sum_h a_h |h\rangle\rangle, \quad (L_n - \bar{L}_{-n}) |h\rangle\rangle = 0 \quad (n \in \mathbb{Z}), \quad (2.39)$$

where a_h are the usual Cardy coefficients determined by the seed theory modular data.

In the twist- k sector, we define the corresponding *twisted Cardy state* $|h\rangle\rangle_{(k)}$ by keeping the coefficients a_h fixed and replacing each untwisted Ishibashi state by its twisted counterpart:

$$|h\rangle\rangle_{(k)} = \sum_n |h, n\rangle_{(k)} \otimes \overline{|h, n\rangle_{(k)}}, \quad |a\rangle_{(k)} = \sum_h a_h |h\rangle\rangle_{(k)}. \quad (2.40)$$

Twisted Ishibashi states $|h\rangle\rangle_{\text{twist-}k}$ live in the sector where k copies of the seed theory are glued into a single long string, and hence satisfy the generalized gluing conditions with fractional Virasoro modes. Hence, n summation includes fractional Virasoro descendants. It should be viewed as the natural image of the seed boundary condition a in the twist- k sector; as a consequence, they satisfy

$${}_{(k)}\langle a | \tilde{q}^{L_0 + \bar{L}_0 - \frac{c}{12}} | b \rangle_{(k)} = Z_{ab}^{(\text{closed})}(\tilde{\tau}/k) = Z_{ab}^{(\text{open})}(k\tau). \quad (2.41)$$

From single cycles to general permutations.

So far we have considered a single cycle of length k . A general twisted sector of the orbifold is labeled by a permutation $g \in S_N$ with cycle decomposition

$$g = \sigma_1 \sigma_2 \cdots \sigma_m, \quad (2.42)$$

where each σ_α is a disjoint cycle of some length k_α , and possibly trivial cycle of length 1 so that k_α s satisfy $\sum_\alpha k_\alpha = N$. Additionally, let us set $\sigma_\alpha \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})$. For a fixed seed boundary condition a , we associate to each cycle σ_α a twist- k_α boundary

block acting on the set of copies permuted by σ_α :

$$|a\rangle_{\sigma_\alpha \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})}. \quad (2.43)$$

The tensor product over all cycles then yields a state in the g -twisted sector,

$$|a\rangle_g = \bigotimes_{\alpha=1}^m |a\rangle_{\sigma_\alpha \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})}. \quad (2.44)$$

To incorporate multiple types of seed boundary conditions, it is convenient to specify which copies of the seed theory are involved in each cycle. Following [49], let us introduce for each block (n_i, N_i) a subgroup $S_{n_i}(N_i) \subset S_N$ acting on n_i consecutive copies starting at position $N_i + 1$, with

$$N_i = \sum_{j < i} n_j. \quad (2.45)$$

In this convention, we may need to denote the building block by specifying the set that g acts on:

$$|a_i\rangle_{g_i \curvearrowright \{N_i+1, \dots, N_i+n_i\}} \quad (2.46)$$

Summing over $h \in S_N$ generates all possible twisted states built from the same seed boundary condition a_i but with different choices of which copies are glued into cycles. We will denote such states compactly by

$$|a_i\rangle_{hg_ih^{-1}} := |a_i\rangle_{hg_ih^{-1} \curvearrowright \{N_i+1, \dots, N_i+n_i\}}, \quad (2.47)$$

with the understanding that g in $|a\rangle_g$ specifies both the cycle lengths and which copies are involved.

2.3.3. Set of Cardy boundary states in symmetric orbifolds

Let n_b for the number of seed Cardy boundary states. The Cardy boundary states in the symmetric orbifold are labeled by the data

$$(\vec{n}, \vec{r}; \vec{a}) = \left\{ (n_i, r_i; a_i) \right\}_{i=1}^{n_b}, \quad (2.48)$$

where

- n_b is the number of seed Cardy boundary states,
- a_i is i -th seed Cardy boundary condition out of n_b ,

- n_i specifies the size of the subset of $\{1, \dots, N\}$, which is the number of seeds associated with a_i ,
- r_i is an irreducible representation of S_{n_i} .

Let χ_{r_i} be the character of r_i . Following [49], the orbifold Cardy boundary states take the form²

$$|\vec{n}, \vec{r}; \vec{a}\rangle = \sum_{h \in S_N} \prod_{i=1}^{n_b} \left(\sum_{g_i \in S_{n_i}} \frac{\chi_{r_i}(g_i)}{n_i!} |a_i\rangle_{hg_i h^{-1}} \right). \quad (2.49)$$

Here $\vec{n} = (n_1, \dots, n_{n_b})$, $\vec{r} = (r_1, \dots, r_{n_b})$, $\vec{a} = (a_1, \dots, a_{n_b})$, and the states $|a_i\rangle_{hg_i h^{-1}}$ are the twisted Cardy states in the sector twisted by $hg_i h^{-1}$ constructed in the previous subsection.

2.3.4. Overlap of boundary states: homogeneous trivial representations

As a simple illustration, let us consider the case in which all copies carry the same seed boundary condition a or b , and we take the trivial representation of S_N . In our notation this corresponds to

$$(\vec{n}, \vec{r}; \vec{a})|_{a_i=a} = (N, \mathbf{1}; a), \quad (\vec{n}, \vec{r}; \vec{a})|_{a_i=b} = (N, \mathbf{1}; b), \quad (2.50)$$

where $\mathbf{1}$ denotes the trivial representation of S_N . The corresponding orbifold Cardy boundary states are obtained from (2.49) by setting $n_1 = N$ and $\chi_1(g) = 1$ for all $g \in S_N$,

$$|N, \mathbf{1}; a\rangle = \sum_{h \in S_N} \sum_{g \in S_N} \frac{1}{N!} |a\rangle_{hgh^{-1}}, \quad |N, \mathbf{1}; b\rangle = \sum_{h \in S_N} \sum_{g \in S_N} \frac{1}{N!} |b\rangle_{hgh^{-1}}. \quad (2.51)$$

In particular, each twisted sector appears exactly once (up to the overall normalization fixed in (2.35)).

The open-string partition function between the homogeneous orbifold boundary states $|N, \mathbf{1}; a\rangle$ and $|N, \mathbf{1}; b\rangle$ is computed by the overlap

$$Z_{(N,a),(N,b)}^{(\text{open})}(\tau) = \langle N, \mathbf{1}; a | \tilde{q}^{L_0 + \bar{L}_0 - \frac{Nc}{12}} | N, \mathbf{1}; b \rangle_{\text{phys}}. \quad (2.52)$$

Using the standard classification of conjugacy classes of S_N in terms of cycle multiplicities $\{m_k\}_{k \geq 1}$,

$$\sum_{k \geq 1} k m_k = N, \quad (2.53)$$

²Note again that there is no overall factor $1/\sqrt{N!}$ in (2.49); instead, the factor $1/N!$ appears in the physical inner product, as in eq. (2.20).

we can write the result as

$$Z_{(N,a),(N,b)}^{(\text{open})}(\tau) = \sum_{\{m_k\}: \sum k m_k = N} \prod_{k \geq 1} \frac{1}{m_k! k^{m_k}} \left[Z_{ab}^{(\text{open})}(k\tau) \right]^{m_k}. \quad (2.54)$$

It is convenient to package these amplitudes into a grand-canonical generating function. Using the combinatorics of symmetric products, the fixed- N expression (2.54) exponentiates to

$$\sum_{N \geq 0} p^N Z_{(N,a),(N,b)}^{(\text{open})}(\tau) = \exp \left[\sum_{k \geq 1} \frac{p^k}{k} Z_{ab}^{(\text{open})}(k\tau) \right]. \quad (2.55)$$

This is the open-string analogue of the DMVV formula, and parallels the grand-canonical boundary partition functions discussed in [50]. Let us expand the open-string partition function $Z_{ab}^{(\text{open})}(\tau)$ as

$$Z_{ab}^{(\text{open})}(\tau) = \sum_{m \in I_{ab}} d_{ab}(m) q^m \quad (2.56)$$

with $d_{ab}(m)$ as positive integers. We can rewrite (2.55) as the following product form:

$$\sum_{N \geq 0} p^N Z_{(N,a),(N,b)}^{(\text{open})}(\tau) = \prod_{m \in I_{ab}} \left(\frac{1}{1 - p q^m} \right)^{d_{ab}(m)}. \quad (2.57)$$

Consequently, the coefficient of p^N , $Z_{(N,a),(N,b)}^{(\text{open})}(\tau)$, has a q -expansion with non-negative integer coefficients, as required by the Cardy consistency condition.

2.4. Topological defects in symmetric orbifolds

Topological line defects offer a useful handle on generalized symmetries in two-dimensional CFTs, and in symmetric-product orbifolds $\text{Sym}^N(\mathcal{C})$ there is a distinguished “universal” class whose definition is essentially group-theoretic. In this section we construct these defects in a form suited for explicit computations, using twisted Ishibashi states, following [51]. See recent developments in [52, 53, 41].

We start from the seed RCFT, review Verlinde lines and their folded-theory boundary-state realization, and then lift the construction to $\text{Sym}^N(\mathcal{C})$ by building conjugacy-class projectors $P_{[g]}$ and checking their orthogonality and idempotency. This leads to universal defect operators $\mathcal{D}_R$ labeled by irreducible representations R of S_N , together with their diagonal action on twisted sectors and their fusion algebra. Finally, we rewrite $\mathcal{D}_R$ as boundary states in the folded orbifold and verify consistency via cylinder overlaps, recovering the symmetric-orbifold torus partition function (DMVV) from

the trivial-defect overlap.

2.4.1. Defects in seed theory and folding trick

Verlinde line in seed CFT

We first review the construction of the trivial topological defect of the seed RCFT and its realization as a boundary state in the folded theory, following the notation of [51]. Consider a rational CFT $\mathcal{C}$ with chiral algebra $\mathcal{A}$ and a diagonal torus partition function

$$Z_{\mathcal{C}}(\tau, \bar{\tau}) = \sum_h \chi_h(\tau) \bar{\chi}_h(\bar{\tau}), \quad (2.58)$$

where h labels the irreducible representations of $\mathcal{A}$ and $\chi_h(\tau)$ is the corresponding character. We choose an orthonormal basis of states built on primary h and its descendants,

$$\left\{ |h, n\rangle, \quad \overline{|h, m\rangle} \right\}, \quad (2.59)$$

where n, m collectively labels holomorphic/anti-holomorphic descendants. Note that Ishibashi states are constructed as

$$|h\rangle\rangle = \sum_n |h, n\rangle \otimes \overline{|h, n\rangle}. \quad (2.60)$$

The projector onto this representation in the full Hilbert space is then

$$P_h = \sum_{n, m} |h, n\rangle \otimes \overline{|h, m\rangle} \langle h, n| \otimes \overline{\langle h, m|}. \quad (2.61)$$

The Verlinde line associated with a primary a is given by

$$\mathcal{D}_a = \sum_h \frac{S_{ah}}{S_{0h}} P_h, \quad (2.62)$$

see eq. (2.2) of [51]. The trivial defect line of the seed theory is obtained by choosing $a = 0$, corresponding to the identity primary. We get the following by simple computation:

$$\mathcal{D}_{\text{triv}} \equiv \mathcal{D}_0 = \sum_h P_h = \mathbf{1}_{\mathcal{H}}. \quad (2.63)$$

Folding trick and boundary state.

It is sometimes useful to describe this defect as a boundary condition. To do so, we apply the folding trick to the line defect which separates two copies of the seed theory.

We consider the folded theory

$$\mathcal{C}^{(-)} \otimes \mathcal{C}^{(+)}, \quad (2.64)$$

where the superscript $(-), (+)$ labels the two copies. Following eq. (2.7) of [51], the $\mathbb{Z}_2$ permutation brane associated with the Verlinde line $\mathcal{D}_a$ is described by the boundary state

$$|\mathcal{D}_a\rangle\rangle = \sum_h \frac{S_{ah}}{S_{0h}} \left(\sum_n |h, n\rangle^{(-)} \otimes \overline{|h, n\rangle}^{(+)} \right) \left(\sum_m |h, m\rangle^{(+)} \otimes \overline{|h, m\rangle}^{(-)} \right). \quad (2.65)$$

Each term in (2.65) is an explicit tensor product of four states,

$$|h, n\rangle^{(-)} \otimes \overline{|h, n\rangle}^{(+)} \otimes |h, m\rangle^{(+)} \otimes \overline{|h, m\rangle}^{(-)}, \quad (2.66)$$

showing that the folded boundary state lives in the tensor product of the holomorphic and anti-holomorphic Hilbert spaces of both copies.

For the trivial defect we again set $a = 0$, so that $S_{0i}/S_{0i} = 1$ and obtain the boundary state

$$|\mathcal{D}_{\text{triv}}\rangle\rangle = \sum_h \left(\sum_n |h, n\rangle^{(-)} \otimes \overline{|h, n\rangle}^{(+)} \right) \left(\sum_m |h, m\rangle^{(+)} \otimes \overline{|h, m\rangle}^{(-)} \right). \quad (2.67)$$

This is the boundary state realization of the trivial topological defect of the seed theory.

Overlap and recovery of torus partition function.

We now consider, in the folded theory, a cylinder whose two ends are both endowed with trivial defect boundary states. One can expect that this matches the torus partition function. See Fig. 2.1. Now let us explicitly calculate

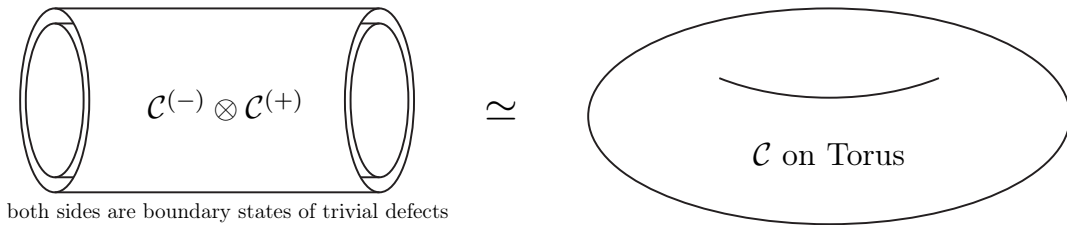


Figure 2.1.: Trivial defects overlap in seed CFT. Its geometry is equivalent to the torus with doubled periodicity of the original cylinder length.

$$Z_{\text{cyl}}^{(\text{triv}, \text{triv})} \left(\frac{\tilde{\tau}}{2} \right) = \langle\langle \mathcal{D}_{\text{triv}} | \sqrt{\tilde{q}}^{L_0^{(-)} + L_0^{(+)} - \frac{c}{12}} \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + \bar{L}_0^{(+)} - \frac{c}{12}} | \mathcal{D}_{\text{triv}} \rangle\rangle, \quad \tilde{q} = e^{2\pi i \tilde{\tau}}, \quad (2.68)$$

where $\tilde{\tau} \in i\mathbb{R}_{>0}$ and $L_0^{(-)}, L_0^{(+)}$ (and similarly for $\bar{L}_0$) act on $(-)$ and $(+)$ copy, respectively. Inserting (2.67) and using orthonormality,

$$\langle h, n | h', m \rangle = \overline{\langle h, n | h', m \rangle} = \delta_{hh'} \delta_{nm}, \quad (2.69)$$

the overlap factorizes into holomorphic and anti-holomorphic pieces:

$$\begin{aligned} Z_{\text{cyl}}^{(\text{triv}, \text{triv})} \left(\frac{\tilde{\tau}}{2} \right) &= \sum_h \left[\sum_n \langle h, n | \sqrt{\tilde{q}}^{L_0 - \frac{c}{24}} | h, n \rangle \overline{\langle h, n |} \sqrt{\tilde{q}}^{\bar{L}_0 - \frac{c}{24}} \overline{| h, n \rangle} \right] \\ &\quad \times \left[\sum_m \overline{\langle h, m |} \sqrt{\tilde{q}}^{\bar{L}_0 - \frac{c}{24}} \overline{| h, m \rangle} \langle h, m | \sqrt{\tilde{q}}^{L_0 - \frac{c}{24}} | h, n \rangle \right] \\ &= \sum_h \chi_h(\tilde{\tau}) \bar{\chi}_h(\bar{\tilde{\tau}}). \end{aligned} \quad (2.70)$$

In the second line we recognized the sums over descendants as the holomorphic and anti-holomorphic characters of the representation i . Therefore the cylinder amplitude with trivial defects on both ends reproduces the torus partition function of the seed theory by doubling the cylinder length:

$$Z_{\text{cyl}}^{(\text{triv}, \text{triv})} \left(\frac{\tilde{\tau}}{2} \right) = Z_{\mathcal{C}}(\tilde{\tau}, \bar{\tilde{\tau}}) = Z_{\mathcal{C}}(\tau, \bar{\tau}), \quad (2.71)$$

showing explicitly how the overlap of boundary states of the trivial defects is translated into the character sum defining the torus partition function.

2.4.2. Defects in symmetric orbifolds

In this section, we construct the projection operator from the language of twisted Ishibashi states introduced in the boundary state section. Also, we see how the physical inner product rule is incorporated to the definition and algebra of the defect.

Let us start from constructing the complete system. Recall the definition of twist- k Ishibashi states:

$$|h\rangle\rangle_{(k)} = \sum_n |h, n\rangle_{(k)} \otimes \overline{|h, n\rangle_{(k)}}. \quad (2.72)$$

Here, n summation includes all possible fractional Virasoro descendants. In that sense, the following definition is not sufficient because states are not projected onto the twisted sector for generic n, m :

$$P_{[g]} \stackrel{?}{=} \sum_h \sum_{n, m} |h, n\rangle_{[g]} \otimes \overline{|h, m\rangle_{[g]}} \langle h, n | \otimes_{[g]} \overline{\langle h, m |}. \quad (2.73)$$

To resolve this problem, we need to introduce a new projector³ Π_g :

$$\Pi_g := \frac{1}{|C(g)|} \sum_{h \in C(g)} h. \quad (2.75)$$

We will describe its action on states later. The precise definition of the projection operator is

$$\begin{aligned} P_{[g]} := \sum_{h^-, h^+ \in S_N} \frac{1}{|C(g)|} \sum_{h, n, m} \Pi_{h^- g(h^-)^{-1}} |h, n\rangle_{h^- g(h^-)^{-1}} \otimes \overline{|h, m\rangle}_{h^- g(h^-)^{-1}} \\ \times_{h^+ g(h^+)^{-1}} \langle h, n| \otimes_{h^+ g(h^+)^{-1}} \overline{\langle h, m|} \Pi_{h^+ g(h^+)^{-1}}. \end{aligned} \quad (2.76)$$

Projector onto twisted sectors.

Let us take a closer look at (2.76), starting from a permutation $g \in S_N$ with cycle decomposition

$$g = \sigma_1 \sigma_2 \cdots \sigma_m, \quad (2.77)$$

where each σ_α is a disjoint cycle of some length k_α , and possibly trivial cycle of length 1 so that k_α s satisfy $\sum_\alpha k_\alpha = N$. Additionally, let us set $\sigma_\alpha \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})$.

For fixed labels h, n, m , we associate to each cycle σ_α a twist- k_α building block acting on the set of copies permuted by σ_α :

$$|h, n\rangle_{\sigma_\alpha} \otimes \overline{|h, m\rangle}_{\sigma_\alpha} \sigma_\alpha \langle h, n| \otimes \sigma_\alpha \overline{\langle h, m|}, \quad (2.78)$$

with σ_α as a shorthand of $\sigma_\alpha \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})$. The tensor product over all cycles then yields a building block in the g -twisted sector,

$$\begin{aligned} \sum_{h, n, m} |h, n\rangle_g \otimes \overline{|h, m\rangle}_g g \langle h, n| \otimes g \overline{\langle h, m|} \\ := \bigotimes_{\alpha=1}^m \sum_{h, n, m} |h, n\rangle_{\sigma_\alpha} \otimes \overline{|h, m\rangle}_{\sigma_\alpha} \sigma_\alpha \langle h, n| \otimes \sigma_\alpha \overline{\langle h, m|}. \end{aligned} \quad (2.79)$$

Note that the first sum $\sum_{h, n, m}$ is an abuse of the notation.

However, actually, this is not enough since for generic h, n, m this tensor product may include non-integer spin that should be removed. To resolve this, let us define a

³ $C(g)$ means followings: let $[g]$ be a conjugacy class in S_N and $g \in S_N$ a fixed representative. Its centralizer is

$$C(g) = \{h \in S_N \mid hg = gh\}, \quad |[g]| = \frac{|S_N|}{|C(g)|} = \frac{N!}{|C(g)|}. \quad (2.74)$$

projection Π_{σ_α} for each twist- k_α building block

$$|h, n\rangle_{\sigma_\alpha \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})} \otimes \overline{|h, m\rangle}_{\sigma_\alpha \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})}.$$

Π_{σ_α} is defined as follows:

$$\begin{aligned} \Pi_{\sigma_\alpha \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})} &:= \frac{1}{k_\alpha} \sum_{b=0}^{k_\alpha-1} \exp \left[2\pi i b \left(L_0^{(i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})} - \bar{L}_0^{(i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})} \right) \right] \\ &= \begin{cases} 1 & \text{if } L_0 - \bar{L}_0 \equiv 0 \in \mathbb{Z} \\ 0 & \text{else} \end{cases}. \end{aligned} \quad (2.80)$$

By acting it, we obtain the following “safe” building block

$$\Pi_{\sigma_\alpha} |h, n\rangle_{\sigma_\alpha} \otimes \overline{|h, m\rangle}_{\sigma_\alpha} \sigma_\alpha \langle h, n| \otimes \sigma_\alpha \overline{\langle h, m|} \Pi_{\sigma_\alpha}, \quad (2.81)$$

and their tensor product gives

$$\begin{aligned} \sum_{h, n, m} \Pi_g |h, n\rangle_g \otimes \overline{|h, m\rangle}_g \langle h, n| \otimes_g \overline{\langle h, m|} \Pi_g \\ := \bigotimes_{\alpha=1}^m \sum_{h, n, m} \Pi_{\sigma_\alpha} |h, n\rangle_{\sigma_\alpha} \otimes \overline{|h, m\rangle}_{\sigma_\alpha} \sigma_\alpha \langle h, n| \otimes \sigma_\alpha \overline{\langle h, m|} \Pi_{\sigma_\alpha}. \end{aligned} \quad (2.82)$$

Note again that the first sum $\sum_{h, n, m}$ and projector Π_g are abuses of the notation.

Finally, we need to rotate both sides respectively, by using $h_\pm \in S_N$:

$$\begin{aligned} \Pi_{h_- \sigma_\alpha h_-^{-1}} |h, n\rangle_{h_- \sigma_\alpha h_-^{-1}} \otimes \overline{|h, m\rangle}_{h_- \sigma_\alpha h_-^{-1}} \\ \times {}_{h_+ \sigma_\alpha h_+^{-1}} \langle h, n| \otimes {}_{h_+ \sigma_\alpha h_+^{-1}} \overline{\langle h, m|} \Pi_{h_+ \sigma_\alpha h_+^{-1}}, \end{aligned} \quad (2.83)$$

with $h_\pm \sigma_\alpha h_\pm^{-1}$ as a shorthand of $h_\pm \sigma_\alpha h_\pm^{-1} \curvearrowright h_\pm(i(\alpha)_1, \dots, i(\alpha)_{k_\alpha})$. This gives the elementary projector $P_{[g]}$ on $[g]$:

$$\begin{aligned} P_{[g]} &:= \sum_{h_\pm \in S_N} \frac{1}{|C(g)|} \sum_{h, n, m} \Pi_{h_- g h_-^{-1}} |h, n\rangle_{h_- g h_-^{-1}} \otimes \overline{|h, m\rangle}_{h_- g h_-^{-1}} \\ &\quad \times {}_{h_+ g h_+^{-1}} \langle h, n| \otimes {}_{h_+ g h_+^{-1}} \overline{\langle h, m|} \Pi_{h_+ g h_+^{-1}} \\ &= \sum_{h_\pm \in S_N} \frac{1}{|C(g)|} \bigotimes_{\alpha=1}^m \sum_{h, n, m} \Pi_{h_- \sigma_\alpha h_-^{-1}} |h, n\rangle_{h_- \sigma_\alpha h_-^{-1}} \otimes \overline{|h, m\rangle}_{h_- \sigma_\alpha h_-^{-1}} \\ &\quad \times {}_{h_+ \sigma_\alpha h_+^{-1}} \langle h, n| \otimes {}_{h_+ \sigma_\alpha h_+^{-1}} \overline{\langle h, m|} \Pi_{h_+ \sigma_\alpha h_+^{-1}}, \end{aligned} \quad (2.84)$$

It is immediate that $P_{[g]}$ is self-adjoint $P_{[g]}^\dagger = P_{[g]}$ and orthogonal $P_{[g_0]} P_{[g_1]} \propto \delta_{[g_0], [g_1]}$.

Idempotency of $P_{[g]}$.

It is instructive to check explicitly that $P_{[g]}$ is idempotent: $P_{[g]}^2 = P_{[g]}$. We need to take inner product once, and hence we have $1/N!$ factor:

$$\begin{aligned}
 P_{[g]}^2 &= P_{[g]} P_{[h_+ g h_+^{-1}]} \\
 &= \frac{1}{N!} \\
 &\quad \times \sum_{h_{\pm} \in S_N} \frac{1}{|C(g)|} \bigotimes_{\alpha=1}^m \sum_{h, n, m} \Pi_{h_- \sigma_{\alpha} h_-^{-1}} |h, n\rangle_{h_- \sigma_{\alpha} h_-^{-1}} \otimes \overline{|h, m\rangle}_{h_- \sigma_{\alpha} h_-^{-1}} \\
 &\quad \times {}_{h_+ \sigma_{\alpha} h_+^{-1}} \langle h, n| \otimes {}_{h_+ \sigma_{\alpha} h_+^{-1}} \overline{\langle h, m|} \Pi_{h_+ \sigma_{\alpha} h_+^{-1}} \\
 &\quad \times \sum_{h'_{\pm} \in S_N} \frac{1}{|C(g)|} \bigotimes_{\alpha=1}^m \sum_{H, N, M} \Pi_{h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-} |H, N\rangle_{h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-} \otimes \overline{|H, M\rangle}_{h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-} \\
 &\quad \times {}_{h'_+ h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-} \langle H, N| \otimes {}_{h'_+ h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-} \overline{\langle H, M|} \Pi_{h'_+ h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-}.
 \end{aligned} \tag{2.85}$$

In the last line, we included extra h'_- factor in subscripts so that the calculation goes simpler. Here, the algebraic inner product

$${}_{h_+ \sigma_{\alpha} h_+^{-1}} \langle h, n| \otimes {}_{h_+ \sigma_{\alpha} h_+^{-1}} \overline{\langle h, m|} \Pi_{h_+ \sigma_{\alpha} h_+^{-1}} \Pi_{h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-} |H, N\rangle_{h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-} \otimes \overline{|H, M\rangle}_{h'_- h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_-}$$

is non-vanishing for all α only if $h'_- \in C(h_+ g h_+^{-1})$, with $\delta_{hH} \delta_{nN} \delta_{mM}$. Thus the summations over h'_-, H, N, M are killed, with producing extra factor $|C(h_+ g h_+^{-1})| = |C(g)|$:

$$\begin{aligned}
 P_{[g]}^2 &= \frac{1}{N!} \sum_{h_{\pm}, h'_+ \in S_N} \frac{1}{|C(g)|} \bigotimes_{\alpha=1}^m \sum_{h, n, m} \Pi_{h_- \sigma_{\alpha} h_-^{-1}} |h, n\rangle_{h_- \sigma_{\alpha} h_-^{-1}} \otimes \overline{|h, m\rangle}_{h_- \sigma_{\alpha} h_-^{-1}} \\
 &\quad \times {}_{h'_+ h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_+} \langle h, n| \otimes {}_{h'_+ h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_+} \overline{\langle h, m|} \Pi_{h'_+ h_+ \sigma_{\alpha} h_+^{-1} h'^{-1}_+}, \tag{2.86}
 \end{aligned}$$

with redefining $h'_+ h'_- \rightarrow h'_+$. Redundant summation over h'_+ produces $N!$ and we obtain

$$\begin{aligned}
 P_{[g]}^2 &= \sum_{h_{\pm} \in S_N} \frac{1}{|C(g)|} \bigotimes_{\alpha=1}^m \sum_{h, n, m} \Pi_{h_- \sigma_{\alpha} h_-^{-1}} |h, n\rangle_{h_- \sigma_{\alpha} h_-^{-1}} \otimes \overline{|h, m\rangle}_{h_- \sigma_{\alpha} h_-^{-1}} \\
 &\quad \times {}_{h_+ \sigma_{\alpha} h_+^{-1}} \langle h, n| \otimes {}_{h_+ \sigma_{\alpha} h_+^{-1}} \overline{\langle h, m|} \Pi_{h_+ \sigma_{\alpha} h_+^{-1}} \\
 &= P_{[g]}. \tag{2.87}
 \end{aligned}$$

Definition of the universal defect operator.

The conjugacy-class projectors $P_{[g]}$ constructed in (2.76) provide an orthogonal resolution of the identity on the orbifold Hilbert space. In particular, any operator which is diagonal in the decomposition by conjugacy classes can be written as a linear com-

ination of the $P_{[g]}$.

We define the defect associated with irreducible representation R of S_N by

$$\begin{aligned} \mathcal{D}_R &= \sum_{[g] \in \text{Conj}(S_N)} \chi_R([g]) P_{[g]} \\ &= \sum_{h_{\pm} \in S_N} \sum_{[g] \in \text{Conj}(S_N)} \frac{\chi_R([g])}{|C(g)|} \sum_{h, n, m} \Pi_{h_- g h_-^{-1}} |h, n\rangle_{h_- g h_-^{-1}} \otimes \overline{|h, m\rangle}_{h_- g h_-^{-1}} \\ &\quad \otimes_{h_+ g h_+^{-1}} \langle h, n| \otimes_{h_+ g h_+^{-1}} \overline{\langle h, m|} \Pi_{h_+ g h_+^{-1}}, \end{aligned} \quad (2.88)$$

By construction, $\mathcal{D}_R$ depend only on the group-theoretic data (S_N, R) and the projectors $P_{[g]}$, and are therefore “universal” in the sense that they do not refer to any further details of the seed CFT.

Action on twisted sectors. Let us consider the action of $\mathcal{D}_R$ on the subspace $\mathcal{H}_{[g]}$. By construction we have

$$P_{[g]}|_{\mathcal{H}_{[g]}} = \mathbf{1}_{\mathcal{H}_{[g]}}, \quad P_{[g']}|_{\mathcal{H}_{[g]}} = 0 \quad \text{for } [g'] \neq [g]. \quad (2.89)$$

Therefore, the defect $\mathcal{D}_R$ acts on the decomposition by conjugacy classes as:

$$\mathcal{D}_R|_{\mathcal{H}_{[g]}} = \chi_R([g]) \mathbf{1}_{\mathcal{H}_{[g]}}, \quad \mathcal{D}_R|\Psi\rangle_{[g]} = \chi_R([g]) |\Psi\rangle_{[g]} \quad \text{for } |\Psi\rangle_{[g]} \in \mathcal{H}_{[g]}. \quad (2.90)$$

In particular, in the identity sector $[e]$ we have

$$\mathcal{D}_R|_{\mathcal{H}_{[e]}} = \chi_R([e]) \mathbf{1}_{\mathcal{H}_{[e]}} = \dim_R \mathbf{1}_{\mathcal{H}_{[e]}}, \quad (2.91)$$

so that the universal defect acts on the orbifold vacuum as

$$\mathcal{D}_R|0\rangle = \dim_R |0\rangle, \quad \langle 0|\mathcal{D}_R = \dim_R \langle 0|, \quad (2.92)$$

where $\dim_R = \chi_R(e)$ is the dimension of the representation R .

Fusion of universal defects. Let us consider the fusion algebra of universal defects $\mathcal{D}_R$. Using idempotency and orthogonality, we find

$$\begin{aligned}\mathcal{D}_{R_1}\mathcal{D}_{R_2} &= \left(\sum_{[g]} \chi_{R_1}([g]) P_{[g]} \right) \left(\sum_{[h]} \chi_{R_2}([h]) P_{[h]} \right) \\ &= \sum_{[g],[h]} \chi_{R_1}([g]) \chi_{R_2}([h]) P_{[g]} P_{[h]} \\ &= \sum_{[g]} \chi_{R_1}([g]) \chi_{R_2}([g]) P_{[g]}. \end{aligned} \quad (2.93)$$

The product of characters can be decomposed by Kronecker coefficient⁴ $N_{R_1 R_2}^{R_3}$ as

$$\chi_{R_1}([g]) \chi_{R_2}([g]) = \sum_{R_3} N_{R_1 R_2}^{R_3} \chi_{R_3}([g]), \quad (2.94)$$

and we obtain the algebra

$$\mathcal{D}_{R_1}\mathcal{D}_{R_2} = \sum_{R_3} N_{R_1 R_2}^{R_3} \mathcal{D}_{R_3}. \quad (2.95)$$

2.4.3. Universal defects as boundary states in folded theory

We now describe the universal defects $\mathcal{D}_R$ of the symmetric orbifold as boundary states in the folded theory, following the same logic as in [51]. We consider the folded theory

$$\left(\text{Sym}^N \mathcal{C} \right)^{(-)} \otimes \left(\text{Sym}^N \mathcal{C} \right)^{(+)}, \quad (2.96)$$

where the superscript $(-), (+)$ labels the two copies. The boundary associated with projector $P_{[g]}$ is described by the boundary state

$$\begin{aligned} |P_{[g]}\rangle\rangle &:= \sum_{h_{\pm} \in S_N} \frac{1}{|C(g)|} \sum_{h,n,m} \Pi_{h-g h_{-}^{-1}}^{(-)} |h, n\rangle_{h-g h_{-}^{-1}}^{(-)} \otimes \overline{|h, m\rangle_{h-g h_{-}^{-1}}^{(-)}} \\ &\quad \otimes \Pi_{h+g h_{+}^{-1}}^{(+)} \overline{|h, n\rangle_{h+g h_{+}^{-1}}^{(+)}} \otimes |h, m\rangle_{h+g h_{+}^{-1}}^{(+)} \\ &= \sum_{h_{\pm} \in S_N} \frac{1}{|C(g)|} \bigotimes_{\alpha=1}^m \sum_{h,n,m} \Pi_{h-\sigma_{\alpha} h_{-}^{-1}}^{(-)} |h, n\rangle_{h-\sigma_{\alpha} h_{-}^{-1}}^{(-)} \otimes \overline{|h, m\rangle_{h-\sigma_{\alpha} h_{-}^{-1}}^{(-)}} \\ &\quad \otimes \Pi_{h+\sigma_{\alpha} h_{+}^{-1}}^{(+)} \overline{|h, n\rangle_{h+\sigma_{\alpha} h_{+}^{-1}}^{(+)}} \otimes |h, m\rangle_{h+\sigma_{\alpha} h_{+}^{-1}}^{(+)}, \end{aligned} \quad (2.97)$$

⁴It is hard to determine this coefficient in large N : see details in [54].

and consequently, we obtain the boundary state corresponding to the defect $\mathcal{D}_R$ as

$$\begin{aligned}
 |\mathcal{D}_R\rangle\rangle &:= \sum_{[g] \in \text{Conj}(S_N)} \chi_R([g]) |P_{[g]}\rangle\rangle \\
 &= \sum_{h_{\pm} \in S_N} \sum_{[g] \in \text{Conj}(S_N)} \frac{\chi_R([g])}{|C(g)|} \sum_{h, n, m} \Pi_{h_- g h_-^{-1}}^{(-)} |h, n\rangle_{h_- g h_-^{-1}}^{(-)} \otimes \overline{|h, m\rangle_{h_- g h_-^{-1}}^{(-)}} \\
 &\quad \otimes \Pi_{h_+ g h_+^{-1}}^{(+)} \overline{|h, n\rangle_{h_+ g h_+^{-1}}^{(+)}} \otimes |h, m\rangle_{h_+ g h_+^{-1}}^{(+)} . \quad (2.98)
 \end{aligned}$$

Overlap and recovery of torus partition function.

We now consider, in the folded theory, a cylinder whose two ends are both endowed with trivial defect boundary states as a consistency check done for the seed theory case in Sec. 2.4.1. Note that the gauge group in the folded theory is $S_N \times S_N$ and the physical inner product introduces $1/(N!)^2$. Now let us explicitly calculate

$$Z_{N; \text{cyl}}^{(\text{triv}, \text{triv})} \left(\frac{\tilde{\tau}}{2} \right) := \langle\langle \mathcal{D}_{\text{triv}} | \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + L_0^{(+)} - \frac{c}{12}} \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + \bar{L}_0^{(+)} - \frac{c}{12}} | \mathcal{D}_{\text{triv}} \rangle\rangle, \quad \tilde{q} = e^{2\pi i \tilde{\tau}} . \quad (2.99)$$

Because of the orthogonality of $P_{[g]}$, it is sufficient to calculate

$$Z_{N; \text{cyl}}^{(\text{triv}, \text{triv})} \left(\frac{\tilde{\tau}}{2} \right) = \sum_{[g] \in \text{Conj}(S_N)} \langle\langle P_{[g]} | \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + L_0^{(+)} - \frac{c}{12}} \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + \bar{L}_0^{(+)} - \frac{c}{12}} | P_{[g]} \rangle\rangle_{\text{phys}} . \quad (2.100)$$

Here, we chose specific group elements for subscripts, but the overall value is essentially unchanged. We can perform the redundant sum over $h'_{\pm}$ and $h_{\pm}$, and it simplifies as

$$\begin{aligned}
 &\langle\langle P_{[g]} | \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + L_0^{(+)} - \frac{c}{12}} \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + \bar{L}_0^{(+)} - \frac{c}{12}} | P_{[g]} \rangle\rangle_{\text{phys}} \\
 &= \frac{1}{|C(g)|} \sum_{h \in C(g)} \bigotimes_{\alpha=1}^m \sum_{h, n, m} \\
 &\quad \overline{\langle h, n |}_{\sigma_{\alpha}} \otimes \langle h, m |_{\sigma_{\alpha}} \Pi_{\sigma_{\alpha}} \tilde{q}^{L_0 + \bar{L}_0 - \frac{Nc}{12}} \overline{|h, n\rangle_{h\sigma_{\alpha}h^{-1}}} \otimes |h, m\rangle_{h\sigma_{\alpha}h^{-1}}, \quad (2.101)
 \end{aligned}$$

where σ_{α} is a shorthand of $\sigma_{\alpha} \curvearrowright (i(\alpha)_1, \dots, i(\alpha)_{k_{\alpha}})$ and $h\sigma_{\alpha}h^{-1}$ is a shorthand of $h\sigma_{\alpha}h^{-1} \curvearrowright h(i(\alpha)_1, \dots, i(\alpha)_{k_{\alpha}})$.

Closer look at permutation $g \in S_N$. Instead of writing g as an abstract product $g = \prod_{\alpha} \sigma_{\alpha}$, we would like to organize the cycles by their length. More precisely, we write

$$g = \prod_{k \geq 1} \prod_{a=1}^{m_k} \sigma_{k,a} . \quad (2.102)$$

Here, for each $k \geq 1$, the symbols $\sigma_{k,1}, \dots, \sigma_{k,m_k}$ denote the cycles of length k in the chosen decomposition of g ; consequently, we have the constraint $\sum_k k m_k = N$. For each fixed k , the m_k cycles of length k are indistinguishable from the point of view of the cycle type of g , and they are permuted by a natural action of the symmetric group S_{m_k} . An element $\varsigma_k \in S_{m_k}$ acts by reordering the labels $a = 1, \dots, m_k$ of the cycles $\sigma_{k,a}$. The centralizer $C(g)$ acts on this family of cycles in two conceptually distinct ways. First, for each individual cycle $\sigma_{k,a}$ there is an internal cyclic permutation of its k entries, generated by a $\mathbb{Z}_k$ action; this corresponds to shifting the labels along the k -cycle $\sigma_{k,a}$. Second, there is a permutation of the m_k distinct cycles of the same length k , generated by the symmetric group S_{m_k} acting on the index $a = 1, \dots, m_k$. Equivalently, the restriction of any $h \in C(g)$ to the k -cycle sector may be written as a pair

$$h|_{k\text{-cycle}} \longleftrightarrow \left((r_{k,1}, \dots, r_{k,m_k}), \varsigma_k \right), \quad r_{k,a} \in \mathbb{Z}_k, \quad \varsigma_k \in S_{m_k}, \quad (2.103)$$

where $r_{k,a}$ encodes the cyclic rotation inside each cycle $\sigma_{k,a}$ and ς_k permutes the m_k cycles of length k among themselves. This is the content of the standard structural isomorphism

$$C(g) \simeq \prod_{k \geq 1} \left(\mathbb{Z}_k^{m_k} \rtimes S_{m_k} \right). \quad (2.104)$$

Consequently, we have $|C(g)| = \prod_{k \geq 1} k^{m_k} m_k!$.

Generating function of overlap. The action of $r_{k,a}$ in this case is trivial, and we obtain

$$\begin{aligned} & \langle \langle P_{[g]} | \sqrt{\tilde{q}}^{L_0^{(-)} + L_0^{(+)} - \frac{c}{12}} \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + \bar{L}_0^{(+)} - \frac{c}{12}} | P_{[g]} \rangle \rangle_{\text{phys}} \\ &= \bigotimes_{k \geq 1} \frac{1}{m_k!} \sum_{\varsigma_k \in S_{m_k}} \sum_{h,n,m} \\ & \quad \overline{\langle h, n |}_{\sigma_\alpha} \otimes \langle h, m |_{\sigma_\alpha} \Pi_{\sigma_\alpha} \tilde{q}^{L_0 + \bar{L}_0 - \frac{Nc}{12}} \overline{| h, n \rangle}_{\varsigma_k \sigma_\alpha \varsigma_k^{-1}} \otimes | h, m \rangle_{\varsigma_k \sigma_\alpha \varsigma_k^{-1}}. \end{aligned} \quad (2.105)$$

One can obtain its m_k k -cycle sector from the following generating function

$$\prod_{a \geq 1} \exp \left[p^{ka} \frac{1}{k} \sum_{b=0}^{k-1} Z_c \left(\frac{a\tilde{\tau} + b}{k} \right) \right], \quad (2.106)$$

by looking at the coefficient of p^{km_k} . To see the full overlap $Z_{N;\text{cyl}}^{(\text{triv}, \text{triv})} \left(\frac{\tilde{\tau}}{2} \right)$, we need to take all $k \geq 1$ into account and take sum over $[g] \in \text{Conj}(S_N)$. This yields the

following generating function

$$\sum_{N \geq 0} p^N Z_{N; \text{cyl}}^{(\text{triv}, \text{triv})} \left(\frac{\tilde{\tau}}{2} \right) = \prod_{k \geq 1} \prod_{a \geq 1} \exp \left[p^{ka} \frac{1}{k} \sum_{b=0}^{k-1} Z_c \left(\frac{a\tilde{\tau} + b}{k} \right) \right], \quad (2.107)$$

where the right-hand side is exactly the generating function of the torus partition function (2.24).

2.5. Computation of correlation functions

In symmetric product orbifold conformal field theories of the form $\mathcal{M}^N/S_N$, correlation functions of twist operators and their excitations can be computed using the covering-space method of Lunin and Mathur[55]. One introduces a branched covering map

$$\Gamma : \Sigma \rightarrow X \quad (2.108)$$

from a Riemann surface Σ , on which the fields of the seed theory on $\mathcal{M}$ are single-valued, to the base manifold X where twist operators are inserted. Excitations of the symmetric orbifold on X are then pulled back along Γ and become local operator insertions in the seed theory living on Σ . Near a twist operator σ_w the map behaves locally as $x \sim z^w$, so that σ_w creates a w -branch point on X with opening angle $2\pi w$ i.e. a conical singularity with excess angle $2\pi(w - 1)$. The associated conical curvature leads, via the Weyl anomaly, to a non-trivial Liouville action on Σ , and the correlators acquire a Liouville factor multiplying the seed-theory partition function on the covering surface.

Given the base space X and a set of correlation functions of twist-sector excitations O_w on X , $\langle O_{w_1} \cdots O_{w_{n_c}} \rangle^X$, the existence of a compatible branched covering $\Gamma : \Sigma \rightarrow X$ is not automatic; moreover, even when such coverings exist they are generally not unique. Pakman, Rastelli, and Razamat developed a diagrammatic classification of these coverings and introduced Feynman-like rules that express twist-field correlators on $X = S^2$ as sums over diagrams, each diagram corresponding to an equivalence class of covering maps [6]. In Section 2.5.1, we briefly describe coefficients for each equivalence class of contributions to the correlation functions. In Section 2.5.2, we review their diagrammatic method in a form adapted to our later applications, and in subsequent Section 2.5.3, we introduce alternative construction of PRR diagrammatics.

For each covering, we need to evaluate the Liouville action and combine it with the seed-theory correlator on Σ . We briefly review this prescription given by Lunin and Mathur [55] in Section 2.5.4.

In Section 2.5.5, we emphasize how in the large- N limit the contributions of the

diagrams are naturally organized by the genus of the covering surface, leading to a genus expansion.

Finally, in Section 2.5.6, we discuss the grand-canonical uplift of correlation functions, in which one introduces a fugacity μ and considers a grand-canonical ensemble that sums over all symmetric product orbifolds $\mathcal{M}^N/S_N$. Our discussion is based on the recent work of Aharony and Urbach [56]: the grand-canonical ensemble of symmetric orbifold CFTs reproduces the string genus expansion to all orders.

In many of our explicit examples we take the base manifold X to be the sphere $X = S^2$. Before go into the detail, let us discuss the normalization of correlation functions.

Before turning to the normalization of correlators, let us make precise what we mean by a (branched) covering map.

Definition of branched covering map. Let X and Σ be compact Riemann surfaces. A holomorphic map

$$\Gamma : \Sigma \rightarrow X \tag{2.109}$$

is called a (branched) covering map of degree N if the following conditions hold:

- There exists a finite subset $B \subset X$ (the set of branch points) such that

$$\Gamma : \Sigma \setminus \Gamma^{-1}(B) \longrightarrow X \setminus B$$

is a topological covering of degree N , i.e. every $x \in X \setminus B$ has exactly N preimages under Γ .

- For every point $p \in \Sigma$ there exist local complex coordinates z on Σ near p and x on X near $\Gamma(p)$ such that, up to multiplication by a nonzero constant,

$$x - \Gamma(p) = z^{w_p} \tag{2.110}$$

for some integer $w_p \geq 1$.

The integer w_p is called the ramification index at p ; points with $w_p > 1$ are ramification points, and their images in X are the branch points of Γ . In the applications to symmetric orbifolds, the branch points coincide with the insertion points of twist operators on X , and the ramification index w_p matches the length w of the corresponding cycle in S_N , so that locally near a w -twist insertion we have $x \sim z^w$ as described above.

2.5.1. Normalization of correlation functions

Let us discuss the normalization of the correlation function at this point, as we did in detail in Sec. 2.2.2. Usually, the orbifold path integral carries the usual factor given by the inverse volume of the gauge group S_N . When one reorganizes the path integral in terms of gauge-invariant configurations, it is natural to associate to each configuration an effective prefactor given by the inverse volume of the subgroup of S_N that leaves this configuration invariant. In the present context, the relevant configurations are branched coverings $\Gamma : \Sigma \rightarrow X$, and the remaining gauge group $\text{Aut}(\Gamma)$.

With this notation, the contribution of a given covering to the *unnormalized* correlator⁵ takes the form

$$Z_{\text{Sym}^N \mathcal{M}}^X [O_{w_1} \cdots O_{w_{n_c}}] = \sum_{\Gamma} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma} [\prod O_i]}{|\text{Aut}(\Gamma)|}, \quad (2.111)$$

where $\hat{Z}_{\mathcal{M}}^{\Sigma}$ denotes the correlation function of the seed CFT on the covering surface including the Liouville factor.

To obtain the usual correlation function on the sphere, we need to divide once by the vacuum partition function, which effectively multiplies all correlators by a universal factor $N!$. In this normalized convention, the contribution of Γ becomes

$$\langle O_{w_1} \cdots O_{w_{n_c}} \rangle_N = \frac{Z_{\text{Sym}^N \mathcal{M}}^{S^2} [O_{w_1} \cdots O_{w_{n_c}}]}{Z_{\text{Sym}^N \mathcal{M}}^{S^2}} = \sum_{\Gamma} \frac{N!}{|\text{Aut}(\Gamma)|} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma} [\prod O_i]}{z^N}, \quad (2.112)$$

with $z = Z_{\mathcal{M}}^{S^2}$. Thus, in our normalization, the coefficient of each covering is given by $N!/|\text{Aut}(\Gamma)|$, where $\text{Aut}(\Gamma)$ is the remaining gauge group associated with Γ .

2.5.2. Pakman-Rastelli-Razamat diagrammatics

Let us draw on $X = S^2$ a double-line graph, made of solid and dashed lines, that passes through the insertion points of all twist excitations; see Fig. 2.2 for an example.

Given a covering map $\Gamma : \Sigma \rightarrow X = S^2$, we can consider the inverse image of this double-line graph under Γ and thus obtain a double-line graph drawn on the covering surface Σ . We now focus on the topological properties of this lifted double-line graph on Σ . By construction, the following properties hold:[6]

- At the preimage of a w -twist insertion point, w strands of the double-line graph join together; away from such points, the strands of the double line neither split nor join.

⁵To be more precise, e.g. torus partition functions can also be understood in this prescription.

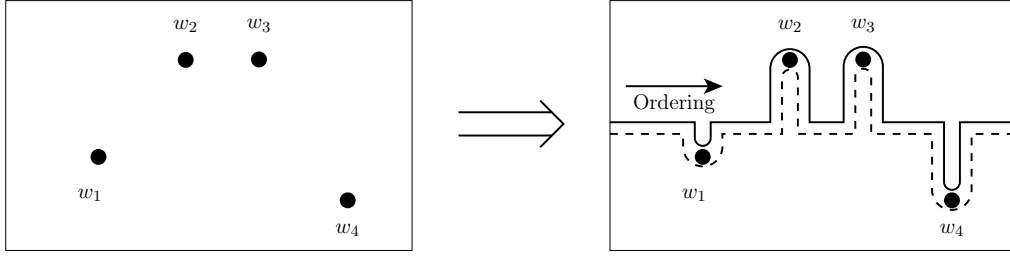


Figure 2.2.: Double-line graph made of solid and dashed lines on base manifold S^2 . The line passes through all the twist insertion points. Ordering of twists is naturally induced.

- Each connected component of the solid (respectively dashed) line is a contractible closed curve on Σ .
- When one moves along a solid line in the counter-clockwise direction and records the twist insertion points encountered, the resulting cyclic order coincides with the partial order in which the twist excitations are arranged on X .
- When one moves along a dashed line in the clockwise direction and records the insertion points encountered, one again obtains the resulting cyclic order as the partial order on X .
- The numbers of solid and dashed closed curves are each equal to the degree of the covering map Γ .

If we retain only the topological information contained in the lifted double-line graph on Σ , we refer to the resulting object as a *PRR diagram*⁶, following Pakman, Rastelli and Razamat [6]. The above discussion shows that inequivalent covering maps give rise to different PRR diagrams. Figure 2.3 illustrates this construction for the correlator $\langle \sigma_3(a)\sigma_2(b)\sigma_2(c) \rangle$ of $\text{Sym}^3 \mathcal{M}$.

Conversely, one can reconstruct a covering surface from a given PRR diagram that satisfies the rules listed above: starting from a PRR diagram satisfying the topological properties, we first count the number of solid (equivalently dashed) loops; this number is the degree d of the covering. We then take d copies of the sphere S^2 , each cut open along the chosen double line. The PRR diagram specifies, for each segment of the double line and for each twist vertex, how the different copies of the cut sphere are to be glued together along their boundaries: whenever two strands of the double line are connected in the diagram, we identify the corresponding boundary segments of the

⁶Note that in this thesis the base manifold is always taken to be the sphere $X = S^2$, and PRR diagrams, as we use the term, are defined only for this spherical case. We will not attempt to generalize the diagrammatics to higher-genus base surfaces.

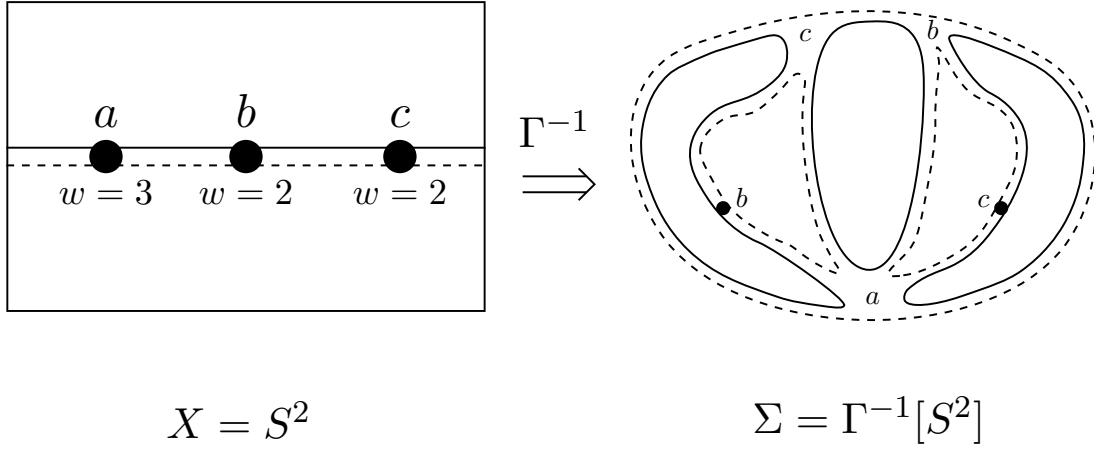


Figure 2.3.: PRR diagram for $\langle \sigma_3(a)\sigma_2(b)\sigma_2(c) \rangle$ of $\text{Sym}^3\mathcal{M}$. We explicitly indicate the trivial preimages of b and c . One can find all the solid/dashed closed curves satisfy the expected natures.

appropriate copies of S^2 . Gluing all segments according to the diagram produces a two-dimensional manifold Σ , possibly disconnected, together with a natural projection $\Gamma : \Sigma \rightarrow S^2$ that locally looks like the identity away from the twist insertions and has the prescribed branching behavior at the twist points. By construction, this Γ is a branched covering of S^2 , and different PRR diagrams lead to non-isomorphic coverings⁷.

In this way, PRR diagrams are in one-to-one correspondence with the branched coverings that contribute to correlators of twists.

PRR rules.

To systematically reproduce PRR diagrams, we summarize the PRR rules in the following form:

- A w -twist insertion is represented by a vertex surrounded by w solid and w dashed strands. See Fig. 2.4 for example.
- A propagator is represented by a double line consisting of one solid and one dashed strand.

⁷More formally, branched covers of the sphere with given branch points can be classified by their monodromies, and Riemann's existence theorem states that such monodromy data and branched covers are in one-to-one correspondence. The PRR diagrams give a convenient graphical representation of this monodromy data.

- Each connected component of the solid (respectively dashed) line is a contractible closed curve on Σ .
- Traversing a solid line counter-clockwise, one encounters the twist insertions in the partial cyclic order as on the base sphere X .
- Traversing a dashed line clockwise, one again encounters the twist insertions in the partial cyclic order as on X .
- The numbers of solid and dashed closed curves are each equal to the degree of the covering map.

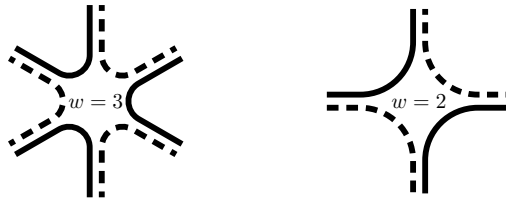


Figure 2.4.: Examples of vertices for PRR diagrams. Vertices are surrounded by w solid and w dashed strands in an alternative order. Away from vertices, these strands pair up to form double-line propagators.

Disconnected covering surface.

In many cases, the completed PRR diagram is realized on a disconnected Riemann surface Σ that is identified as a covering surface. In general, a degree- N covering can be decomposed as a disjoint union

$$\Sigma = \bigsqcup_{\alpha} \Sigma_{\alpha}^{\sqcup k_{\alpha}}, \quad \Gamma|_{\Sigma_{\alpha}} : \Sigma_{\alpha} \rightarrow S^2 \text{ of degree } n_{\alpha},$$

with $\sum_{\alpha} k_{\alpha} n_{\alpha} = N$. Some components may be *trivial* in the sense that $\Sigma_{\alpha} = S^2_{\text{trivial}} = X$ and $\Gamma|_{\Sigma_{\alpha}}$ is the identity map of degree one. In typical situations one can write

$$\Sigma = \left(S^2_{\text{trivial}} \right)^{\sqcup N-n} \sqcup \Sigma_{\text{nontrivial}}, \quad (2.113)$$

where $\Gamma|_{\Sigma_{\text{nontrivial}}}$ has degree n . On the trivial S^2 components, the corresponding PRR diagram is just a double-line circle; such diagrams do not carry any non-trivial information and will be systematically omitted in what follows.

Example: $\langle \sigma_2 \sigma_2 \sigma_2 \sigma_2 \rangle$.

As a simple example, consider the correlator $\langle \sigma_2(x_1) \sigma_2(x_2) \sigma_2(x_3) \sigma_2(x_4) \rangle_N$. The associated covering surface decomposes into a non-trivial component that carries the branched cover, together with several trivial spherical components on which Γ is the identity. The PRR diagram we will draw corresponds only to the non-trivial component; an illustration is shown in Figs. 2.5 and 2.6.

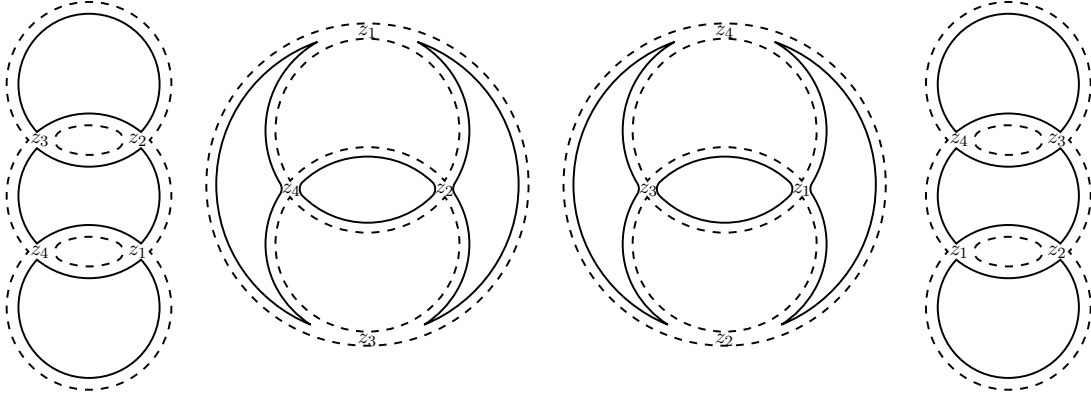


Figure 2.5.: Genus 0 PRR diagrams to $\langle \sigma_2(x_1) \sigma_2(x_2) \sigma_2(x_3) \sigma_2(x_4) \rangle$.
The degree of $\Gamma|_{\Sigma_{\text{nontrivial}}}$ is 3.

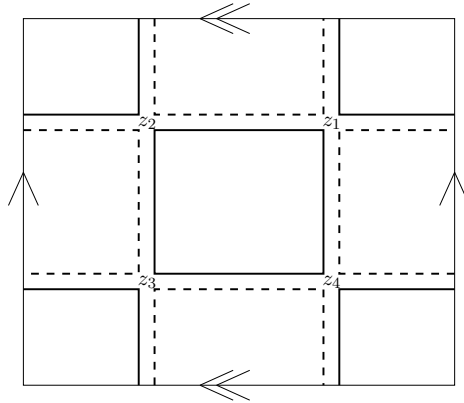


Figure 2.6.: Genus 1 PRR diagram to $\langle \sigma_2(x_1) \sigma_2(x_2) \sigma_2(x_3) \sigma_2(x_4) \rangle$.
The degree of $\Gamma|_{\Sigma_{\text{nontrivial}}}$ is 2.

Remaining gauge group $\text{Aut}(\Gamma)$.

Previously we have seen that a degree- N covering can decompose into a disjoint union of trivial and non-trivial components in eq. (2.113). Here, we explain how the remaining gauge group of a covering, and hence its volume $|\text{Aut}(\Gamma)|$, can be determined directly from the symmetries of the corresponding PRR diagram.

Let us first clarify the definition of the remaining gauge group $\text{Aut}(\Gamma)$ ⁸ of a covering Γ :

$$\text{Aut}(\Gamma) \equiv \left\{ h : \Sigma \rightarrow \Sigma \mid \Gamma \circ h = \Gamma \right\}. \quad (2.114)$$

Diagrammatically, $\text{Aut}(\Gamma)$ is the group of permutations of sheets and connected components that leave the PRR diagram invariant.

A first source of symmetry comes from the trivial components in (2.113). The $(N - n)$ trivial spheres S_{trivial}^2 are indistinguishable: any permutation of these components defines an automorphism of the covering. Consequently,

$$\text{Aut}(\Gamma) = S_{N-n} \times \text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}}) \quad (2.115)$$

This factor can be read off directly from the PRR diagram as soon as the number of trivial components is known: each trivial sphere contributes a double-line circle, and permutations of these circles act trivially on the non-trivial part of the diagram.

Even when $\Sigma_{\text{nontrivial}}$ is connected, the covering may still possess non-trivial automorphisms. A simple example is provided by a two-point function of a w -twist excitation and its conjugate. In this case, the non-trivial part of the covering consists of a single w -sheeted sphere, and the PRR diagram is invariant under a cyclic rotation of the w sheets. This cyclic symmetry is realized as $\text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}}) = \mathbb{Z}_w$, and contributes an additional factor

$$|\text{Aut}(\Gamma)| = w (N - w)!. \quad (2.116)$$

In the PRR diagram, this symmetry is visible as a cyclic relabeling of the w identical strands around the two twist vertices that leaves the diagram unchanged.

More generally, the non-trivial covering surface may decompose further into several connected components of the same topology and with identical PRR diagrams. If there are k such identical non-trivial components, say $\Sigma_{\text{nontrivial}} = (\Sigma_\alpha)^{\sqcup k_\alpha}$, permutations among them define automorphisms of Γ . Therefore

$$\text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}}) = S_{k_\alpha} \times \text{Aut}(\Gamma|_{\Sigma_\alpha}). \quad (2.117)$$

This situation arises, for instance, in multi-particle configurations where the covering contains several copies of the same graph.

Collecting these contributions, the symmetry factor of a given covering can be obtained as the order of its automorphism group:

$$|\text{Aut}(\Gamma)| = (N - n)! \times \text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}}), \quad (2.118)$$

⁸In the mathematical literature it is often called as the group of deck transformations that preserve the covering map.

where the internal part includes factors such as the cyclic symmetry w in two-point functions and permutation factors $k!$ for identical non-trivial components, as in (2.116) and (2.117). In practice, all of these factors can be read off from the PRR diagram by identifying permutations of strands and components that leave the diagram invariant.

Using this decomposition, let us see how eq. (2.112) can be written explicitly. It takes the following form:

$$\langle O_{w_1} \cdots O_{w_{n_c}} \rangle_N = \sum_{\Gamma} \frac{N!}{(N-n)!} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma_{\text{nontrivial}}} [\prod O_i]}{|\text{Aut}(\Gamma)_{\Sigma_{\text{nontrivial}}}|} z^n. \quad (2.119)$$

Here, all of the N -dependence is contained in the prefactor $\frac{N!}{(N-n)!}$, while the remaining factor is independent of N .

Gauge fixing of twists.

On the sphere, it is often useful to separate explicitly the overall gauge volume in order to count inequivalent gauge fixings. For a given covering Γ , the contribution to the unnormalized correlator can be written schematically as

$$Z_{\text{Sym}^N \mathcal{M}}^{S^2} [O_{w_1} \cdots O_{w_{n_c}}] = \frac{1}{N!} \sum_{\Gamma} \frac{N!}{|\text{Aut}(\Gamma)|} \hat{Z}_{\mathcal{M}}^{\Sigma} [\prod O_i], \quad (2.120)$$

where the factor $N!/\text{Sym}(\Gamma)$ can be interpreted as the number of essentially different gauge fixings $\mathbf{g}_1, \dots, \mathbf{g}_{n_c}$ that give rise to the same covering Γ .

First, to see the gauge-fixing through the PRR diagrammatics, we bijectively assign integers in $\underline{N} := \{1, \dots, N\}$ to solid circles in the diagram. We call a PRR diagram whose solid circles are labeled in such a way *gauge fixed*. For each vertex in the gauge fixed PRR diagram, we can assign permutation $\mathbf{g} = (a_1 \dots a_w) \in S_N$ by circumnavigating the associated vertex of the diagram anti-clockwise and reading off the encountered $\underline{N}$ labels $(a_i)_{i=1}^w$ of the traversed solid lines. We refer to these permutations as *gauge fixed vertices*.

From the orbit-stabilizer theorem, the prefactor in (2.120) can be interpreted as the number of inequivalent gauge fixing of the corresponding PRR diagram:

$$\frac{N!}{|\text{Aut}(\Gamma)|} = |\mathbf{g}_1, \dots, \mathbf{g}_{n_c}|, \quad (2.121)$$

where the r.h.s. is the size of the orbit

$$[\mathbf{g}_1, \dots, \mathbf{g}_{n_c}] := \{(g\mathbf{g}_1g^{-1}, \dots, g\mathbf{g}_{n_c}g^{-1}) | g \in S_N\}. \quad (2.122)$$

This group action ggg^{-1} corresponds to the relabeling of solid circles in the gauge fixed

PRR diagram.

In that sense, twist correlators are sometimes denoted as

$$Z_{\text{Sym}^N \mathcal{M}}^{S^2}[O_{w_1} \cdots O_{w_{n_c}}] = \sum_{\mathbf{g}_i \in [(1 \dots w_i)] \subset S_N} Z_{\text{Sym}^N \mathcal{M}}^{S^2}[O_{\mathbf{g}_1} \cdots O_{\mathbf{g}_{n_c}}], \quad (2.123)$$

or

$$\langle O_{w_1} \cdots O_{w_{n_c}} \rangle = \sum_{\mathbf{g}_i \in [(1 \dots w_i)] \subset S_N} \langle O_{\mathbf{g}_1} \cdots O_{\mathbf{g}_{n_c}} \rangle, \quad (2.124)$$

with the assumption that inappropriate combinations of $\mathbf{g}_i$ yield vanishing contributions.

2.5.3. Alternative PRR diagrammatics

Let us consider an alternative diagrammatics, starting from drawing lines on $X = S^2$ in a different way. We first fix the location of a double line along some reference line, say the real axis, and then deform one of the lines so that it passes through all twist insertions; see Fig. 2.7.

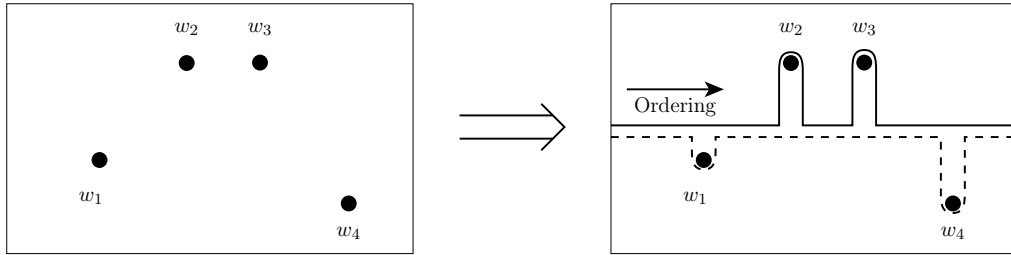


Figure 2.7.: Alternative construction of a double-line graph made of solid and dashed lines on the base manifold S^2 . The solid (dashed) line is deformed so that it passes through twist insertion points, while the dashed (solid) line stays on the reference line, respectively. The ordering of twists is then naturally induced.

Given a covering map $\Gamma : \Sigma \rightarrow X = S^2$, we can again consider the inverse image of this double-line graph under Γ and thus obtain a double-line graph drawn on the covering surface Σ . These diagrams consist of the following objects: see Fig. 2.8.

Alternative PRR rules and equivalence to original PRR diagrammatics.

Corresponding alternative PRR rules are following:

- A w -twist insertion at the upper (lower) half-plane is represented by a vertex



Figure 2.8.: Examples of vertices for the alternative PRR diagrams. Vertices corresponding to upper (lower) half-plane twist insertions are surrounded by w solid (dashed) strands. The two rightmost figures are *redundant vertices*, surrounded by three strands. Away from vertices, these strands pair up to form double-line propagators.

surrounded by w solid (dashed) strands, respectively. See the two leftmost figures in Fig. 2.8 for example.

- w redundant vertices are introduced for each w -twist insertion. A redundant vertex is surrounded by two solid (dashed) strands and one dashed (solid) strands, respectively. See the two rightmost figures in Fig. 2.8 for example.
- A propagator is represented by a double line.
- Each connected component of the solid (respectively dashed) line is a contractible closed curve on Σ .
- Traversing a solid line counter-clockwise, one encounters the twist insertions and the redundant vertices in the partial cyclic order as on the base sphere X .
- Traversing a dashed line clockwise, one again encounters the twist insertions and the redundant vertices in the partial cyclic order as on X .
- The numbers of solid and dashed closed curves are each equal to the degree of the covering map.

The resulting diagram is equivalent to the original PRR diagram. By performing the contraction of redundant vertices, one can recover the original PRR diagram; conversely, one can obtain the alternative PRR diagram from the original one. See Fig. 2.9.

2.5.4. Lunin-Mathur covering-space prescription

In this subsection, we briefly review how, for a given branched covering $\Gamma : \Sigma \rightarrow X$, the quantity $\hat{Z}_{\mathcal{M}}^{\Sigma}$ appearing in (2.111) is computed from the seed theory on $\mathcal{M}$. By definition, $\hat{Z}_{\mathcal{M}}^{\Sigma}$ is the seed-theory correlator on the covering surface Σ with metric induced from X , including the Liouville factor generated by the Weyl anomaly. In

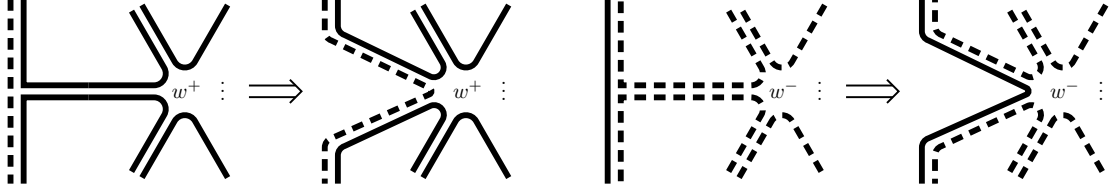


Figure 2.9.: Contraction of redundant vertices in the alternative PRR diagrammatics. For a w^+ (left) or w^- (right) twist insertion, each segment of solid-solid (respectively dashed-dashed) double line is accompanied by redundant vertices. By a *contraction* we mean moving each redundant vertex along the double line towards the twist vertex as close as possible. If one contracts all the redundant vertices, one recovers a single $w^\pm$ vertex, which coincides with the one appearing in the original PRR diagrammatics.

particular, once a corresponding covering Γ is determined, $\hat{Z}_{\mathcal{M}}^\Sigma$ can also be completely determined. Let us start with the generic case, arbitrary base Riemann surface X .

Pullback metric on the cover.

Let X be equipped with a metric

$$ds_X^2 = g_{x\bar{x}}(x, \bar{x}) dx d\bar{x}. \quad (2.125)$$

The covering map $\Gamma : \Sigma \rightarrow X$ induces a metric on Σ via the pullback,

$$ds_\Sigma^2 = \Gamma^*g = g_{x\bar{x}}(\Gamma(z), \overline{\Gamma(z)}) \left| \frac{\partial \Gamma}{\partial z} \right|^2 dz d\bar{z}. \quad (2.126)$$

Near a ramification point $p \in \Sigma$ lying above a twist insertion at $x_\alpha \in X$, as defined in eq. (2.110), we can choose local coordinates z on Σ and x on X such that

$$x - x_\alpha = a_\alpha (z - z_\alpha)^{w_\alpha} + \mathcal{O}((z - z_\alpha)^{w_\alpha+1}), \quad a_\alpha \neq 0, \quad (2.127)$$

so that circling once around z_α in the z -plane corresponds to circling w_α times around x_α in the x -plane. In particular, $\partial_z \Gamma(z) \sim z^{w_\alpha-1}$ near the ramification point, and the pullback metric (2.126) has a conical behavior consistent with the presence of a w_α -twist.

It is convenient to compare Γ^*g to a fiducial reference metric $\hat{g}_{z\bar{z}}(z, \bar{z}) dz d\bar{z}$ on Σ . We write

$$ds_\Sigma^2 = e^{\phi(z, \bar{z})} \hat{g}_{z\bar{z}}(z, \bar{z}) dz d\bar{z}, \quad \phi(z, \bar{z}) = \log \left(g_{x\bar{x}}(\Gamma(z), \overline{\Gamma(z)}) \left| \partial_z \Gamma(z) \right|^2 \right) - \log \hat{g}_{z\bar{z}}(z, \bar{z}). \quad (2.128)$$

The Weyl factor ϕ encodes all the local information about the ramification of Γ and

thus about the twist insertions on X .

Liouville action and seed partition function.

Let $Z_{\mathcal{M}}^{\Sigma}[g; \prod O_i]$ denote the correlation function of seed-theory operators $\prod O_i$ inserted on Σ , computed with respect to the metric g . The corresponding correlator with the physical metric Γ^*g induced from X is given by

$$Z_{\mathcal{M}}^{\Sigma}[\Gamma^*g; \prod O_i] = \exp(S_L[\phi; \hat{g}]) Z_{\mathcal{M}}^{\Sigma}[\hat{g}; \prod O_i], \quad (2.129)$$

where S_L is the Liouville action

$$S_L[\phi; \hat{g}] = \frac{c}{96\pi} \int_{\Sigma} d^2z \sqrt{\hat{g}} \left(\hat{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + 2\hat{R} \phi \right), \quad (2.130)$$

with c the central charge of the seed CFT and $\hat{R}$ the Ricci scalar of $\hat{g}$.

In the notation of (2.111), we therefore identify

$$\hat{Z}_{\mathcal{M}}^{\Sigma}[\prod O_i] := Z_{\mathcal{M}}^{\Sigma}[\Gamma^*g; \prod O_i] = \exp(S_L[\phi; \hat{g}]) Z_{\mathcal{M}}^{\Sigma}[\hat{g}; \prod O_i], \quad (2.131)$$

where the operator insertions $\prod O_i$ are the pullbacks to Σ of the twist-sector excitations $\prod O_{w_i}$ on X .

Evaluation on the case $X = S^2$.

From now on we specialize to the case where the base is the sphere $X = S^2$ equipped with the standard round metric. Let $\Gamma : \Sigma \rightarrow S^2$ be a degree- N branched covering contributing to a given twist correlator, and introduce UV discs $|x - x_{\alpha}| < \varepsilon$ around each finite branch point $x_{\alpha} \in S^2$ and an IR cutoff $|x| < \Lambda$ around infinity. Pulling these circles back to Σ gives small discs $|z - z_r| < \rho_r$ around the ramification points $z_r \in \Sigma$, and $|z - z_s| < R_s$ around the points around preimages of infinity. In local coordinates we can write

$$x - x_{\alpha} = a_r (z - z_r)^{w_r} + \mathcal{O}((z - z_r)^{w_r+1}), \quad r \in R_{\text{fin}}[\Gamma], \quad (2.132)$$

near the finite branch points, while near infinity one has

$$x = -\frac{\xi_s}{z - z_s}, \quad s \in R_{\infty}[\Gamma]. \quad (2.133)$$

where $R_{\text{fin}}[\Gamma]$ and $R_{\infty}[\Gamma]$ denote the sets of ramification points lying above finite branch points and preimages of infinity, respectively. The points z_s are regular from the twist

point of view (no ramification), but they lie above the IR boundary at large $|x|$ and therefore contribute to the Liouville action through the outer cutoff. In particular, the divergence at infinity produces a universal factor $Z_\delta = \delta^{-c/3}$ together with a finite covering-dependent factor involving $\prod_\ell |\xi_\ell|^{-c/6}$, as in eqs. (2.22)–(2.29) of [57]. In our conventions the universal divergence is absorbed into the overall factor z^N . where $R_{\text{fin}}[\Gamma]$ and $R_\infty[\Gamma]$ denote the sets of ramification points lying above finite branch points and above infinity, respectively, and w_r, w_s are the corresponding ramification indices. The coefficients a_r and b_s encode the local behaviour of Γ at the ramification points.

With these cutoffs in place, the Liouville action on (Σ, Γ^*g) can be evaluated by excising the small discs and the outer cap and using integration by parts, as in [55, 57, 4]. The bulk contribution depends only on the Euler characteristic $\chi(\Sigma)$ and can be absorbed into curvature counterterms and into an overall normalization of twist operators, while the non-trivial dependence on the covering map localizes on the cutoff circles and is expressed entirely in terms of the local data of the covering map w_r, w_s, a_r, b_s .

Once a UV/IR regularization scheme and local counterterms are fixed uniformly for the seed theory, the contribution of a given covering Γ to the quantity $\hat{Z}_{\mathcal{M}}^\Sigma$ in (2.111) can be written in the form

$$\hat{Z}_{\mathcal{M}}^\Sigma \left[\prod O_i \right] = z^N \mathcal{F}_\Gamma \prod_{r \in R_{\text{fin}}[\Gamma]} |a_r|^{-\frac{c}{12} \left(1 - \frac{1}{w_r} \right)} \prod_{s \in R_\infty[\Gamma]} |b_s|^{-\frac{c}{6}} Z_{\mathcal{M}}^\Sigma[\hat{g}; \prod O_i], \quad (2.134)$$

where $z := Z_{\mathcal{M}}^{S^2}$ is the vacuum partition function of the seed theory on the round sphere⁹, and $Z_{\mathcal{M}}^\Sigma[\hat{g}; \prod O_i]$ is the seed correlator on Σ computed with respect to a smooth reference metric $\hat{g}$. The prefactor $\mathcal{F}_\Gamma$ collects terms that can be included in twist normalizations: in what follows we choose the normalization of single-cycle twist operators once and for all so that the prefactor in (2.134) reduces to $\mathcal{F}_\Gamma = 1$.

Consequently, the bare two-point function on S^2 is

$$Z_{\text{Sym}^N \mathcal{M}}^{S^2}[\sigma_w(0)\sigma_w(a)] = \frac{z^N}{w(N-w)!} \frac{1}{|a|^{2(h+\bar{h})}}, \quad (2.135a)$$

with covering map

$$x = \Gamma(z) = a \frac{z^w}{z^w - (1-z)^w}, \quad (2.135b)$$

where the combinatorial analysis of Section 2.5.1 is included..

⁹ z^{N-n} comes from the trivial component $(S_{\text{trivial}}^2)^{\sqcup N-n}$, while remaining z^n comes from the non-trivial Liouville factor of pullback metric.

2.5.5. Large- N limit and genus expansion

We begin by recalling the normalization of bulk twist operators. We introduce normalized twists by

$$\hat{\sigma}_w := \mathcal{N}_w \sigma_w, \quad \mathcal{N}_w := \sqrt{\frac{w(N-w)!}{N!}}. \quad (2.136a)$$

With this choice, the two-point function of w -cycle twists is unit-normalized:

$$\langle \hat{\sigma}_w(x_1) \hat{\sigma}_w(x_2) \rangle_N = \frac{1}{|x_1 - x_2|^{2(h+\bar{h})}}, \quad h = \bar{h} = \frac{c}{24} \left(w - \frac{1}{w} \right). \quad (2.137)$$

We now turn to the N -dependence of correlation functions of normalized twists. Recall that the correlator of *unnormalized* twists can be written as

$$\langle \sigma_{w_1} \cdots \sigma_{w_{n_c}} \rangle_N = \sum_{\Gamma} \frac{N!}{(N-n)!} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma_{\text{nontrivial}}}}{|\text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}})| z^n}, \quad (2.138)$$

as in eq. (2.119). Replacing each σ_{w_i} by the *normalized* operator $\hat{\sigma}_{w_i}$, we obtain

$$\left\langle \prod_{i=1}^{n_c} \hat{\sigma}_{w_i}(x_i) \right\rangle_N = \sum_{\Gamma} C_{\Gamma}(N) \frac{\hat{Z}_{\mathcal{M}}^{\Sigma_{\text{nontrivial}}}}{|\text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}})| z^n}, \quad (2.139)$$

where all the N -dependence of the contribution of a given covering Γ has been collected into

$$C_{\Gamma}(N) = \frac{N!}{(N-n)!} \prod_{i=1}^{n_c} \mathcal{N}_{w_i} = \frac{N!}{(N-n)!} \prod_{i=1}^{n_c} \sqrt{\frac{w_i(N-w_i)!}{N!}}. \quad (2.140)$$

The remaining factor in (2.139) is independent of N .

In the large- N limit with all w_i kept fixed, Stirling's approximation gives

$$\mathcal{N}_{w_i} \sim N^{-\frac{w_i}{2}}, \quad \prod_{i=1}^{n_c} \mathcal{N}_{w_i} \sim N^{-\frac{1}{2} \sum_{i=1}^{n_c} w_i}. \quad (2.141)$$

On the other hand, the combinatorial prefactor behaves as

$$\frac{N!}{(N-n)!} \sim N^n. \quad (2.142)$$

Thus, up to an overall $O(1)$ factor independent of N , the N -dependence of the contribution from a fixed covering Γ is

$$C_{\Gamma}(N) \sim N^{n - \frac{1}{2} \sum_{i=1}^{n_c} w_i}. \quad (2.143)$$

The exponent in (2.143) can be expressed purely in terms of the topology of the covering. Let the nontrivial covering surface $\Sigma_{\text{nontrivial}}$ have degree n , genus g , and n_c branch points with cycle lengths w_i . For a covering of the sphere, the Riemann-Hurwitz formula gives the following equation:

$$n - \frac{1}{2} \sum_{i=1}^{n_c} w_i = 1 - g - \frac{n_c}{2}. \quad (2.144)$$

Combining (2.143) with (2.144), we find

$$C_\Gamma(N) \sim N^{1-g-\frac{n_c}{2}}. \quad (2.145)$$

Introducing the string coupling via $g_s^2 \sim N^{-1}$, this can be rewritten as

$$C_\Gamma(N) \sim g_s^{-2+2g+n_c}. \quad (2.146)$$

In other words, after normalizing the twists as in (2.136), the dependence on the covering degree n and the twist lengths w_i reorganizes itself into the simple topological combination $-2+2g+n_c$. This is precisely the string-theoretic genus counting expected for a closed-string background of genus g with n_c operator insertions.

2.5.6. Grand-canonical uplift of symmetric orbifold correlators

So far we have worked in the canonical ensemble at fixed N , where correlation functions of (normalized) twist operators in $\text{Sym}^N \mathcal{M}$ admit a genus expansion in the large- N limit with an effective string coupling $g_s^2 \sim N^{-1}$. However, Aharony and Urbach emphasized in [56] that this canonical description does not fully match perturbative string theory once subleading powers of $1/N$ and disconnected coverings are taken into account. See related work in [58]. The basic issue can be stated as follows.

For normalized correlators, covering-space expressions such as (2.119) involve combinatorial factors $N!/(N-n)!$ multiplying the contribution of a covering with n active copies. When expanded at large N , these factors produce subleading powers of N whose coefficients are not determined solely by the topology (g, n_c) of the covering: they depend in detail on how many copies are active, and on the distribution of cycle lengths. In other words, at fixed N , the contribution of a disconnected covering does not factorize into a simple product of independent weights for its connected components, and the canonical partition function of normalized correlators is therefore not exactly the exponential of a sum over connected coverings. The clean, purely topological genus counting expected from string theory only emerges in the strict $N \rightarrow \infty$

limit.

The grand-canonical uplift, obtained by summing over N with a fugacity μ , resolves this problem. Let us now take a closer look at how this works. Let $\mathcal{Z}^X[J]$ denote the generating functional of twist-field correlators in $\text{Sym}^N \mathcal{M}$ on X , where $J_w(x)$ couples to the single-cycle twist operator $\sigma_w(x)$. The grand-canonical generating functional is defined by

$$\begin{aligned}\mathcal{Z}_\mu^X[J] &:= \sum_{N=0}^{\infty} \mu^N Z_{\text{Sym}^N \mathcal{M}}^X[J], \\ Z_{\text{Sym}^N \mathcal{M}}^X[J] &:= Z_{\text{Sym}^N \mathcal{M}}^X \left[\exp \left(\sum_{w=1}^{\infty} \int_X d^2x J_w(x) \sigma_w(x) \right) \right].\end{aligned}\quad (2.147)$$

In the covering-space representation reviewed above, each fixed- N correlator is a sum over branched coverings $\Gamma : \Sigma \rightarrow X$ with prescribed ramification data. For definiteness, consider a correlator of twist-sector vacuum σ_{w_i} :

$$Z_{\text{Sym}^N \mathcal{M}}^X [\sigma_{w_1} \cdots \sigma_{w_{n_c}}] = \sum_{\Gamma} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma}}{|\text{Aut}(\Gamma)|}. \quad (2.148)$$

Applying a decomposition of each covering surface Σ into its connected components

$$\Sigma = \bigsqcup_{\alpha} \Sigma_{\alpha}^{\sqcup k_{\alpha}}, \quad \Gamma|_{\Sigma_{\alpha}} : \Sigma_{\alpha} \rightarrow X \text{ of degree } n_{\alpha}, \quad \sum_{\alpha} k_{\alpha} n_{\alpha} = N, \quad (2.149)$$

the correlator can be reorganized as

$$Z_{\text{Sym}^N \mathcal{M}}^X [\sigma_{w_1} \cdots \sigma_{w_{n_c}}] = \sum_{\Gamma} \prod_{\alpha} \frac{1}{k_{\alpha}!} \left(\frac{\hat{Z}_{\mathcal{M}}^{\Sigma_{\alpha}}}{|\text{Aut}(\Gamma|_{\Sigma_{\alpha}})|} \right)^{k_{\alpha}}. \quad (2.150)$$

Once we sum over N as in (2.147), the constraint that the total number of copies participating in a given covering cannot exceed N is removed: a connected covering of degree n_{α} simply contributes with a weight $\mu^{n_{\alpha}}$. It is then convenient to focus on the logarithm of $\mathcal{Z}_{\mu}^X[J]$, which collects *connected* coverings. Introducing a formal string coupling g_s by attaching a factor g_s^w to each insertion of a w -cycle twist in the source¹⁰, one can write

$$\log \mathcal{Z}_{\mu}^X[J] = \sum_{\{w_i\}} \exp \left(\sum_{w=1}^{\infty} \int_X d^2x J_w(x) \right) \bigg|_{\{w_i\}} \sum_{\substack{\Gamma: \Sigma \rightarrow X \\ \text{connected}}} \mu^{n(\Gamma)} g_s^{w(\Gamma)} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma}}{|\text{Aut}(\Gamma)|}, \quad (2.151)$$

where:

¹⁰Here g_s is introduced as a bookkeeping parameter that keeps track of the total ramification. Its relation to the canonical large- N expansion will be made explicit below in the specialization to $X = S^2$.

- The sum $\{w_i\}$ runs over finite multisets of positive integers, specifying the lengths of the inserted single-cycle twists. In particular, if $\{w_i\}$ contains m_w copies of the twist length w , we may write $\{w_i\} = \{1^{m_1}, 2^{m_2}, \dots\}$.
- For a fixed multiset $\{w_i\}$ with multiplicities m_w , the notation

$$\exp \left(\sum_{w=1}^{\infty} \int_X d^2x J_w(x) \right) \Big|_{\{w_i\}}$$

denotes the coefficient of the monomial

$$\prod_{w=1}^{\infty} \frac{1}{m_w!} \left(\int_X d^2x J_w(x) \right)^{m_w}$$

in the expansion of the source exponential. In other words, it picks out the contribution with exactly m_w insertions of twists of length w for each w .

- In the second sum, $\Gamma : \Sigma \rightarrow X$ runs over connected branched coverings whose ramification profile is precisely encoded by the multiset $\{w_i\}$ selected in the first sum, together with the locations of the corresponding branch points, encoded in the integration variables x .
- $n(\Gamma) = \deg(\Gamma)$ is the degree of the covering, and $w(\Gamma)$ is the total ramification $w(\Gamma) = \sum_i w_i$, associated with the twist profile $\{w_i\}$ of Γ .

Note that the factor $1/|\text{Aut}(\Gamma)|$ must appear in the exponent to avoid overcounting equivalent coverings.

Riemann-Hurwitz formula and the identification of g_s .

The topology of a connected covering $\Gamma : \Sigma \rightarrow X$ is constrained by the Riemann-Hurwitz formula for generic coverings

$$\chi(\Sigma) = n(\Gamma) \chi(X) - \sum_i (w_i - 1), \quad \chi(X) = 2 - 2G, \quad (2.152)$$

where G is the genus of X and w_i is the local branching order at each branch point. This can be written as

$$n(\Gamma)(2G - 2) + w(\Gamma) = -2 + 2g(\Sigma) + n_c. \quad (2.153)$$

Comparing (2.153) with the g_s -dependence in (2.151), we see that if we choose

$$\mu = g_s^{2G-2}, \quad (2.154)$$

then the combined factor satisfies the following genus-expansion form:

$$\mu^{n(\Gamma)} g_s^{w(\Gamma)} = g_s^{n(\Gamma)(2G-2)+w(\Gamma)} = g_s^{-2+2g(\Sigma)+n_c}. \quad (2.155)$$

In other words, for each fixed base surface X of genus G , it is natural to identify the fugacity μ with the $(2G - 2)$ -th power of the string coupling. With this choice, the weight of a connected covering in the exponent becomes

$$g_s^{-2+2g(\Sigma)+n_c} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma}}{|\text{Aut}(\Gamma)|}, \quad (2.156)$$

which depends only on the topology of the covering surface and the number of insertions, as expected in a string-theoretic genus expansion.

The case $X = S^2$.

Let us now specialize to the case $X = S^2$, for which $G = 0$ and $\chi(X) = 2$. Recall that we denote the vacuum partition function of the seed theory on the sphere by $Z_{\mathcal{M}}^{S^2} = z$. Recall that, in the absence of any insertions ($J = 0$), the orbifold partition function is given by $Z_{\text{Sym}^N \mathcal{M}}^{S^2}[0] = z^N/N!$ as in (2.19). The grand-canonical vacuum partition function becomes

$$\mathcal{Z}_{\mu}^{S^2}[0] = \sum_{N=0}^{\infty} \mu^N Z_{\text{Sym}^N \mathcal{M}}^{S^2}[0] = \sum_{N=0}^{\infty} \frac{(\mu z)^N}{N!} = \exp(\mu z). \quad (2.157)$$

The expectation value of N in this vacuum ensemble is then

$$\overline{N} = \mu \partial_{\mu} \mathcal{Z}_{\mu}^{S^2}[0] = \mu z. \quad (2.158)$$

In particular, for $X = S^2$, the natural choice (2.154) yields

$$\mu = g_s^{-2}, \quad \overline{N} \propto g_s^{-2}, \quad (2.159)$$

so that in the weak-coupling regime $g_s \rightarrow 0$ we have

$$g_s^2 \sim \frac{1}{\overline{N}}, \quad (2.160)$$

in precise agreement with the canonical large- N scaling $g_s^2 \sim N^{-1}$ used in Sec. 2.5.5. In particular, with $g_s^2 \sim N^{-1}$, this assignment also reproduces the scaling of the twist normalizations in (2.141), since each w -cycle insertion carries a factor $g_s^w \sim N^{-w/2}$.

Let us consider normalized correlation functions of the grand-canonical theory

$$\left\langle \prod_{i=1}^{n_c} \sigma_{w_i}(x_i) \right\rangle_{g_s}^{S^2} = \left[\prod_{i=1}^{n_c} \frac{\delta}{\delta J_{w_i}(x_i)} \right] \frac{\mathcal{Z}_\mu^{S^2}[J]}{\mathcal{Z}_\mu^{S^2}[0]} \Big|_{J=0}. \quad (2.161)$$

To leading order in g_s , this can be approximated by fixing N to its expectation value $\bar{N} = \mu z$. Indeed, both the numerator and the denominator in (2.161) are sums over N of the form

$$\mathcal{Z}_\mu^{S^2}[J] = \sum_{N=0}^{\infty} \mu^N Z_{\text{Sym}^N \mathcal{M}}^{S^2}[J], \quad \mathcal{Z}_\mu^{S^2}[0] = \sum_{N=0}^{\infty} \mu^N Z_{\text{Sym}^N \mathcal{M}}^{S^2}[0] = \sum_{N=0}^{\infty} \frac{(\mu z)^N}{N!}. \quad (2.162)$$

For large μz the latter sum is sharply peaked at $N \simeq \bar{N} = \mu z$ with width $\Delta N \sim \sqrt{\bar{N}}$. Approximating the sums by their largest terms (saddle-point approximation) gives

$$\mathcal{Z}_\mu^{S^2}[0] \sim \frac{(\mu z)^{\bar{N}}}{\bar{N}!} \times \sqrt{2\pi \mu z}, \quad (2.163)$$

$$\left[\prod_{i=1}^{n_c} \frac{\delta}{\delta J_{w_i}(x_i)} \right] \mathcal{Z}_\mu^{S^2}[J] \Big|_{J=0} \sim \mu^{\bar{N}} Z_{\text{Sym}^{\bar{N}} \mathcal{M}}^{S^2} \left[\prod_{i=1}^{n_c} \sigma_{w_i}(x_i) \right] \times \sqrt{2\pi \mu z}, \quad (2.164)$$

so that the Gaussian prefactors $\sqrt{2\pi \mu z}$ cancel in the normalized ratio (2.161). We therefore obtain to leading order,

$$\left\langle \prod_{i=1}^{n_c} \sigma_{w_i}(x_i) \right\rangle_{g_s}^{S^2} \simeq \left\langle \prod_{i=1}^{n_c} \hat{\sigma}_{w_i}(x_i) \right\rangle_{\bar{N}}^{S^2}, \quad \bar{N} = \mu z. \quad (2.165)$$

In particular, the normalized grand-canonical correlators (2.161) reproduce, to leading order in $g_s \sim N^{-1/2}$, the canonical large- N behavior of normalized twist correlators discussed in Sec. 2.5.5.

Part II.

Bottom-up Approaches

3. Entropy in Symmetric Orbifolds

In this chapter, we take a bottom-up viewpoint and analyze concrete observables in symmetric orbifold CFTs. Rather than starting from a putative semiclassical bulk dual and working down, we will exploit the microscopic structure of the orbifold theory itself to gain insight into the emergence of semiclassical physics.

In this chapter, we develop a framework for exact computations by “breaking down” the symmetric orbifold into cyclic orbifold sectors. As an application, we compute Rényi entropies for thermal states in the symmetric orbifold, making explicit use of the cyclic orbifold description.

Let us first introduce the expected answer from the holographic perspective.

Entanglement entropy and plateaux in the BTZ phase.

We consider a large- c holographic CFT₂ on a spatial circle of circumference L at finite temperature $T = 1/\beta$. For sufficiently high temperature, above the Hawking–Page transition, the dominant bulk saddle of AdS₃ gravity is the non-rotating BTZ black hole [59]. We study the entanglement entropy of a single interval A of length $\ell \in (0, L)$ using the Ryu–Takayanagi (RT) prescription [60, 61]; see also the reviews [62, 63].

RT surfaces and competing saddles. In the BTZ geometry the RT surface for a single interval A is a bulk geodesic anchored at the endpoints of A . One relevant homology class consists of geodesics that connect the two endpoints without wrapping the horizon. The corresponding entanglement entropy takes the standard finite-temperature form

$$S_{\text{conn}}(\ell) = \frac{c}{3} \log \left(\frac{\beta}{\pi \varepsilon} \sinh \frac{\pi \ell}{\beta} \right), \quad (3.1)$$

where ε is a UV cutoff. This agrees with the universal single-interval result in 2D CFT at temperature $1/\beta$ [64, 65, 66, 67] and can be derived holographically in the BTZ background [68]. However, because the spatial direction is compact, there is a second natural homology class: a configuration consisting of

- a geodesic homologous to the complement interval A^c of length $L - \ell$, and

- the BTZ horizon, a closed codimension-two surface in the bulk.

The Bekenstein–Hawking entropy of the BTZ black hole equals the thermal entropy S_{th} of the CFT on the circle,

$$S_{\text{BH}} = S_{\text{th}} = \frac{\pi c L}{3\beta}, \quad (3.2)$$

as follows from the Cardy formula [69]. The RT candidate in this “disconnected” homology class has entanglement entropy

$$S_{\text{disc}}(\ell) = S_{\text{BH}} + S_{\text{conn}}(L - \ell) = \frac{\pi c L}{3\beta} + \frac{c}{3} \log \left(\frac{\beta}{\pi \varepsilon} \sinh \frac{\pi(L - \ell)}{\beta} \right). \quad (3.3)$$

At the classical level, the entanglement entropy of A is given by the minimal area,

$$S(\ell) = \min \{ S_{\text{conn}}(\ell), S_{\text{disc}}(\ell) \}. \quad (3.4)$$

The competition between the connected and disconnected RT surfaces underlies the entanglement plateau phenomenon in finite-volume holographic CFTs [70].

Small intervals, plateau, and transition point. To make the plateau structure more explicit, consider a small interval A of length ℓ and its complement A^c of length $L - \ell$. In the small- ℓ regime we have

$$S(\ell) = S_{\text{conn}}(\ell), \quad (3.5)$$

while for the large complement A^c the minimal RT surface is the disconnected one,

$$S(L - \ell) = S_{\text{disc}}(L - \ell) = S_{\text{BH}} + S_{\text{conn}}(\ell). \quad (3.6)$$

Therefore, in this regime

$$S(L - \ell) - S(\ell) = S_{\text{BH}}, \quad (3.7)$$

and the thermal entropy can be recovered from the difference of entanglement entropies of a region and its complement. This reproduces, at the classical holographic level, the universal relation between thermal and entanglement entropy established on the CFT side in [71]. The endpoint of the plateau on the small-interval side is characterized by the transition point ℓ_* at which the two RT saddles have equal area,

$$S_{\text{conn}}(\ell_*) = S_{\text{disc}}(\ell_*) \iff S_{\text{conn}}(\ell_*) = S_{\text{BH}} + S_{\text{conn}}(L - \ell_*). \quad (3.8)$$

Using (3.1) and (3.2), this yields the transcendental equation

$$\sinh\left(\frac{\pi(L - \ell_*)}{\beta}\right) = \exp\left(\frac{\pi L}{\beta}\right) \sinh\left(\frac{\pi \ell_*}{\beta}\right), \quad (3.9)$$

which determines ℓ_* as a function of β/L . In the classical large- c limit this leads to an exact entanglement plateau for $0 < \ell < \ell_*$, while subleading corrections that smooth out the plateau have been analyzed in detail in [72].

3.1. Exact computations via cyclic orbifolds

Here we review how twisted sectors of a symmetric product CFT can be decomposed into products of cyclic orbifold theories. The key point is that the conjugacy classes of the symmetric group S_N are labeled by cycle structures, and each cycle of length n gives rise to a $\mathbb{Z}_n$ cyclic orbifold sector. This decomposition allows us to reduce computations in the full symmetric orbifold to those in simpler cyclic orbifold theories.

3.1.1. Cyclic orbifold CFT

Since any element of the symmetric group can be decomposed into a product of cyclic permutations, it is natural to expect that there should be a corresponding relation between symmetric orbifold CFTs and cyclic orbifold CFTs. In this section, we present the relation in terms of Hecke operators that we gave in [1]. This relation allows us to reduce problems in symmetric orbifold CFTs to problems in cyclic orbifold CFTs. In general, the more complicated the orbifold group is, the harder it is to obtain exact results for the corresponding orbifold CFT. The relation developed in this section may thus be regarded as a new method for solving problems in symmetric orbifold CFTs.

Let us define the cyclic orbifold CFT in the similar way to the symmetric orbifold:

$$\text{Cyc}^N \mathcal{C} := C^{\otimes N} / \mathbb{Z}_N. \quad (3.10)$$

Square-free Hecke operator

While it had long been known that the torus partition function of a symmetric orbifold CFT can be written in terms of Hecke operators, only partial results were available for the torus partition function of a cyclic orbifold CFT (for example, for prime N , it is given by (3.15) [73]). In order to obtain an explicit expression for the torus partition function for general N , we introduced in [1] a deformation of the Hecke operator, called the square-free Hecke operator.

Let us briefly review some aspects of Hecke operators. Consider the case in which k is divisible by a square a^2 . In this case, the Hecke operator T_k of order k contains, as a subleading piece, $\frac{1}{a^2}$ times the Hecke operator of order $\frac{k}{a^2}$. This fact can be seen as follows. Starting from the definition (2.27) of T_k , if we restrict the summation range of i to (divisors of k/a^2) $\times a$ and that of j to multiples of a , we obtain

$$\begin{aligned} T_k Z(\tau) &= \frac{1}{k} \sum_{i|k} \sum_{j=0}^{i-1} Z\left(\frac{k}{i^2}\tau + \frac{j}{i}\right) \\ &\supset \frac{1}{k} \sum_{i'|\frac{k}{a^2}} \sum_{j'=0}^{i'-1} Z\left(\frac{k}{(ai')^2}\tau + \frac{aj'}{ai'}\right) \end{aligned} \quad (3.11)$$

$$= \frac{1}{a^2} \cdot \frac{1}{k/a^2} \sum_{i'|\frac{k}{a^2}} \sum_{j'=0}^{i'-1} Z\left(\frac{k/a^2}{i'^2}\tau + \frac{j'}{i'}\right), \quad (3.12)$$

where we have introduced i', j' by $(i, j) = (ai', aj')$. The last line is nothing but $\frac{1}{a^2} \cdot T_{\frac{k}{a^2}} Z(\tau)$.

From the above discussion, we see that a generic Hecke operator decomposes into several independent pieces, each of which is closed under modular transformations. The square-free Hecke operator T_k^{sf} of order k is defined as the largest among these independent pieces. It can be constructed by subtracting off the redundant parts from T_k , leading to the following recursive definition:

$$T_k^{\text{sf}} Z(\tau) = T_k Z(\tau) - \sum_{\substack{a \in \mathbb{Z}_{>1} \\ a^2 | k}} \frac{1}{a^2} T_{\frac{k}{a^2}}^{\text{sf}} Z(\tau). \quad (3.13)$$

This definition reduces to the usual definition (2.27) of the Hecke operator when k has no square factor. This is the reason for the name “square-free”.

Indeed, starting for instance from terms such as $\frac{1}{k} Z(k\tau)$ and repeatedly applying modular S - and modular T -transformations, one can check that all terms in $T_k^{\text{sf}} Z(\tau)$ are reproduced.

Torus partition function of cyclic orbifold CFT

The torus partition function of a cyclic orbifold CFT takes the following form:

$$\begin{aligned} Z_{\text{Cyc}^N \mathcal{C}}(\tau) &= \sum_{l=0}^{N-1} \text{Tr}_{(l)} \left(\frac{\sum_{l'=1}^N g^{l'} q^{H_L} \bar{q}^{H_R}}{N} \right) \\ &= \frac{1}{N} \sum_{l, l'} \text{Tr}_{(l)} \left(g^{l'} q^{H_L} \bar{q}^{H_R} \right), \end{aligned} \quad (3.14)$$

where g is a generator of $\mathbb{Z}_N$, and l labels the twisted sectors. Moreover, H_L and H_R are the Hamiltonians of $\mathcal{C}^{\otimes N}$. In the l -twisted sector, an operator living in the j -th copy of the CFT is mapped, under $z \rightarrow z + 1$, to an operator in the $(j + l)$ -th copy of the CFT.

When N is prime, this can be written in terms of the Hecke operator [73]:

$$Z_{\text{Cyc}^N \mathcal{C}}(\tau) = \frac{1}{N} Z(\tau)^N + (N - 1) T_N Z(\tau) \quad (3.15)$$

$$= \frac{1}{N} \left(Z(\tau)^N + (N - 1) \left(Z(N\tau) + \sum_{j=0}^{N-1} Z\left(\frac{\tau + j}{N}\right) \right) \right). \quad (3.16)$$

For prime N , the expression (3.15) of $Z_{\text{Cyc}^N \mathcal{C}}(\tau)$ can be generalized to arbitrary N by using the square-free Hecke operator [1]:

$$Z_{\text{Cyc}^N \mathcal{C}}(\tau) = \sum_{k=1}^N \frac{1}{\gcd(N, k)} T_{\frac{N}{\gcd(N, k)}}^{\text{sf}} \left(Z(\tau)^{\gcd(N, k)} \right) \quad (3.17)$$

$$= \sum_{d|N} \frac{\phi(N/d)}{d} T_{\frac{N}{d}}^{\text{sf}} \left(Z(\tau)^d \right), \quad (3.18)$$

where $\phi(n)$ is the Euler totient function, which for a positive integer n counts the number of integers between 1 and n that are coprime to n . When N is prime, we have $T_N^{\text{sf}} = T_N$, so that (3.18) indeed reduces to the natural extension of (3.15). A proof of this construction is given in [1].

Example: $\mathbb{Z}_8$ cyclic orbifold. As a simple but nontrivial example, let us consider the case $N = 8$. The divisors of 8 are 1, 2, 4, 8. We have

- for $d = 8$, $\phi(8/8) = 1$,
- for $d = 4$, $\phi(8/4) = 1$,
- for $d = 2$, $\phi(8/2) = 2$,
- for $d = 1$, $\phi(8/1) = 4$,

and therefore

$$Z_{\text{Cyc}^8 \mathcal{C}}(\tau) = \frac{1}{8} T_1^{\text{sf}} \left(Z(\tau)^8 \right) + \frac{1}{4} T_2^{\text{sf}} \left(Z(\tau)^4 \right) + \frac{2}{2} T_4^{\text{sf}} \left(Z(\tau)^2 \right) + \frac{4}{1} T_8^{\text{sf}} \left(Z(\tau)^1 \right), \quad (3.19)$$

as claimed.

3.1.2. Symmetric orbifold from cyclic orbifolds

In this subsection, we reconstruct the twisted sectors of a symmetric orbifold CFT in terms of the twisted sectors of a cyclic orbifold CFT. This allows us to reduce the analysis of symmetric orbifold CFTs on the torus to problems in the simpler cyclic orbifold CFT. The partition function of the cyclic orbifold CFT of order k can be written, using (3.14), as

$$Z_{\text{Cyc}^k \mathcal{C}}(\tau, \bar{\tau}) = \frac{1}{k} \sum_{l, l'=1}^i \text{Tr}_{(l)} \left(g^{l'} q^{H_L} \bar{q}^{H_R} \right). \quad (3.20)$$

By comparing this with the expression (3.18) in terms of the square-free Hecke operators, we obtain the correspondence

$$\frac{\phi(k/d)}{d} T_{\frac{k}{d}}^{\text{sf}} \left(Z(\tau, \bar{\tau})^d \right) = \frac{1}{k} \sum_{\substack{l, l'=1 \\ \gcd(k, l, l')=d}}^k \text{Tr}_{(l)} \left(g^{l'} q^{H_L} \bar{q}^{H_R} \right). \quad (3.21)$$

In particular, for $d = 1$ we find

$$T_k^{\text{sf}} Z(\tau, \bar{\tau}) = \frac{1}{k\phi(k)} \sum_{\substack{l, l'=1 \\ \gcd(k, l, l')=1}}^k \text{Tr}_{(l)} \left(g^{l'} q^{H_L} \bar{q}^{H_R} \right), \quad (3.22)$$

which shows that we can compute the square-free Hecke operator $T_k^{\text{sf}} Z(\tau, \bar{\tau})$ by restricting to those twisted sectors of the cyclic orbifold CFT for which $\gcd(k, l, l') = 1$.

We can now reconstruct the ordinary Hecke operator from the square-free Hecke operators as

$$\begin{aligned} T_k Z(\tau, \bar{\tau}) &= T_k^{\text{sf}} Z(\tau, \bar{\tau}) + \sum_{\substack{a \in \mathbb{Z}_{>1} \\ a^2 | k}} \frac{1}{a^2} T_{k/a^2}^{\text{sf}} Z(\tau, \bar{\tau}) \\ &= \sum_{\substack{a \in \mathbb{Z}_{>0} \\ a^2 | k}} \frac{1}{a^2} T_{k/a^2}^{\text{sf}} Z(\tau, \bar{\tau}). \end{aligned} \quad (3.23)$$

Combining this with the result above, we see that the Hecke operator $T_k Z(\tau, \bar{\tau})$ can be reconstructed by considering the cyclic orbifold CFTs of order k/a^2 .

Substituting this into the Hecke-operator representation (2.29) of the partition function of the symmetric orbifold CFT of order N ,

$$Z_{\text{Sym}^N \mathcal{C}}(\tau, \bar{\tau}) = \sum_{\substack{\text{partition of } N \\ (\dots, k, \dots)}} \prod_{k=1}^N \frac{1}{(\#k)!} \left(T_k Z(\tau, \bar{\tau}) \right)^{\#k},$$

we obtain the desired reconstruction. In Section 3.4 we will carry out this computation explicitly.

3.2. Rényi entropy in seed theory

Entanglement entropy is quite universal as long as we work on the plane. However, if we consider it on different states, it highly depends on the detail of the CFT we are interested in. In large N symmetric orbifolds, we have phase transitions.

In this section, we review a detailed entropy calculations of a two-dimensional Dirac fermion CFT[74, 75, 76, 77, 1], which we indicate as $\mathcal{D}$. We implement some computational techniques for the replica method. By using them, we consider Rényi entropy on sphere/torus. The analysis on the torus is more non-trivial than that on the Riemann sphere since it is sensitive to the global properties of the CFT. In the limit where the subsystem is small, only local properties are emphasized and the result on the Riemann sphere is recovered.

Replica method in generic CFT

Let us briefly review the replica method in two-dimensional CFT. Let the endpoints of subsystem l be $x = x_1, x_2$. We take n -replica copy, and consequently for each primary field ϕ it has copy $\phi_{[a]}$ for $a = 1, \dots, n$. For the fields $\phi_{[a]}$ and a sufficiently small real number ϵ , following boundary conditions are imposed:

$$\phi_{[a]}(x_1 + \epsilon e^{2\pi i}) = \phi_{[a+1]}(x_1 + \epsilon), \quad (3.24a)$$

$$\phi_{[a]}(x_2 + \epsilon e^{2\pi i}) = \phi_{[a-1]}(x_2 + \epsilon), \quad (3.24b)$$

so that the fields are cyclically permuted when encircling the endpoints. The operators that implement such boundary conditions are twist operators, denoted by $\sigma(x_1, \bar{x}_1)$ and $\bar{\sigma}(x_2, \bar{x}_2)$, and their expectation value gives $\text{Tr}_A[\rho_A^n]$:

$$\text{Tr}_A[\rho_A^n] = \langle \sigma(x_1, \bar{x}_1) \bar{\sigma}(x_2, \bar{x}_2) \rangle. \quad (3.25)$$

The conformal dimensions of these twist operators are given by $\left(\frac{c}{24} \left(n - \frac{1}{n}\right), \frac{c}{24} \left(n - \frac{1}{n}\right)\right)$. Therefore, the scaling dimension Δ is given by $\Delta = \frac{c}{12} \left(n - \frac{1}{n}\right)$. This allows us to determine the leading behavior in the limit where $|x_1 - x_2| = l$ is sufficiently small:

$$\text{Tr}_A[\rho_A^n] \propto l^{-\frac{c}{6} \left(n - \frac{1}{n}\right)}. \quad (3.26)$$

Introducing a cutoff ϵ , the leading terms of the n -th Rényi entropy and of the entanglement entropy are

$$S_A^{(n)}(l) = \frac{c}{6} \left(1 + \frac{1}{n}\right) \log \frac{l}{\epsilon}, \quad (3.27)$$

$$S_A(l) = \frac{c}{3} \log \frac{l}{\epsilon}. \quad (3.28)$$

Consistency condition for the replica method on the torus.

For the replica method or Rényi entropies of subsystems on the torus, there is a geometric interpretation summarized as follows:

- When the subsystem A is sufficiently small (the limit $l \rightarrow 0$), each replica sheet factorizes into the original torus.
- When the subsystem A is sufficiently large (the limit $l \rightarrow 1$), each replica sheet becomes a torus with the modular parameter τ stretched by a factor of n in the τ direction.

Thus, the n -th Rényi entropy is expected to satisfy

$$\lim_{\sigma \rightarrow 0} [S_A^{(n)}(1 - l) - S_A^{(n)}(l)] = \frac{1}{1 - n} \log \frac{Z(n\tau, n\bar{\tau})}{Z(\tau, \bar{\tau})^n}. \quad (3.29)$$

In the limit $n \rightarrow 1$, this reduces to the ordinary thermal entropy $\log Z(\tau) - \tau \frac{\partial}{\partial \tau} \log Z(\tau)$.

3.2.1. Computational techniques for Dirac fermion

To implement the replica method, we first perform a discrete Fourier transform of the Dirac fermion fields. This allows us to translate the monodromies of the fields around the endpoints of the subsystem into phase rotations. We then encode the monodromy information in appropriate twist operators by bosonization.

Discrete Fourier transform.

Let $\psi_{L,[a]}(x)$ denote the left-moving (holomorphic) Dirac fermion living on the a -th replica sheet. Around the twist operator $\sigma(x_0)$, the field transforms as

$$\psi_{L,[a]}(x_0 + \epsilon e^{2\pi i}) \rightarrow e^{\pi i \frac{n-1}{n}} \psi_{L,[a+1]}(x_0 + \epsilon), \quad (3.30)$$

where the phase factor is introduced as follows. Considering the local structure of the Riemann surface around x_0 , the configuration with n connected replica sheets can be

locally described by a single z -plane via the conformal map

$$(z - z_0)^n = x - x_0. \quad (3.31)$$

Recalling that the conformal transformation of a field with conformal weight $(1/2, 0)$ is given by $\psi_L(z)(dz)^{1/2} = \psi_{L,[a]}(x)(dx)^{1/2}$, we obtain

$$\begin{aligned} \psi_L(z) &= \sqrt{\frac{dx}{dz}} \psi_{L,[a]}(x) \\ &= \sqrt{n}(x - x_0)^{\frac{1}{2} \frac{n-1}{n}} \psi_{L,[a]}(x), \end{aligned} \quad (3.32)$$

so that $\psi_L(z)$ is non-singular. Considering the transformation $x = x_0 + \epsilon \rightarrow x_0 + \epsilon e^{2\pi i}$, we see that the field acquires a phase factor $e^{\pi i \frac{n-1}{n}}$.

We now perform the following discrete Fourier transform. Let p is the label of operators after transformation¹, $p = -\frac{n-1}{2}, -\frac{n+1}{2}, \dots, \frac{n-1}{2}$:

$$\tilde{\psi}_{L,p} = \frac{1}{\sqrt{n}} \sum_{a=1}^n e^{-2\pi i \frac{a}{n} (p - \frac{n-1}{2})} \psi_{L,[a]}. \quad (3.33)$$

Under the transformation $x = x_0 + \epsilon \rightarrow x_0 + \epsilon e^{2\pi i}$, the right-hand side becomes

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{a=1}^n e^{-2\pi i \frac{a}{n} (p - \frac{n-1}{2})} \psi_{L,[a]}(x_0 + \epsilon e^{2\pi i}) &= \frac{1}{\sqrt{n}} \sum_{a=1}^n e^{-2\pi i \frac{a}{n} (p - \frac{n-1}{2})} e^{\pi i \frac{n-1}{n}} \psi_{L,[a+1]}(x_0 + \epsilon) \\ &= e^{2\pi i \frac{p}{n}} \frac{1}{\sqrt{n}} \sum_{a=1}^n e^{-2\pi i \frac{a+1}{n} (p - \frac{n-1}{2})} \psi_{L,[a+1]}(x_0 + \epsilon), \end{aligned} \quad (3.34)$$

so that

$$\tilde{\psi}_{L,p}(x_0 + \epsilon e^{2\pi i}) = e^{2\pi i \frac{p}{n}} \tilde{\psi}_{L,p}(x_0 + \epsilon). \quad (3.35)$$

Thus, the twist operator σ inserted at $x = x_0$ must implement the above transformation.

Bosonization and construction of replica-twist operators

Each $\tilde{\psi}_{L,p}$ can now be regarded as an independent Dirac fermion. We wish to construct operators that induce the above phase rotations. To this end, for each p we bosonize

¹This type of discrete Fourier transform, for which the replica method works even for even n , was introduced in [1].

$\tilde{\psi}_{L;p}$ by introducing n free scalar fields $\tilde{\varphi}_{L;p}$ and setting

$$\tilde{\psi}_{L;p} = e^{i\tilde{\varphi}_{L;p}}. \quad (3.36)$$

The left-moving twist operator σ_L can then be written as

$$\sigma_L(x) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sigma_{L;p}(x) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{i\frac{p}{n}\tilde{\varphi}_{L;p}(x)}, \quad (3.37)$$

where we have introduced the twist operator $\sigma_{L;p}$ for each scalar field as $\sigma_{L;p}(x) = e^{i\frac{p}{n}\tilde{\varphi}_{L;p}(x)}$. Using the OPE $\tilde{\varphi}_{L;p}(x)\tilde{\varphi}_{L;p}(0) \sim -\log x$, we find that each $\psi_{L;p}$ has the OPE

$$\tilde{\psi}_{L;p}(x)\sigma_{L;p}(x_0) \sim (x-x_0)^{\frac{p}{n}} \tilde{\psi}_{L;p}(x_0)\sigma_{L;p}(x_0), \quad (3.38)$$

so that under $x = x_0 + \epsilon \rightarrow x_0 + \epsilon e^{2\pi i}$, the desired phase factor $e^{2\pi i\frac{p}{n}}$ is indeed reproduced².

Right-moving sector and replica-twist conjugate.

For the right-moving sector, we perform the similar procedure and obtain for $p = -\frac{n-1}{2}, -\frac{n+1}{2}, \dots, \frac{n-1}{2}$,

$$\tilde{\psi}_{R;p} = \frac{1}{\sqrt{n}} \sum_{a=1}^n e^{-2\pi i\frac{a}{n}(p+\frac{n-1}{2})} \psi_{R;[a]}, \quad (3.41)$$

which satisfies

$$\tilde{\psi}_{R;p}(\bar{x}_0 + \epsilon e^{-2\pi i}) = e^{2\pi i\frac{p}{n}} \tilde{\psi}_{R;p}(\bar{x}_0 + \epsilon). \quad (3.42)$$

²In this construction, we must take the complex conjugate to bosonize as $\tilde{\bar{\psi}}_{L;p} = e^{-i\tilde{\varphi}_{L;p}}$. Considering the OPE of this with $\sigma_{L;p}$, the phase factor under $x = x_0 + \epsilon \rightarrow x_0 + \epsilon e^{2\pi i}$ becomes $e^{-2\pi i\frac{p}{n}}$. To match this behavior, we must modify the discrete Fourier transform as follows: for $p = -\frac{n-1}{2}, -\frac{n+1}{2}, \dots, \frac{n-1}{2}$,

$$\tilde{\bar{\psi}}_{L;p} = \frac{1}{\sqrt{n}} \sum_{a=1}^n e^{2\pi i\frac{a}{n}(p+\frac{n-1}{2})} \bar{\psi}_{L;[a]}. \quad (3.39)$$

Under $\bar{\psi}_{L;[a]}(x_0 + \epsilon e^{2\pi i}) \rightarrow e^{\pi i\frac{n-1}{n}} \bar{\psi}_{L;[a+1]}(x_0 + \epsilon)$, one can then verify that

$$\tilde{\bar{\psi}}_{L;p}(x_0 + \epsilon e^{2\pi i}) = e^{-2\pi i\frac{p}{n}} \tilde{\bar{\psi}}_{L;p}(x_0 + \epsilon). \quad (3.40)$$

We bosonize as³

$$\tilde{\psi}_{R;p} = e^{i\tilde{\varphi}_{R;p}}, \quad (3.43)$$

and the twist operator $\sigma_{R;p}$ for each p can be taken as $\sigma_{R;p} = e^{-i\frac{p}{n}\tilde{\varphi}_{R;p}}$. From the OPE

$$\tilde{\psi}_{R;p}(\bar{x})\sigma_{R;p}(\bar{x}_0) \sim (\bar{x} - \bar{x}_0)^{-\frac{p}{n}} \tilde{\psi}_{R;p}(\bar{x}_0)\sigma_{R;p}(\bar{x}_0), \quad (3.44)$$

we confirm that the boundary conditions are correctly implemented for $\bar{x} = \bar{x}_0 + \epsilon \rightarrow \bar{x}_0 + \epsilon e^{-2\pi i}$. Combining the left- and right-moving parts, the full replica-twist operator is given by

$$\sigma(x, \bar{x}) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sigma_{L;p}(x)\sigma_{R;p}(\bar{x}). \quad (3.45)$$

We also need to construct the twist operator $\bar{\sigma}$ that induces the transformation $[a] \rightarrow [a+1]$ when encircling x_0 in the opposite direction ($x = x_0 + \epsilon \rightarrow x_0 + \epsilon e^{-2\pi i}$). This can be obtained simply by taking the complex conjugate of the twist operator constructed above, yielding

$$\bar{\sigma}(x, \bar{x}) = \bar{\sigma}_L(x)\bar{\sigma}_R(\bar{x}), \quad (3.46a)$$

$$\bar{\sigma}_L(x) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \bar{\sigma}_{L;p}(x), \quad \bar{\sigma}_{L;p}(x) = e^{-i\frac{p}{n}\tilde{\varphi}_{L;p}(x)}, \quad (3.46b)$$

$$\bar{\sigma}_R(\bar{x}) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \bar{\sigma}_{R;p}(\bar{x}), \quad \bar{\sigma}_{R;p}(\bar{x}) = e^{i\frac{p}{n}\tilde{\varphi}_{R;p}(\bar{x})}, \quad (3.46c)$$

respectively.

3.2.2. Computation on the sphere

We are now ready to compute $\text{Tr}_A[\rho_A^n]$. Let the endpoints of the subsystem A be x_1, x_2 with $|x_1 - x_2| = l$. Then we have

$$\text{Tr}_A[\rho_A^n] = \langle \sigma(x_1, \bar{x}_1) \bar{\sigma}(x_2, \bar{x}_2) \rangle. \quad (3.47)$$

³There is a sign ambiguity in bosonization. Here we chose $\tilde{\psi}_{R;p} = e^{i\tilde{\varphi}_{R;p}}$, but we could equally well choose $e^{-i\tilde{\varphi}_{R;p}}$. This sign change corresponds to T-duality in the compact scalar description, so the resulting theory is equivalent.

Using the explicit construction of the twist operators, this becomes

$$\langle \sigma(x_1, \bar{x}_1) \bar{\sigma}(x_2, \bar{x}_2) \rangle = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle e^{+i\frac{p}{n}\tilde{\varphi}_{L;p}(x_1)} e^{-i\frac{p}{n}\tilde{\varphi}_{L;p}(x_2)} \rangle \langle e^{-i\frac{p}{n}\tilde{\varphi}_{R;p}(\bar{x}_1)} e^{+i\frac{p}{n}\tilde{\varphi}_{R;p}(\bar{x}_2)} \rangle. \quad (3.48)$$

Hence the expectation value of the twist operators is reduced to the expectation value of vertex operators of the scalar fields. Applying Wick's theorem, we obtain

$$\langle e^{+i\frac{p}{n}\tilde{\varphi}_{L;p}(x_1)} e^{-i\frac{p}{n}\tilde{\varphi}_{L;p}(x_2)} \rangle \propto (x_1 - x_2)^{-\frac{p^2}{n^2}}, \quad (3.49)$$

$$\langle e^{-i\frac{p}{n}\tilde{\varphi}_{R;p}(\bar{x}_1)} e^{+i\frac{p}{n}\tilde{\varphi}_{R;p}(\bar{x}_2)} \rangle \propto (\bar{x}_1 - \bar{x}_2)^{-\frac{p^2}{n^2}}, \quad (3.50)$$

and thus

$$\text{Tr}_A[\rho_A^n] = \langle \sigma(x_1, \bar{x}_1) \bar{\sigma}(x_2, \bar{x}_2) \rangle \propto |x_1 - x_2|^{-\frac{1}{6}(n-\frac{1}{n})}, \quad (3.51)$$

where we used the summation formula

$$\sum_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{p^2}{n^2} = \frac{1}{12} \left(n - \frac{1}{n} \right). \quad (3.52)$$

We can now compute the n -th Rényi entropy of the subsystem A . Introducing a cutoff ϵ , we find

$$\begin{aligned} S_A^{(n)}(l) &= \frac{1}{1-n} \log \text{Tr}_A[\rho_A^n] \\ &= \frac{1}{6} \left(1 + \frac{1}{n} \right) \log \frac{l}{\epsilon}. \end{aligned} \quad (3.53)$$

Taking the limit $n \rightarrow 1$, we obtain the entanglement entropy

$$S_A(l) = \frac{1}{3} \log \frac{l}{\epsilon}. \quad (3.54)$$

3.2.3. Computation on the torus

The computation on the torus corresponds to that in a finite-volume and finite-temperature system. In this section, we compute $\text{Tr}_A[\rho_A^n]$ on the torus, focusing for simplicity on the NS sector. Let the endpoints of the subsystem A be x_1, x_2 (with

$x_1 - x_2 = l$). We start from

$$\begin{aligned} \text{Tr}_A[\rho_A^n] &= \langle \sigma(x_1, \bar{x}_1) \bar{\sigma}(x_2, \bar{x}_2) \rangle \\ &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle e^{+i\frac{p}{n}\tilde{\varphi}_{L;p}(x_1)} e^{-i\frac{p}{n}\tilde{\varphi}_{R;p}(\bar{x}_1)} e^{-i\frac{p}{n}\tilde{\varphi}_{L;p}(x_2)} e^{+i\frac{p}{n}\tilde{\varphi}_{R;p}(\bar{x}_2)} \rangle, \end{aligned} \quad (3.55)$$

and compute the expectation value on the torus. The details of the computation are given in Appendix A.2. The result can be read off from eq. (A.42):

$$\begin{aligned} &\langle \mathcal{O}_{0,m}(x_1, \bar{x}_1) \mathcal{O}_{0,-m}(x_2, \bar{x}_2) \rangle \\ &= \frac{1}{Z_{\mathcal{D}}(\tau, \bar{\tau})} \left(\frac{\partial \theta_1(0|\tau)}{\theta_1(x_{12}|\tau)} \right)^{2h_{0,m}} \overline{\left(\frac{\partial \theta_1(0|\tau)}{\theta_1(x_{12}|\tau)} \right)^{2\bar{h}_{0,m}}} \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} \left| \frac{\theta_a(mx_{12}|\tau)}{\eta(\tau)} \right|^2, \end{aligned} \quad (3.56)$$

in which we set $m = p/n$ and $x_{12} = l$, and keep only the NS sector (the contribution of θ_3). This yields

$$\begin{aligned} \text{Tr}_A[\rho_A^n] &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left(\frac{\partial \theta_1(0|\tau)}{\theta_1(l|\tau)} \right)^{\frac{p^2}{n^2}} \overline{\left(\frac{\partial \theta_1(0|\tau)}{\theta_1(l|\tau)} \right)^{\frac{p^2}{n^2}}} \left| \frac{\theta_3\left(\frac{pl}{n}|\tau\right)}{\theta_3(\tau)} \right|^2 \\ &= \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(l|\tau)} \right|^{\frac{1}{6}\left(n-\frac{1}{n}\right)} \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left| \frac{\theta_3\left(\frac{pl}{n}|\tau\right)}{\theta_3(\tau)} \right|^2, \end{aligned} \quad (3.57)$$

from which we can compute the Rényi entropy:

$$\begin{aligned} S_A^{(n)}(l) &= \frac{1}{1-n} \log \text{Tr}_A[\rho_A^n] \\ &= S_A^{(n)1}(l) + S_A^{(n)2}(l), \end{aligned} \quad (3.58a)$$

where $S_A^{(n)1}(l)$ denotes the contribution from θ_1 and $S_A^{(n)2}(l)$ denotes the remaining contribution:

$$S_A^{(n)1}(l) = \frac{1}{6} \left(1 + \frac{1}{n} \right) \log \left| \frac{\theta_1(l|\tau)}{\partial \theta_1(0|\tau)} \right|, \quad (3.58b)$$

$$S_A^{(n)2}(l) = \frac{1}{1-n} \log \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left| \frac{\theta_3\left(\frac{pl}{n}|\tau\right)}{\theta_3(\tau)} \right|^2. \quad (3.58c)$$

Taking the limit $n \rightarrow 1$ in this expression, we can obtain the entanglement entropy. The computation can be carried out following [74], using the expansion $\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$.

Consistency condition for the replica method on the torus.

Finally, we verify that the Rényi entropy in the NS sector of the Dirac fermion on the torus, eq. (3.58), satisfies the relation (3.29). The contributions from $S_A^{(n)1}(l)$ cancel, and we find

$$\begin{aligned}
 \lim_{\sigma \rightarrow 0} [S_A^{(n)}(1-l) - S_A^{(n)}(l)] &= S_A^{(n)2}(1) - S_A^{(n)2}(0) \\
 &= \frac{1}{1-n} \log \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left| \frac{\theta_3\left(\frac{p}{n} \middle| \tau\right)}{\theta_3(\tau)} \right|^2 \\
 &= \frac{1}{1-n} \log \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left| \frac{\theta_3\left(\frac{p}{n} \middle| \tau\right) / \eta(\tau)}{\theta_3(\tau) / \eta(\tau)} \right|^2, \tag{3.59}
 \end{aligned}$$

which can be simplified using eq. (A.18),

$$\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\theta_3\left(\frac{p}{n} \middle| \tau\right)}{\eta(\tau)} = \frac{\theta_3(n\tau)}{\eta(n\tau)},$$

to reproduce eq. (3.29):

$$\lim_{l \rightarrow 0} [S_A^{(n)}(1-l) - S_A^{(n)}(l)] = \frac{1}{1-n} \log \frac{Z_{\mathcal{D}}^{\text{NS}}(n\tau, n\bar{\tau})}{Z_{\mathcal{D}}^{\text{NS}}(\tau, \bar{\tau})^n}, \tag{3.60}$$

$$Z_{\mathcal{D}}^{\text{NS}}(\tau, \bar{\tau}) = \left| \frac{\theta_3(\tau)}{\eta(\tau)} \right|^2. \tag{3.61}$$

3.3. Rényi entropy in Dirac fermion cyclic orbifolds on torus

In Sec. 3.1 we provided a clue for reducing the analysis of symmetric orbifold CFTs to that of cyclic orbifold CFTs. In this chapter, we explicitly analyze the cyclic orbifold CFT whose seed CFT is a Dirac fermion CFT, focusing on Rényi entropies. The results in this chapter are based on our paper [1].

In this chapter, both twists arising from orbifolding the CFT and twists arising from the replica method will appear. We therefore refer to them separately as *orbifold twists* and *replica twists*. We will mostly restrict our interest to the NS sector. We use a capital letter N when referring to the degree of the symmetric or cyclic orbifold CFT, and a lower-case letter n when referring to the number of replica sheets.

3.3.1. Twisted sector in torus partition function

We start from the Dirac-fermion CFT and consider its cyclic orbifold CFT of degree N . We denote the Dirac fermions of the j -th copy of the CFT by $\psi_{L,R}^j$ and $\bar{\psi}_{L,R}^j$, where $j = 1, 2, \dots, N$. Let g be the generator of $\mathbb{Z}_N$, whose action is given by

$$g : (\psi_{L,R}^j, \bar{\psi}_{L,R}^j) \rightarrow (\psi_{L,R}^{j+1}, \bar{\psi}_{L,R}^{j+1}), \quad (3.62)$$

Then we write the torus partition function (3.14) as

$$\begin{aligned} Z_{\text{Cyc}^N \mathcal{D}}(\tau, \bar{\tau}) &= \frac{1}{N} \sum_{l,l'} \text{Tr}_{(l)} (g^{l'} q^{H_L} \bar{q}^{H_R}) \\ &= \frac{1}{N} \sum_{l,l'} Z_{(l,l')}(\tau, \bar{\tau}). \end{aligned} \quad (3.63)$$

Thus, in order to obtain $Z_{\text{Cyc}^N \mathcal{D}}(\tau, \bar{\tau})$, it suffices to compute $Z_{(l,l')}(\tau, \bar{\tau})$. Here $Z_{(l,l')}(\tau, \bar{\tau})$ is the torus partition function of N copies of the Dirac-fermion CFT with the following boundary conditions induced by the action of g :

$$\psi_L^j(z+1) = \psi_L^{j+l}(z), \quad (3.64a)$$

$$\psi_L^j(z+\tau) = \psi_L^{j+l'}(z), \quad (3.64b)$$

$$\psi_R^j(\bar{z}+1) = \psi_R^{j+l}(\bar{z}), \quad (3.64c)$$

$$\psi_R^j(\bar{z}+\bar{\tau}) = \psi_R^{j+l'}(\bar{z}). \quad (3.64d)$$

Discrete Fourier transformation of orbifold twist

Once the structure of the orbifold twists is written out in this way, we see that it closely resembles the construction of replica-twist operators in Sec. 3.2.1. We again perform a discrete Fourier transform:

$$\tilde{\psi}_{L,R}^{(k)} = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-2\pi i \frac{kj}{N}} \psi_{L,R}^j, \quad (k = 0, 1, \dots, N-1), \quad (3.65)$$

and for $\bar{\psi}$ we simply take complex conjugation⁴. This transformation turns the action of the generator g into a phase rotation:

$$g : \tilde{\psi}_{L,R}^{(k)} \rightarrow e^{2\pi i \frac{k}{N}} \tilde{\psi}_{L,R}^{(k)}. \quad (3.68)$$

We now bosonize the fermions obtained by the discrete Fourier transform:

$$\tilde{\psi}_L^{(k)}(z) = e^{i\tilde{\varphi}_L^{(k)}(z)}, \quad \tilde{\psi}_R^{(k)}(\bar{z}) = e^{i\tilde{\varphi}_R^{(k)}(\bar{z})}. \quad (3.69)$$

In this way, the fermions entering the computation of $Z_{(l,l')}(\tau, \bar{\tau})$ are mapped to compact bosons $\tilde{\varphi}_L^{(k)}, \tilde{\varphi}_R^{(k)}$ ($k = 0, 1, \dots, N-1$) of fermionic radius $\hat{R} = \sqrt{2}$ with the following boundary conditions:

$$\tilde{\varphi}_L^{(k)}(z+1) = \tilde{\varphi}_L^{(k)}(z) + 2\pi \frac{kl}{N}, \quad (3.70a)$$

$$\tilde{\varphi}_L^{(k)}(z+\tau) = \tilde{\varphi}_L^{(k)}(z) + 2\pi \frac{kl'}{N}, \quad (3.70b)$$

$$\tilde{\varphi}_R^{(k)}(\bar{z}+1) = \tilde{\varphi}_R^{(k)}(\bar{z}) + 2\pi \frac{kl}{N}, \quad (3.70c)$$

$$\tilde{\varphi}_R^{(k)}(\bar{z}+\bar{\tau}) = \tilde{\varphi}_R^{(k)}(\bar{z}) + 2\pi \frac{kl'}{N}. \quad (3.70d)$$

The details of various computations in such theories are presented in Appendix A.2. The results can be obtained from Sec. A.2.3 by setting $\eta = kl/N, \eta' = kl'/N$:

$$Z_{(l,l')}(\tau, \bar{\tau}) = \prod_{k=0}^{N-1} Z_{(l,l';k)}(\tau, \bar{\tau}), \quad (3.71)$$

$$Z_{(l,l';k)}(\tau, \bar{\tau}) = \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} e^{-2\pi\tau_2 \left(\frac{kl}{N}\right)^2} \left| \frac{\theta_a \left(\frac{kl}{N}\tau - \frac{kl'}{N} \middle| \tau \right)}{\eta(\tau)} \right|^2, \quad (3.72)$$

and hence

$$Z_{\text{Cyc}^N \mathcal{D}}(\tau, \bar{\tau}) = \frac{1}{N} \sum_{l,l'=0}^{N-1} \prod_{k=0}^{N-1} \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} e^{-2\pi\tau_2 \left(\frac{kl}{N}\right)^2} \left| \frac{\theta_a \left(\frac{kl}{N}\tau - \frac{kl'}{N} \middle| \tau \right)}{\eta(\tau)} \right|^2, \quad (3.73)$$

follows.

⁴For $\bar{\psi}$ we just take the complex conjugate,

$$\bar{\tilde{\psi}}_{L,R}^{(k)} = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{2\pi i \frac{kj}{N}} \bar{\psi}_{L,R}^j, \quad (k = 0, 1, \dots, N-1), \quad (3.66)$$

$$g : \bar{\tilde{\psi}}_{L,R}^{(k)} \rightarrow e^{-2\pi i \frac{k}{N}} \bar{\tilde{\psi}}_{L,R}^{(k)}, \quad (3.67)$$

which is a convenient convention.

Restricting to the NS sector of the seed CFT, the corresponding torus partition function Z^{NS} is obtained by taking only the θ_3 contribution above:

$$Z_{(l,l';k)}^{\text{NS}}(\tau, \bar{\tau}) = e^{-2\pi\tau_2(\frac{kl}{N})^2} \left| \frac{\theta_3\left(\frac{kl}{N}\tau - \frac{kl'}{N}\middle|\tau\right)}{\eta(\tau)} \right|^2, \quad (3.74)$$

$$Z_{(l,l')}^{\text{NS}}(\tau, \bar{\tau}) = \prod_{k=0}^{N-1} e^{-2\pi\tau_2(\frac{kl}{N})^2} \left| \frac{\theta_3\left(\frac{kl}{N}\tau - \frac{kl'}{N}\middle|\tau\right)}{\eta(\tau)} \right|^2, \quad (3.75)$$

$$Z_{\text{Cyc}^N\mathcal{D}}^{\text{NS}}(\tau, \bar{\tau}) = \frac{1}{N} \sum_{l,l'=0}^{N-1} \prod_{k=0}^{N-1} e^{-2\pi\tau_2(\frac{kl}{N})^2} \left| \frac{\theta_3\left(\frac{kl}{N}\tau - \frac{kl'}{N}\middle|\tau\right)}{\eta(\tau)} \right|^2, \quad (3.76)$$

Since we are in the NS sector, modular invariance under the S -transformation is expected, and it is easy to confirm that this indeed holds.

By focusing on the modular transformation properties of the above torus partition function, it can be related to the representation (3.18) in terms of the square-free Hecke operator:

$$\begin{aligned} & \frac{\phi(N/d)}{d} T_{\frac{N}{d}}^{sf} \left(Z_{\mathcal{D}}(\tau, \bar{\tau})^d \right) \\ &= \frac{1}{N} \sum_{\substack{l,l'=0 \\ \gcd(l,l')=d}}^{N-1} \prod_{k=0}^{N-1} \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} e^{-2\pi\tau_2(\frac{kl}{N})^2} \left| \frac{\theta_a\left(\frac{kl}{N}\tau - \frac{kl'}{N}\middle|\tau\right)}{\eta(\tau)} \right|^2. \end{aligned} \quad (3.77)$$

3.3.2. Replica method and Rényi entropy

We now construct replica-twist operators and apply the replica method to compute Rényi entropies in the cyclic orbifold CFT of the Dirac fermion.

As will become clear in the course of the computation, there is a subtlety when applying the replica method to the full orbifold CFT. The orbifold-twist boundary conditions do not have to be identical on each replica sheet; in fact, one must sum over contributions with different orbifold-twist boundary conditions on different replica sheets. This is not specific to the Dirac-fermion cyclic orbifold CFT but is expected to hold for general orbifold CFTs.

Discrete Fourier transform and replica-twist operators

Consider $\text{Cyc}^N\mathcal{D}$ with n -replica copy. The fields of interest are

$$\psi_{L,[a]}^j(z), \quad \bar{\psi}_{L,[a]}^j(z), \quad \psi_{R,[a]}^j(\bar{z}), \quad \bar{\psi}_{R,[a]}^j(\bar{z}), \quad (3.78)$$

where $j = 1, 2, \dots, N$ labels which copy of $\mathcal{D}^{\otimes N}$ the field belongs to, and $a = 1, 2, \dots, n$ labels which replica sheet the field lives on. As in the analysis of a single Dirac fermion, we wish to write down the replica-twist operators and compute $\text{Tr}_A[\rho_A^n]$ from their correlation functions.

To translate shifts of the indices into phase rotations, we perform discrete Fourier transforms in both a and j . The discrete Fourier transform over a is carried out in the same way as in eqs. (3.33), (3.39), and (3.41), while that over j is done as in eqs. (3.65) and (3.66). The transformed fields are

$$\tilde{\psi}_{L,p}^{(k)}(z), \quad \bar{\tilde{\psi}}_{L,p}^{(k)}(z), \quad \tilde{\psi}_{R,p}^{(k)}(\bar{z}), \quad \bar{\tilde{\psi}}_{R,p}^{(k)}(\bar{z}), \quad (3.79)$$

where $k = 0, 1, 2, \dots, N-1$ and $p = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$. The phase transformations are as follows. When going around a replica-twist operator $\sigma(x_0, \bar{x}_0)$,

$$\tilde{\psi}_{L,p}^{(k)}(x_0 + \epsilon e^{2\pi i}) = e^{2\pi i \frac{p}{n}} \tilde{\psi}_{L,p}^{(k)}(x_0 + \epsilon), \quad (3.80a)$$

$$\bar{\tilde{\psi}}_{L,p}^{(k)}(x_0 + \epsilon e^{2\pi i}) = e^{-2\pi i \frac{p}{n}} \bar{\tilde{\psi}}_{L,p}^{(k)}(x_0 + \epsilon), \quad (3.80b)$$

$$\tilde{\psi}_{R,p}^{(k)}(\bar{x}_0 + \epsilon e^{-2\pi i}) = e^{2\pi i \frac{p}{n}} \tilde{\psi}_{R,p}^{(k)}(\bar{x}_0 + \epsilon), \quad (3.80c)$$

$$\bar{\tilde{\psi}}_{R,p}^{(k)}(\bar{x}_0 + \epsilon e^{-2\pi i}) = e^{-2\pi i \frac{p}{n}} \bar{\tilde{\psi}}_{R,p}^{(k)}(\bar{x}_0 + \epsilon), \quad (3.80d)$$

and the action of the generator $\mathbb{Z}_N$ on these fields is

$$g : \tilde{\psi}_{L,p}^{(k)} \rightarrow e^{2\pi i \frac{k}{N}} \tilde{\psi}_{L,p}^{(k)}, \quad (3.81a)$$

$$g : \bar{\tilde{\psi}}_{L,p}^{(k)} \rightarrow e^{-2\pi i \frac{k}{N}} \bar{\tilde{\psi}}_{L,p}^{(k)}, \quad (3.81b)$$

$$g : \tilde{\psi}_{R,p}^{(k)} \rightarrow e^{2\pi i \frac{k}{N}} \tilde{\psi}_{R,p}^{(k)}, \quad (3.81c)$$

$$g : \bar{\tilde{\psi}}_{R,p}^{(k)} \rightarrow e^{-2\pi i \frac{k}{N}} \bar{\tilde{\psi}}_{R,p}^{(k)}, \quad (3.81d)$$

respectively.

We then bosonize the $N \times n$ fermionic fields obtained above:

$$\tilde{\psi}_{L,p}^{(k)}(z) = e^{i\tilde{\varphi}_{L,p}^{(k)}(z)}, \quad \tilde{\psi}_{R,p}^{(k)}(\bar{z}) = e^{i\tilde{\varphi}_{R,p}^{(k)}(\bar{z})}. \quad (3.82)$$

As in Sec. 3.2.1, we can now construct replica-twist operators which reproduce the

boundary conditions (3.80):

$$\sigma(z, \bar{z}) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \prod_{k=0}^{N-1} e^{+i\frac{p}{n}(\tilde{\varphi}_{L,p}^{(k)}(z) - \tilde{\varphi}_{R,p}^{(k)}(\bar{z}))}, \quad (3.83a)$$

$$\bar{\sigma}(z, \bar{z}) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \prod_{k=0}^{N-1} e^{-i\frac{p}{n}(\tilde{\varphi}_{L,p}^{(k)}(z) - \tilde{\varphi}_{R,p}^{(k)}(\bar{z}))}. \quad (3.83b)$$

Furthermore, for the compact scalars $\tilde{\varphi}_{L,p}^{(k)}(z), \tilde{\varphi}_{R,p}^{(k)}(\bar{z})$ introduced above with fermionic radius, the action of g in eq. (3.81) is translated into

$$g : \tilde{\varphi}_{L,p}^{(k)} \rightarrow \tilde{\varphi}_{L,p}^{(k)} + 2\pi \frac{k}{N}, \quad (3.84a)$$

$$g : \tilde{\varphi}_{R,p}^{(k)} \rightarrow \tilde{\varphi}_{R,p}^{(k)} + 2\pi \frac{k}{N}. \quad (3.84b)$$

Note that the replica-twist operators constructed in this way are invariant under the action of g .

Rényi entropy in the (l, l') -sector

We now use the replica-twist operators constructed above to compute the Rényi entropy in a given (l, l') -sector.

First, we compute $\text{Tr}_A^{(l, l')}[\rho_A^n]$, which is $\text{Tr}_A[\rho_A^n]$ restricted to the (l, l') -sector. We take the endpoints of the subsystem A to be $x_1 = L, x_2 = 0$. Using eq. A.36 with $m = \frac{p}{n}$ and keeping only the θ_3 contribution, we obtain

$$\begin{aligned} \text{Tr}_A^{(l, l')}[\rho_A^n] &= \langle \sigma(L) \bar{\sigma}(0) \rangle_{(l, l')} \\ &= \prod_{p=-(n-1)/2}^{(n-1)/2} \prod_{k=0}^{N-1} \left\langle e^{+i\frac{p}{n}(\tilde{\varphi}_{L,p}^{(k)}(L) - \tilde{\varphi}_{R,p}^{(k)}(L))} e^{-i\frac{p}{n}(\tilde{\varphi}_{L,p}^{(k)}(0) - \tilde{\varphi}_{R,p}^{(k)}(0))} \right\rangle_{(\eta=\frac{kl}{N}, \eta'=\frac{kl'}{N})} \\ &= \frac{1}{(Z_{(l, l')}^{\text{NS}})^n} \prod_{p=-(n-1)/2}^{(n-1)/2} \prod_{k=0}^{N-1} \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{4h_{0,p/n}} \frac{e^{-2\pi \frac{k^2 l^2}{N^2} \tau_2}}{|\eta(\tau)^2|} \cdot \left| \theta_3 \left(\frac{kl}{N} \tau - \frac{kl'}{N} + \frac{p}{n} L |\tau \right) \right|^2, \end{aligned} \quad (3.85)$$

where $h_{0,p/n} = \frac{p^2}{2n^2}$.

This yields the n -th Rényi entropy in the (l, l') -sector:

$$\begin{aligned} S_{A(l, l')}^{(n)}(L) &= \frac{1}{1-n} \log \text{Tr}_A^{(l, l')}[\rho_A^n] \\ &= S_{A(l, l')}^{(n)1}(L) + S_{A(l, l')}^{(n)2}(L) \end{aligned} \quad (3.86a)$$

where $S_{A(l,l')}^{(n)1}(L)$ is the contribution from θ_1 , and $S_{A(l,l')}^{(n)2}(L)$ is the contribution from the remaining part:

$$S_{A(l,l')}^{(n)1}(L) = \frac{N}{6} \left(1 + \frac{1}{n}\right) \log \left| \frac{\theta_1(L|\tau)}{\partial\theta_1(0|\tau)} \right|, \quad (3.86b)$$

$$\begin{aligned} S_{A(l,l')}^{(n)2}(L) &= \frac{1}{1-n} \log \prod_{p=-(n-1)/2}^{(n-1)/2} \prod_{k=0}^{N-1} \frac{e^{-2\pi \frac{k^2 l^2}{N^2} \tau_2}}{|\eta(\tau)^2|} \cdot \left| \theta_3 \left(\frac{kl}{N} \tau - \frac{kl'}{N} + \frac{p}{n} L | \tau \right) \right|^2 \\ &\quad - \frac{1}{1-n} \log \left(Z_{(l,l')}^{\text{NS}} \right)^n. \end{aligned} \quad (3.86c)$$

We now compute the thermal Rényi entropy and see how the relation (3.29) is modified. The contribution from $S_{A(l,l')}^{(n)1}(L)$ cancels, and we find

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \left[S_{A(l,l')}^{(n)}(1 - \sigma) - S_{A(l,l')}^{(n)}(\sigma) \right] &= S_{A(l,l')}^{(n)2}(1) - S_{A(l,l')}^{(n)2}(0) \\ &= \frac{1}{1-n} \log \prod_{p=-(n-1)/2}^{(n-1)/2} \prod_{k=0}^{N-1} \frac{e^{-2\pi \frac{k^2 l^2}{N^2} \tau_2}}{|\eta(\tau)^2|} \cdot \left| \theta_3 \left(\frac{kl}{N} \tau - \frac{kl'}{N} + \frac{p}{n} | \tau \right) \right|^2 \\ &\quad - \frac{1}{1-n} \log \left(Z_{(l,l')}^{\text{NS}} \right)^n, \end{aligned} \quad (3.87)$$

which can be compactly written using the n -tuple-angle formula (A.18),

$$\frac{\theta_3(\alpha | n\tau)}{\eta(n\tau)} = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\theta_3 \left(\frac{\alpha}{n} + \frac{p}{n} | \tau \right)}{\eta(\tau)}$$

as

$$\lim_{\sigma \rightarrow 0} \left[S_{A(l,l')}^{(n)}(1 - \sigma) - S_{A(l,l')}^{(n)}(\sigma) \right] = \frac{1}{1-n} \log \frac{Z_{(l,nl')}^{\text{NS}}(n\tau, n\bar{\tau})}{\left(Z_{(l,l')}^{\text{NS}} \right)^n}. \quad (3.88)$$

In the numerator, the orbifold twist along the τ -direction is multiplied by n , which is geometrically natural. If we could correctly sum over the orbifold twists and compute the full $\text{Tr}_A[\rho_A^n]$, then the ensuing Rényi entropies would be expected to satisfy the relation (3.29).

Towards the complete computation of Rényi entropy

When we consider the n -th Rényi entropy on the torus, the covering surface becomes a genus- n surface. In particular, in an orbifold CFT setup like the present one, we need to take sum of different orbifold-twist boundary conditions on each replica-copy handle. Indeed, the above computation in fact corresponds to the contribution in which the (l, l') boundary condition is imposed on *all* replica-copy handles. Therefore,

if we proceed with the computation in a naive way, we find that it does not satisfy the consistency condition (3.29):

$$\begin{aligned}
 S_A^{(n)}(L) &= \frac{1}{1-n} \log \text{Tr}_A[\rho_A^n] \\
 &\stackrel{?}{=} \frac{1}{1-n} \log \frac{\frac{1}{N} \sum_{l,l'=0}^{N-1} \left(Z_{(l,l')}^{\text{NS}} \right)^n \cdot \text{Tr}_A^{(l,l')}[\rho_A^n]}{\left(Z_{\text{Cyc}^N \mathcal{D}}^{\text{NS}} \right)^n} \\
 &= \frac{1}{1-n} \log \frac{\frac{1}{N} \sum_{l,l'=0}^{N-1} \prod_{p=-(n-1)/2}^{(n-1)/2} \prod_{k=0}^{N-1} \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{4h_{0,p/n}} \frac{e^{-2\pi \frac{k^2 l^2}{N^2} \tau_2}}{|\eta(\tau)^2|} \cdot \left| \theta_3 \left(\frac{kl}{N} \tau - \frac{kl'}{N} + \frac{p}{n} L|\tau \right) \right|^2}{\left(Z_{\text{Cyc}^N \mathcal{D}}^{\text{NS}} \right)^n} \\
 &= \frac{N}{6} \left(1 + \frac{1}{n} \right) \log \left| \frac{\theta_1(L|\tau)}{\partial \theta_1(0|\tau)} \right| \\
 &\quad + \frac{1}{1-n} \log \frac{\frac{1}{N} \sum_{l,l'=0}^{N-1} \prod_{p=-(n-1)/2}^{(n-1)/2} \prod_{k=0}^{N-1} \frac{e^{-2\pi \frac{k^2 l^2}{N^2} \tau_2}}{|\eta(\tau)^2|} \cdot \left| \theta_3 \left(\frac{kl}{N} \tau - \frac{kl'}{N} + \frac{p}{n} L|\tau \right) \right|^2}{\left(Z_{\text{Cyc}^N \mathcal{D}}^{\text{NS}} \right)^n}.
 \end{aligned} \tag{3.89}$$

Substituting this into the consistency condition (3.29), we obtain

$$\begin{aligned}
 &\lim_{\sigma \rightarrow 0} \left[S_A^{(n)}(1-\sigma) - S_A^{(n)}(\sigma) \right] \\
 &\stackrel{?}{=} \frac{1}{1-n} \log \left[\frac{1}{N} \sum_{l,l'=0}^{N-1} \prod_{p=-(n-1)/2}^{(n-1)/2} \prod_{k=0}^{N-1} \frac{e^{-2\pi \frac{k^2 l^2}{N^2} \tau_2}}{|\eta(\tau)^2|} \cdot \left| \theta_3 \left(-\frac{k}{N} l' + \frac{k}{N} l\tau + \frac{p}{n} |\tau \right) \right|^2 \right] \\
 &\quad - \frac{1}{1-n} \log \left[\frac{1}{N} \sum_{l,l'=0}^{N-1} \prod_{p=-(n-1)/2}^{(n-1)/2} \prod_{k=0}^{N-1} \frac{e^{-2\pi \frac{k^2 l^2}{N^2} \tau_2}}{|\eta(\tau)^2|} \cdot \left| \theta_3 \left(-\frac{k}{N} l' + \frac{k}{N} l\tau |\tau \right) \right|^2 \right] \\
 &= \frac{1}{1-n} \log \left[\frac{\frac{1}{N} \sum_{l,l'=0}^{N-1} Z_{(l,nl')}^{\text{NS}}(n\tau, n\bar{\tau})}{\frac{1}{N} \sum_{l,l'=0}^{N-1} \left(Z_{(l,l')}^{\text{NS}}(\tau) \right)^n} \right].
 \end{aligned} \tag{3.90}$$

which does *not* agree with the expected relation

$$\lim_{\sigma \rightarrow 0} \left[S_A^{(n)}(1-\sigma) - S_A^{(n)}(\sigma) \right] = \frac{1}{1-n} \log \left[\frac{Z_{\text{Cyc}^N \mathcal{D}}^{\text{NS}}(n\tau, n\bar{\tau})}{\left(Z_{\text{Cyc}^N \mathcal{D}}^{\text{NS}}(\tau, \bar{\tau}) \right)^n} \right]. \tag{3.91}$$

We therefore refer to this *diagonal* contribution to the Rényi entropy computed by (3.89) as the *diagonal Rényi entropy*, and denote it by $S_{A,\text{diag}}^{(n)}(L)$. In terms of the

diagonal Rényi entropy, we can summarize our results as

$$S_{A,\text{diag}}^{(n)}(L) = \frac{1}{1-n} \log \frac{\frac{1}{N} \sum_{l,l'=0}^{N-1} \left(Z_{(l,l')}^{\text{NS}} \right)^n \cdot \text{Tr}_A^{(l,l')}[\rho_A^n]}{\left(Z_{\text{Cyc}^N \mathcal{D}}^{\text{NS}} \right)^n}, \quad (3.92)$$

$$\lim_{\sigma \rightarrow 0} \left[S_{A,\text{diag}}^{(n)}(1-\sigma) - S_{A,\text{diag}}^{(n)}(\sigma) \right] = \frac{1}{1-n} \log \left[\frac{\frac{1}{N} \sum_{l,l'=0}^{N-1} Z_{(l,nl')}^{\text{NS}}(n\tau, n\bar{\tau})}{\frac{1}{N} \sum_{l,l'=0}^{N-1} \left(Z_{(l,l')}^{\text{NS}}(\tau) \right)^n} \right], \quad (3.93)$$

respectively.

For general N and n , carrying out the replica computation including non-diagonal contributions is left as a future problem. However, in the next subsection we perform the complete replica computation, including non-diagonal contributions, for the special case $N = n = 2$.

3.3.3. Second Rényi entropy in $N = 2$ orbifold

In this section we present the complete computation of the second Rényi entropy in the orbifold CFT of a Dirac fermion with $\mathbb{Z}_2 = S_2$. We must sum over partition functions with orbifold boundary conditions as depicted schematically in Fig. 3.1. We label the orbifold boundary conditions on the first replica sheet by (l_1, l'_1) and on the second replica sheet by (l_2, l'_2) , and denote the corresponding partition function by $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}}(L)$. Computational details are given in Appendix A.3.

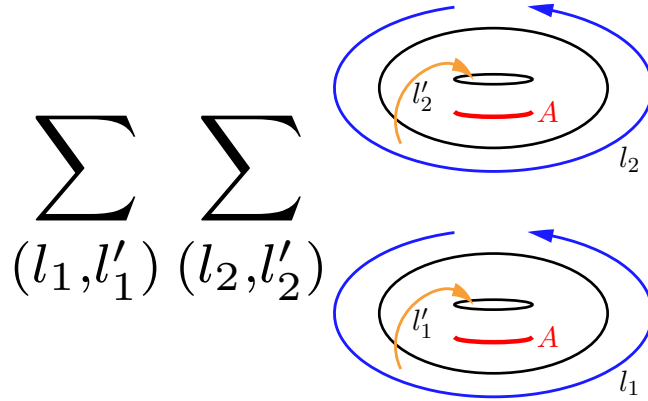


Figure 3.1.: Schematic picture of the computation of the second Rényi entropy for subsystem A in a cyclic orbifold CFT on the torus. One must sum over all possible orbifold boundary conditions (l_i, l'_i) on each replica sheet.

We have $\text{Tr}_A[\rho_A^2]$ as

$$\begin{aligned} & \left(Z_{\text{Cyc}^2\mathcal{D}}^{\text{NS}} \right)^2 \cdot \text{Tr}_A[\rho_A^2] \\ &= \left| \frac{\partial\theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{\frac{1}{4}} \frac{1}{|\eta(\tau)|^4} \left| \theta_3\left(\frac{1}{4}L|\tau\right) \right|^2 \left| \theta_3\left(-\frac{1}{4}L|\tau\right) \right|^2 \cdot \left(\frac{1}{2}\right)^2 \sum_{l_1, l'_1=0}^1 \sum_{l_2, l'_2=0}^1 Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L), \end{aligned}$$

where $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L)$ satisfying $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L) = Z_{(l_2, l'_2)(l_1, l'_1)}^{\text{NS}, k=1}(L)$ is given by

$$Z_{(l, l')(l, l')}^{\text{NS}, k=1}(L) = \left| \frac{\partial\theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{\frac{1}{4}} \prod_{p=-\frac{1}{2}, \frac{1}{2}} e^{-2\pi \frac{l^2}{2^2} \tau_2} \left| \theta_3\left(\frac{l}{2}\tau - \frac{l'}{2} + \frac{p}{2}L|\tau\right) \right|^2, \quad (3.94a)$$

$$Z_{(0,0)(1,0)}^{\text{NS}, k=1}(L) = \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{L}{2}|\frac{\tau}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{1}{2}|\frac{\tau}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{L+1}{2}|\frac{\tau}{2})} \right|^{-\frac{1}{4}} \frac{|\theta_2(\frac{\tau}{2})|^2}{|\eta(\frac{\tau}{2})|^2}, \quad (3.94b)$$

$$Z_{(0,1)(1,1)}^{\text{NS}, k=1}(L) = 0, \quad (3.94c)$$

$$Z_{(0,0)(0,1)}^{\text{NS}, k=1}(L) = \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(L|2\tau)} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(\tau|2\tau)} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(L+\tau|2\tau)} \right|^{-\frac{1}{4}} \frac{|\theta_4(2\tau)|^2}{|\eta(2\tau)|^2}, \quad (3.94d)$$

$$Z_{(1,0)(1,1)}^{\text{NS}, k=1}(L) = 0, \quad (3.94e)$$

$$Z_{(0,0)(1,1)}^{\text{NS}, k=1}(L) = 0, \quad (3.94f)$$

$$Z_{(0,1)(1,0)}^{\text{NS}, k=1}(L) = \left| \frac{\partial\theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{L}{2}|\frac{\tau+1}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{1}{2}|\frac{\tau+1}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{L+1}{2}|\frac{\tau+1}{2})} \right|^{-\frac{1}{4}} \frac{|\theta_2(\frac{\tau+1}{2})|^2}{|\eta(\frac{\tau+1}{2})|^2}. \quad (3.94g)$$

We can confirm that the result satisfy the consistency (3.29).

3.4. Rényi entropy in Dirac fermion symmetric orbifolds on torus

We now analyze the partition function of the symmetric orbifold CFT of Dirac fermions, and use the resulting structure of orbifold-twist sectors to compute diagonal Rényi entropies⁵.

3.4.1. Twisted sector from cyclic orbifolds

In this subsection we apply the analysis of Sec. 3.1.2 to $\text{Sym}^N \mathcal{D}$. As stated in eq. (3.76), the partition function of the i -th cyclic orbifold of Dirac fermions (restricted to the

⁵A complete computation of Rényi entropies including non-diagonal contributions is left for future work. For $N = 2$, the cyclic and symmetric orbifold CFTs are equivalent, so the case $N = n = 2$ has already been analyzed in the previous section.

NS sector) is

$$Z_{\text{Cyc}^i \mathcal{D}}^{\text{NS}}(\tau, \bar{\tau}) = \frac{1}{i} \sum_{l, l'=1}^i \prod_{k=0}^{i-1} \frac{e^{i\pi \frac{k^2 l^2}{i^2} (\tau - \bar{\tau})}}{|\eta(\tau)|^2} \left| \theta_3 \left(\frac{kl'}{i} + \frac{kl}{i} \tau | \tau \right) \right|^2. \quad (3.95)$$

By comparing this with the representation (3.18) in terms of the square-free Hecke operator, we find the correspondence

$$\frac{\phi(i/d)}{d} \left(T_{\frac{i}{d}}^{sf} \left(Z_{\mathcal{D}}(\tau, \bar{\tau})^d \right) \right)^{\text{NS}} = \frac{1}{i} \sum_{\substack{l, l'=1 \\ \gcd(i, l, l')=d}}^i \prod_{k=0}^{i-1} \frac{e^{i\pi \frac{k^2 l^2}{i^2} (\tau - \bar{\tau})}}{|\eta(\tau)|^2} \left| \theta_3 \left(\frac{kl'}{i} + \frac{kl}{i} \tau | \tau \right) \right|^2. \quad (3.96)$$

In particular, for $d = 1$ we obtain

$$\left(T_i^{sf} Z_{\mathcal{D}}(\tau, \bar{\tau}) \right)^{\text{NS}} = \frac{1}{i\phi(i)} \sum_{\substack{l, l'=1 \\ \gcd(i, l, l')=1}}^i \prod_{k=0}^{i-1} \frac{e^{i\pi \frac{k^2 l^2}{i^2} (\tau - \bar{\tau})}}{|\eta(\tau)|^2} \left| \theta_3 \left(\frac{kl'}{i} + \frac{kl}{i} \tau | \tau \right) \right|^2. \quad (3.97)$$

Using this correspondence, we will now write the NS-sector partition function of the Dirac-fermion symmetric orbifold CFT in terms of Jacobi theta functions. The partition function of the symmetric orbifold CFT of degree N can be expressed in terms of the usual Hecke operators as

$$Z_{\text{Sym}^N \mathcal{D}}(\tau) = \sum_{\substack{\text{partition of } N \\ (\dots, i, \dots)}} \prod_{i=1}^N \frac{1}{(\#i)!} \left(T_i Z(\tau) \right)^{\#i},$$

as in eq. (2.29), where the sum runs over partitions of N and $\#i$ denotes the number of occurrences of i in the partition. The usual Hecke operator can be reconstructed from the square-free Hecke operators as

$$\begin{aligned} T_i Z(\tau) &= T_i^{sf} Z(\tau) + \sum_{\substack{a \in \mathbb{Z}_{>1} \\ a^2 | i}} \frac{1}{a^2} T_{i/a^2}^{sf} Z(\tau) \\ &= \sum_{\substack{a \in \mathbb{Z}_{>0} \\ a^2 | i}} \frac{1}{a^2} T_{i/a^2}^{sf} Z(\tau), \end{aligned} \quad (3.98)$$

so to compute $(T_i Z_{\mathcal{D}}(\tau, \bar{\tau}))^{\text{NS}}$ it is enough to evaluate $(T_{i/a^2}^{sf} Z_{\mathcal{D}}(\tau, \bar{\tau}))^{\text{NS}}$.

Case $a = 1$. For $a = 1$ we can use eq. (3.97) directly:

$$\left(T_i^{sf} Z_{\mathcal{D}}(\tau, \bar{\tau})\right)^{\text{NS}} = \frac{1}{i\phi(i)} \sum_{\substack{l, l'=1, \dots, i \\ \gcd(l, l', i)=1}} \prod_{k=0}^{i-1} \frac{e^{i\pi \frac{k^2 l^2}{i^2} (\tau - \bar{\tau})}}{|\eta(\tau)|^2} \left| \theta_3 \left(\frac{kl'}{i} + \frac{kl}{i} \tau | \tau \right) \right|^2, \quad (3.99)$$

as before.

Case $a > 1$. When i has a nontrivial square factor, we must add the contribution $T_{i/a^2} Z(\tau)$ to reconstruct $T_i Z(\tau)$. This can be computed by considering the cyclic orbifold CFT of degree i/a^2 . Replacing i by i/a^2 in eq. (3.97), we obtain

$$\left(T_{i/a^2}^{sf} Z_{\mathcal{D}}(\tau, \bar{\tau})\right)^{\text{NS}} = \frac{a^2}{i\phi(i/a^2)} \sum_{\substack{l, l'=1, \dots, i/a^2 \\ \gcd(l, l', i/a^2)=1}} \prod_{k=0}^{i/a^2-1} \frac{e^{i\pi \frac{a^2 k^2 l^2}{i^2} (\tau - \bar{\tau})}}{|\eta(\tau)|^2} \left| \theta_3 \left(\frac{a^2 kl'}{i} + \frac{a^2 kl}{i} \tau | \tau \right) \right|^2. \quad (3.100)$$

At first sight, this expression seems to involve only i/a^2 fermions (or equivalently, i/a^2 Jacobi theta functions). However, using eq. A.20, we can interpret this product as describing i fermions.⁶ In this case, eq. (A.20) becomes

$$\begin{aligned} & e^{i\pi \frac{a^2 k^2 l^2}{i^2} (\tau - \bar{\tau})} \left| \frac{\theta_3 \left(\frac{a^2 kl'}{i} + \frac{a^2 kl}{i} \tau | \tau \right)}{\eta(\tau)} \right|^2 \\ &= \prod_{\kappa=-\frac{a-1}{2}}^{\frac{a-1}{2}} \prod_{\lambda=-\frac{a-1}{2}}^{\frac{a-1}{2}} e^{i\pi \left(\frac{\lambda}{a} + \frac{a\kappa l}{i} \right)^2 (\tau - \bar{\tau})} \left| \frac{\theta_3 \left(\frac{\kappa}{a} + \frac{a\kappa l'}{i} + \left(\frac{\lambda}{a} + \frac{a\kappa l}{i} \right) \tau | \tau \right)}{\eta(\tau)} \right|^2, \end{aligned}$$

and thus

$$\begin{aligned} & \left(T_{i/a^2}^{sf} Z_{\mathcal{D}}(\tau, \bar{\tau})\right)^{\text{NS}} \\ &= \frac{a^2}{i\phi(i/a^2)} \sum_{\substack{l, l'=1, \dots, i/a^2 \\ \gcd(l, l', i/a^2)=1}} \prod_{k=0}^{i/a^2-1} \prod_{\kappa=-\frac{a-1}{2}}^{\frac{a-1}{2}} \prod_{\lambda=-\frac{a-1}{2}}^{\frac{a-1}{2}} e^{i\pi \left(\frac{\lambda}{a} + \frac{a\kappa l}{i} \right)^2 (\tau - \bar{\tau})} \left| \frac{\theta_3 \left(\frac{\kappa}{a} + \frac{a\kappa l'}{i} + \left(\frac{\lambda}{a} + \frac{a\kappa l}{i} \right) \tau | \tau \right)}{\eta(\tau)} \right|^2. \end{aligned} \quad (3.101)$$

Setting $a = 1$ reproduces the previous result.

⁶Of course, for the purpose of computing the partition function alone, this reinterpretation is not strictly necessary, as we can treat each Jacobi theta function independently. Here we introduce this interpretation for use in the next subsection.

Combining eqs. (3.99) and (3.101), we find

$$(T_i Z_{\mathcal{D}}(\tau, \bar{\tau}))^{\text{NS}} = \frac{1}{i} \sum_{\substack{a \in \mathbb{Z}_{>0} \\ a^2 | i}} \frac{1}{\phi(i/a^2)} \sum_{\substack{l, l'=1, \dots, i/a^2 \\ \gcd(l, l', i/a^2)=1}} \prod_{k=0}^{i/a^2-1} \prod_{\kappa=-\frac{a-1}{2}}^{\frac{a-1}{2}} \prod_{\lambda=-\frac{a-1}{2}}^{\frac{a-1}{2}} e^{i\pi \left(\frac{\lambda}{a} + \frac{akl}{i}\right)^2 (\tau - \bar{\tau})} \left| \frac{\theta_3 \left(\frac{\kappa}{a} + \frac{akl'}{i} + \left(\frac{\lambda}{a} + \frac{akl}{i} \right) \tau \middle| \tau \right)}{\eta(\tau)} \right|^2. \quad (3.102)$$

This expression involves i twisted Dirac fermions. Inserting T_i^{NS} into eq. (2.29) and summing over partitions of N yields the partition function of the symmetric orbifold CFT.

3.4.2. Diagonal contribution to the Rényi entropy

The above results allow us to compute the diagonal Rényi entropy in the NS sector for the symmetric orbifold CFT. As we have seen several times, when we apply the replica method using bosonization to twisted Dirac fermions, the torus partition function transforms as

$$\begin{aligned} & e^{i\pi\beta^2(\tau-\bar{\tau})} \left| \frac{\theta_3(\alpha + \beta\tau|\tau)}{\eta(\tau)} \right|^2 \\ \rightarrow & \prod_{p=-(n-1)/2}^{(n-1)/2} e^{i\pi\beta^2(\tau-\bar{\tau})} \left| \frac{\partial\theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{4h_{0,p/n}} \cdot \left| \frac{\theta_3\left(\alpha + \beta\tau + \frac{p}{n}L|\tau\right)}{|\eta(\tau)^2|} \right|^2, \end{aligned} \quad (3.103)$$

where $h_{0,p/n} = \frac{p^2}{2n^2}$.

We are now ready to perform the computation. As in the previous sections, we can factor out the contribution from θ_1 :

$$\begin{aligned} \left(Z_{\text{Sym}^N \mathcal{D}}^{\text{NS}} \right)^n \langle \sigma(L) \bar{\sigma}(0) \rangle \Big|_{\text{diagonal part}} &= \left(\prod_{p=-(n-1)/2}^{(n-1)/2} \left| \frac{\partial\theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{4h_{0,p/n}} \right)^N \\ &\times \left(\sum_{\text{partition of } N} \prod_{i=1}^N \frac{1}{(\#i)!} \langle \langle \sigma(L) \bar{\sigma}(0) \rangle \rangle_i^{\#i} \right) \\ &= \left| \frac{\partial\theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{\frac{N}{6} \left(n - \frac{1}{n} \right)} \cdot \left(\sum_{\text{partition of } N} \prod_{i=1}^N \frac{1}{(\#i)!} \langle \langle \sigma(L) \bar{\sigma}(0) \rangle \rangle_i^{\#i} \right), \end{aligned} \quad (3.104)$$

where $\langle\langle\sigma(L)\bar{\sigma}(0)\rangle\rangle_i$ denotes

$$\begin{aligned} \langle\langle\sigma(L)\bar{\sigma}(0)\rangle\rangle_i = \frac{1}{i} \sum_{\substack{a \in \mathbb{Z} > 0 \\ a^2 | i}} \frac{1}{\phi(i/a^2)} \sum_{\substack{l, l' = 1, \dots, i/a^2 \\ (l, l', i) = 1}} \prod_{k=0}^{i/a^2 - 1} \prod_{\kappa = -\frac{a-1}{2}}^{\frac{a-1}{2}} \prod_{\lambda = -\frac{a-1}{2}}^{\frac{a-1}{2}} \prod_{p = -\frac{n-1}{2}}^{\frac{n-1}{2}} \\ e^{i\pi \left(\frac{\lambda}{a} + \frac{akl}{i}\right)^2 (\tau - \bar{\tau})} \left| \frac{\theta_3 \left(\frac{\kappa}{a} + \frac{akl'}{i} + \left(\frac{\lambda}{a} + \frac{akl}{i} \right) \tau + \frac{p}{n} L | \tau \right)}{\eta(\tau)} \right|^2. \end{aligned} \quad (3.105)$$

This allows us to compute the diagonal Rényi entropy $S_{A,\text{diag}}^{(n)}$:

$$S_{A,\text{diag}}^{(n)}(L) = S_{A,\text{diag}}^{(n)1}(L) + S_{A,\text{diag}}^{(n)2}(L) + S_{A,\text{diag}}^{(n)3}(L), \quad (3.106a)$$

where $S_{A,\text{diag}}^{(n)1}(l)$ denotes the contribution from θ_1 and others denote the remaining contributions:

$$S_{A,\text{diag}}^{(n)1}(L) = \frac{N}{6} \left(1 + \frac{1}{n} \right) \log \left| \frac{\theta_1(L|\tau)}{\partial \theta_1(0|\tau)} \right|, \quad (3.106b)$$

$$S_{A,\text{diag}}^{(n)2}(L) = \frac{1}{1-n} \log \left(\sum_{\text{partition of } N} \prod_{i=1}^N \frac{1}{(\#i)!} \langle\langle\sigma(L)\bar{\sigma}(0)\rangle\rangle_i^{\#i} \right), \quad (3.106c)$$

$$S_{A,\text{diag}}^{(n)3}(L) = \frac{n}{1-n} \log \frac{1}{Z_{\text{Sym}^N \mathcal{D}}^{\text{NS}}}. \quad (3.106d)$$

The first term correctly captures the universal contribution to the entropy, as expected. The other term is a quantity that is highly specific to this theory. Of course, this quantity also does *not* satisfy the consistency condition (3.29).

Outlook.

A caveat is that our present torus computation retains only the diagonal contribution. However, the off-diagonal terms are expected to drop out in the $n \rightarrow 1$ limit relevant for the entanglement entropy, so the diagonal closed-form result may already contain the essential information for the continuation. Precisely for this reason, it is worthwhile to investigate the result (3.106) more carefully and to understand how the diagonal structure reorganizes under analytic continuation. A natural next step is therefore to control the continuation $n \rightarrow 1$ starting from (3.106). Achieving this would make it possible to extract the entanglement entropy and, in particular, to clarify how subleading contributions behave in comparison with semiclassical-holographic expectations (3.4).

4. Spectral form factors in symmetric orbifolds

In this chapter, we adopt a statistical perspective on symmetric orbifolds. A symmetric orbifold theory becomes semiclassical only after turning on a finite amount of twist-two marginal deformations, which are expected to push many excitations above the threshold. Here, by analyzing the spectral form factor, we examine to what extent the undeformed symmetric orbifold matches semiclassical expectations or deviates from them.

A spectral form factor for a CFT $\mathcal{C}$ is defined through its torus partition function $Z_{\mathcal{C}}(\tau, \bar{\tau})$ as

$$\text{SFF}_{\mathcal{C}}(\beta, t) = \frac{Z_{\mathcal{C}}\left(\frac{i(\beta+it)}{2\pi}, -\frac{i(\beta+it)}{2\pi}\right) Z_{\mathcal{C}}\left(\frac{i(\beta-it)}{2\pi}, -\frac{i(\beta-it)}{2\pi}\right)}{\left(Z_{\mathcal{C}}\left(\frac{i\beta}{2\pi}, -\frac{i\beta}{2\pi}\right)\right)^2}. \quad (4.1)$$

We also use the following progressive time average of SFF, following [5]:

$$\overline{\text{SFF}}_{\mathcal{C}}(\beta, t) := \frac{1}{t} \int_{\frac{1}{2}t}^{\frac{3}{2}t} dt' \text{SFF}_{\mathcal{C}}(\beta, t'). \quad (4.2)$$

4.1. Short review on spectral form factors

The spectral form factor (SFF) is a standard probe of late-time spectral correlations in quantum many-body systems and holography, defined from the analytically continued torus partition function (or thermal trace) as

$$\text{SFF}(\beta, t) = \frac{|Z(\beta + it)|^2}{Z(\beta)^2}. \quad (4.3)$$

A schematic expectation for chaotic theories is the dip-ramp-plateau structure shown in Fig. 4.1: after an initial decay, the SFF reaches a minimum around the dip time t_d , then exhibits a ramp-like growth, and finally saturates at the plateau around the plateau time t_p ; see e.g. [78, 79, 80] for detailed discussions; see also [81].

As a canonical example of chaotic spectral statistics, let us recall how random matrix theory (RMT) explains the dip-ramp-plateau shape. We write the thermal partition

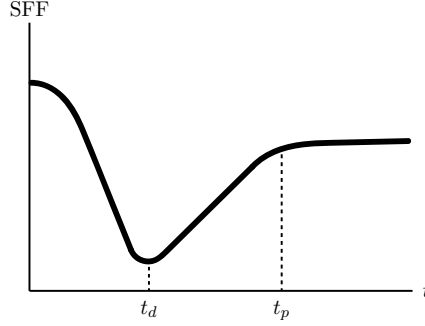


Figure 4.1.: Schematic figure of spectral form factor.

function as

$$Z(\beta) = \sum_i e^{-\beta E_i}. \quad (4.4)$$

This is equivalently expressed by introducing the spectral density

$$\rho(E) = \sum_i \delta(E - E_i), \quad Z(\beta) = \int dE \rho(E) e^{-\beta E}. \quad (4.5)$$

Using this, the averaged SFF can be viewed as (essentially) the Fourier transform of a two-point function of the spectral density:

$$\begin{aligned} \left\langle |Z(\beta + it)|^2 \right\rangle &= \left\langle \sum_{i,j} e^{-\beta(E_i + E_j)} e^{-it(E_i - E_j)} \right\rangle \\ &\simeq \int dE dE' e^{-\beta(E + E')} e^{-it(E - E')} \langle \rho(E) \rho(E') \rangle, \end{aligned} \quad (4.6)$$

where $\langle \dots \rangle$ denotes a suitable averaging (ensemble/energy-window average, possibly supplemented by a mild time average) [78, 79]. Decomposing into disconnected and connected parts,

$$\langle \rho(E) \rho(E') \rangle = \langle \rho(E) \rangle \langle \rho(E') \rangle + \langle \rho(E) \rho(E') \rangle_c, \quad (4.7)$$

the disconnected term controls the early-time decay, while the *connected* term $\langle \rho(E) \rho(E') \rangle_c$ encodes genuine level correlations and is responsible for the ramp.

Emergence of the linear ramp. The connected two-point function $\langle \rho(E)\rho(E') \rangle_c$ in the canonical random-matrix universality classes takes the universal sine-kernel form¹ [78, 82, 80]

$$\langle \rho(E)\rho(E') \rangle_c \propto \frac{\pi}{L} \delta(E - E') - \left(\frac{\sin[L(E - E')]}{L(E - E')} \right)^2, \quad (4.8)$$

where L is the rank (matrix size) of the RMT. Using the standard identity of triangular function

$$\int_{-\infty}^{\infty} ds e^{-2\pi i s t} \left(\frac{\sin(as)}{as} \right)^2 = \begin{cases} \frac{\pi}{|a|} \left(1 - \frac{\pi|t|}{|a|} \right), & |t| \leq \frac{|a|}{\pi}, \\ 0, & |t| \geq \frac{|a|}{\pi}, \end{cases} \quad (4.9)$$

we obtain that the connected contribution (in $\beta \rightarrow 0$ limit) gives rise to a linear ramp-like growth. In a generic theory, whether a linear ramp appears is controlled by the short-distance behavior of the connected two-level correlator $\langle \rho(E)\rho(E') \rangle_c$ in the energy range that dominates the thermal trace at the chosen temperature. If, in that relevant energy band, $\langle \rho(E)\rho(E') \rangle_c$ has essentially a similar functional form to in RMT (up to rescalings by the local mean level spacing), then the theory exhibits an approximately linear ramp, until the corresponding plateau scale is reached. In holographic settings, this logic is often invoked in the following form: in a CFT expected to admit a semiclassical AdS dual, the energy window associated with black-hole microstates (e.g. above the black-hole threshold in $\text{AdS}_3/\text{CFT}_2$) is expected to display increasingly random-matrix-like short-distance spectral correlations after an appropriate averaging procedure [79, 78]. Under this expectation, the corresponding contribution to the SFF should exhibit an approximately linear ramp at late times, prior to the eventual plateau.

Plateau time. Following the viewpoint emphasized in [79], the plateau time is set by the inverse mean level spacing in the energy window (the dominant microcanonical band) that dominates the thermal trace at inverse temperature β . Indeed, writing

$$|Z(\beta + it)|^2 = \sum_{a,b} e^{-\beta(E_a + E_b)} e^{-it(E_a - E_b)}, \quad (4.10)$$

one sees that a long-time average suppresses oscillatory terms with $E_a \neq E_b$, so the late-time saturation is controlled by the discreteness of the spectrum. As a result, the

¹More precisely, the sine-kernel expression written here is the standard result for the unitary class (GUE). However, the orthogonal and symplectic symmetry classes (GOE/GSE) exhibit the same qualitative linear ramp followed by saturation; see e.g. [82, 80].

time scale for reaching the plateau is naturally of order the Heisenberg time,

$$t_p \sim t_H \sim \frac{2\pi}{\Delta E}, \quad (4.11)$$

where ΔE is the mean level spacing in the relevant energy shell. Equivalently, if $\Omega(E)$ denotes the microcanonical density of states, then $\Delta E \sim \Omega(E_\beta)^{-1}$ and hence

$$t_p \sim 2\pi \Omega(E_\beta) \sim \exp(S(E_\beta)), \quad (4.12)$$

with $S(E) := \log \Omega(E)$ and E_β the typical energy at temperature $1/\beta$. Thus, in systems with a large thermal entropy, the plateau time is exponentially large in the entropy, $t_p \sim e^S$. For a 2d CFT on a circle of circumference 2π , modular invariance implies the scale of the thermal entropy as

$$\log Z(\beta) \simeq \frac{\pi^2 c}{3\beta} \quad (\beta \ll 2\pi), \quad S(\beta) = (1 - \beta \partial_\beta) \log Z(\beta) \simeq \frac{2\pi^2 c}{3\beta}. \quad (4.13)$$

Therefore the plateau time is parametrically large,

$$t_p \sim \exp\left(\frac{2\pi^2 c}{3\beta}\right), \quad (4.14)$$

so in practice it is often far beyond direct numerical access in large- c (including holographic) regimes. For symmetric orbifolds one has $c = N c_{\text{seed}}$, so the same estimate suggests an even more rapid growth of t_p with N .

Mini-recurrences and early-time structure in 2d CFTs. In 2d CFTs with strong modular structure and associated large degeneracies, the analytically continued partition function can develop pronounced quasi-periodic peaks (“mini-recurrences”) at special times, often near $t \simeq 2\pi n$. Such peaks can generate an apparent dip and a ramp-like growth, even in non-chaotic (e.g. rational) theories [83, 84, 85]; in particular, the mere presence of an early-time dip is *not* a reliable diagnostic of RMT universality. In addition, the presence of mini-recurrences implies that the unaveraged SFF is highly oscillatory and so does not admit a robust characterization in terms of a smooth dip-ramp-plateau curve. Therefore, in 2d CFTs, it is typically essential to introduce some averaging or smearing to extract stable late-time features. In this work we will in particular adopt a progressive time average (PTA), defined in (4.2). This is also consistent with the general fact that the ramp and plateau are not self-averaging in a single realization [79]. See related work in [86].

Plateau values and degeneracies. The plateau of a time-averaged SFF is controlled by the terms with exactly matched energies in the double sum

$$\sum_{a,b} e^{-\beta(E_a+E_b)} e^{-it(E_a-E_b)}.$$

In a spectrum with no energy degeneracies this reduces to the usual diagonal estimate $Z(2\beta)/Z(\beta)^2$. In two-dimensional CFTs, however, Virasoro symmetry implies systematic degeneracies from descendants. Consequently, even after time averaging there remain many contributions with $E_a = E_b$ but $a \neq b$, and the resulting plateau is generically enhanced relative to $Z(2\beta)/Z(\beta)^2$ [83, 78].

4.2. Spectral form factors in symmetric orbifolds

In this section, we take the seed theory to be a compact-boson CFT $\mathcal{C}(R)$ at various radii, including the fermion radius $R = \sqrt{2}$. It is well understood that symmetric product orbifolds at the orbifold point do not exhibit fully chaotic behavior; see [31, 87]. Nevertheless, it remains important to quantify *how* and *how much* they deviate from the canonical chaotic benchmark, and conversely which aspects (if any) can still mimic the familiar dip-ramp-plateau phenomenology. Here we address this question from a complementary angle by using the progressively time-averaged SFF, which provides a direct probe of late-time spectral correlations.

Before going into details, let us briefly summarize the seed CFT input used in our numerics. The partition function of a compact boson $\phi \sim \phi + 2\pi R$ is given by [88]

$$Z_{\mathcal{C}(R)}(\tau, \bar{\tau}) = \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})} \sum_{n,w \in \mathbb{Z}} e^{2\pi i \tau \left(\frac{n}{2R} + \frac{wR}{2}\right)^2} e^{-2\pi i \bar{\tau} \left(\frac{n}{2R} - \frac{wR}{2}\right)^2}. \quad (4.15)$$

When evaluating this partition function numerically, we truncate the sums in (4.15) to $-30 \leq n, w \leq 30$. At rational radii the seed SFF is exactly periodic²³, and its period T can be determined analytically: $\frac{T}{2\pi} = \max\{2R^2, 2/R^2\}$. See Fig. 4.2.

4.2.1. Plots of SFFs of symmetric orbifolds

In this section, we present explicit plots of SFFs in symmetric orbifolds. We begin with plots with fixed seed theory and inverse temperature β . The log-log plot of the progressive time averaged SFFs for $N = 1, 4, \dots, 16$ is shown in Figs. 4.3 and 4.4, with

²If R is away from a rational point, the SFF becomes *almost* periodic. See [2] for details.

³If the seed theory has periodicity T , the SFF of its symmetric orbifold also exhibits periodic behavior. However, the resulting period can be parametrically larger than T and is naturally tied to $\text{lcm}(1, 2, \dots, N)T$ (up to an overall integer multiple); see [1].

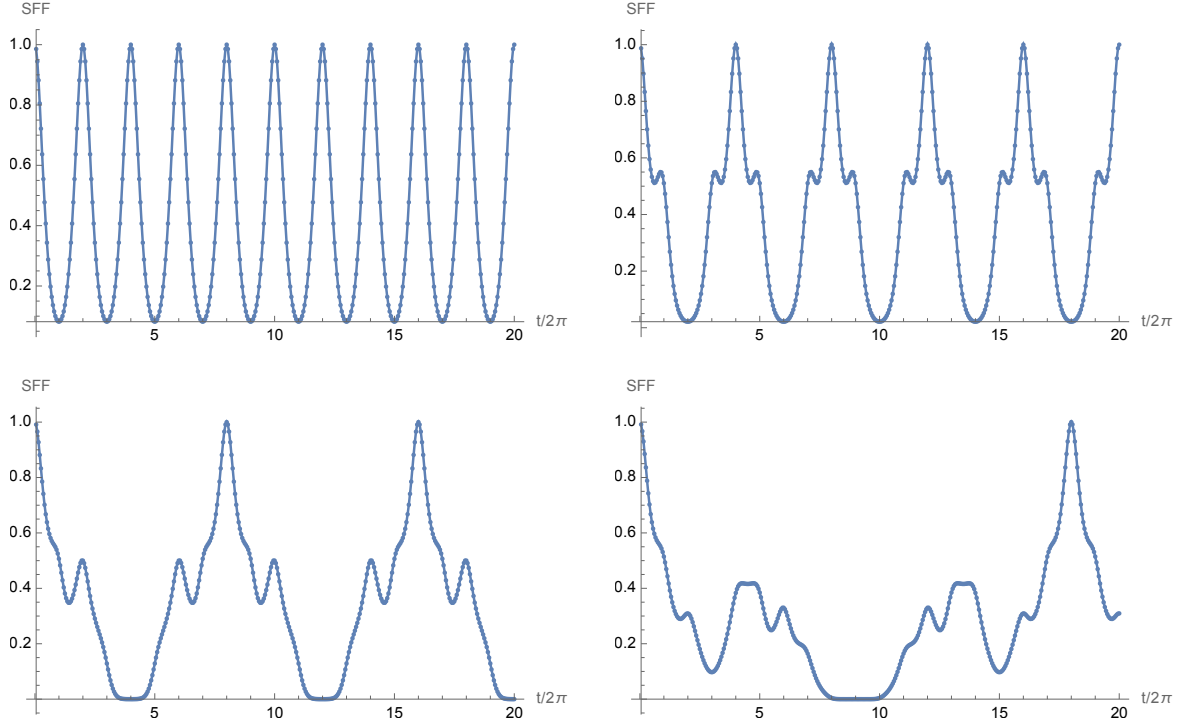


Figure 4.2.: SFFs of compact boson CFTs as a function of $\frac{t}{2\pi}$. From upper left: R is 1, $\sqrt{2}$, and from lower left: R is 2, 3, respectively. Here the inverse temperature is set as $\beta/2\pi = 0.6$.

$R = \sqrt{2}, 3$, $\frac{\beta}{2\pi} = 0.6$ and $\frac{t}{2\pi} \in (0, 100)$. Note that SFFs are normalized as in (4.1), so that they all start from 1 for $t = 0$.

The following qualitative features are immediately visible:

- The “dip” can be seen as the first step. Increasing N , the dip gets deeper. However, the dip time remains $O(1)$, independent of N .
- Unlike the canonical smooth linear ramp suggested by random-matrix universality, the late-time growth is dominated by multiple pronounced steps⁴.
- Because the ramp is step-like rather than smooth, the approach to the plateau is not monotonic and the plateau onset time is difficult to read off unambiguously from the plots.

The step locations are clearly seed-dependent: comparing the $R = \sqrt{2}$ and $R = 3$ seeds in Figs. 4.3 and 4.4, the step times shift appreciably. This is further illustrated by fixing $N = 16$ and varying the seed radius in Fig. 4.5.

Moreover, the step times are found to be essentially independent of the temperature, as shown in Fig. 4.6. The near β -independence of the step locations suggests that they

⁴In [89], this step structure in the ramp regime is also discussed for string gases. It was argued that the positions of steps are related to different winding numbers of the Euclidean Horowitz-Polchinski solutions.

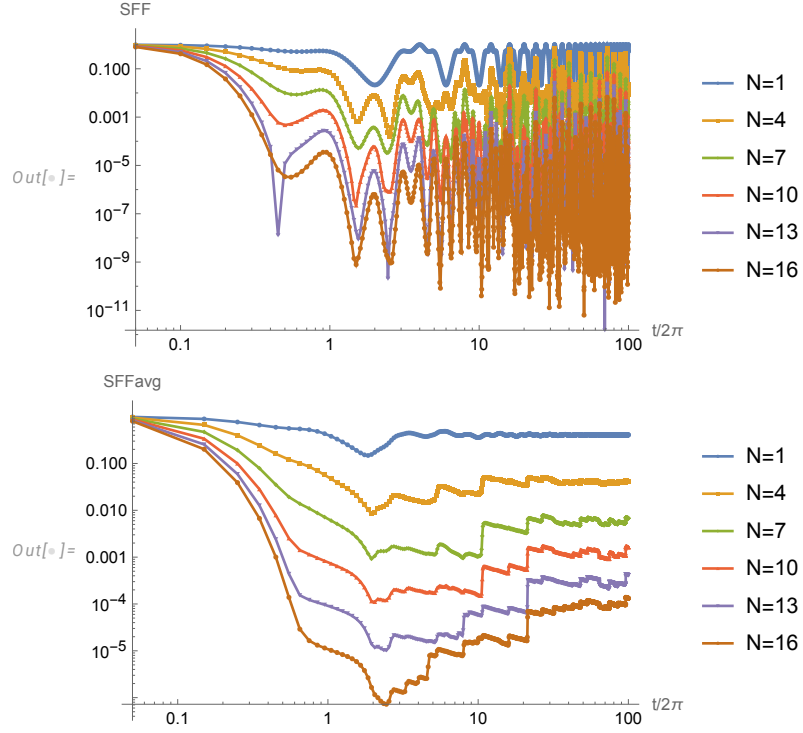


Figure 4.3.: SFFs of fermion radius compact boson symmetric orbifolds with increasing N . The upper panel shows the raw SFF, while the lower panel shows the time-averaged SFF. Here the inverse temperature is set as $\beta/2\pi = 0.6$.

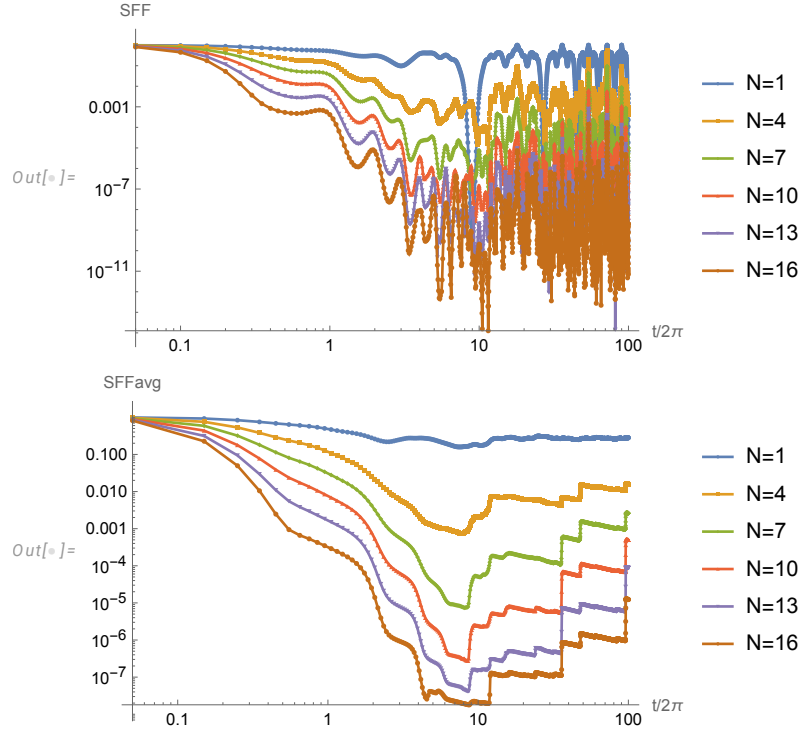


Figure 4.4.: SFFs of $R = 3$ compact boson symmetric orbifolds with increasing N . The upper panel shows the raw SFF, while the lower panel shows the time-averaged SFF. Here the inverse temperature is set as $\beta/2\pi = 0.6$.

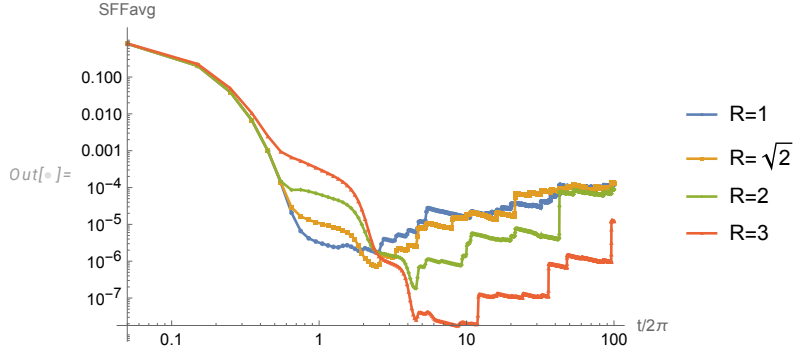


Figure 4.5.: Time-averaged SFFs of various radii compact boson symmetric orbifolds with $N = 16$. R is taken to be $1, \sqrt{2}, 2, 3$. Here the inverse temperature is set as $\beta/2\pi = 0.6$.

are set primarily by discrete energy gaps in the light spectrum (entering through phases $e^{-it\Delta E}$), while β mainly controls the relative weights.

4.2.2. Interpretation of the step-like ramp

Let us summarize the qualitative behavior of the progressive-time-averaged SFF of a symmetric product orbifold $\text{Sym}^N \mathcal{C}(R)$.

For $N > 1$ we observe a clear dip followed by a late-time growth which resembles a “ramp” in the sense of the slope-dip-ramp-plateau paradigm. However, the ramp is *not* smooth and approximately linear. Instead, $\overline{\text{SFF}}(\beta, t)$ exhibits a pronounced *step-like* structure: it stays near quasi-plateaux for extended intervals and then jumps upward to the next level.

A key diagnostic is provided by the location of these steps. Empirically, their characteristic times are

- independent of β ,
- independent of N once N is sufficiently large,
- *dependent* on the choice of the seed theory $\mathcal{C}(R)$, and
- absent for $N = 1$.

This pattern indicates that the step-like behavior is not governed by universal chaotic (random-matrix) physics of the heavy spectrum, but is instead controlled by structured contributions specific to the orbifold construction. In particular, the observed steps are naturally explained by quasi-periodic peaks originating from the *light sector*: light seed excitations, light twist-vacuum states, and/or light twist-sector excitations, together with their cross-term interference in (4.1). In this picture, the averaging window in

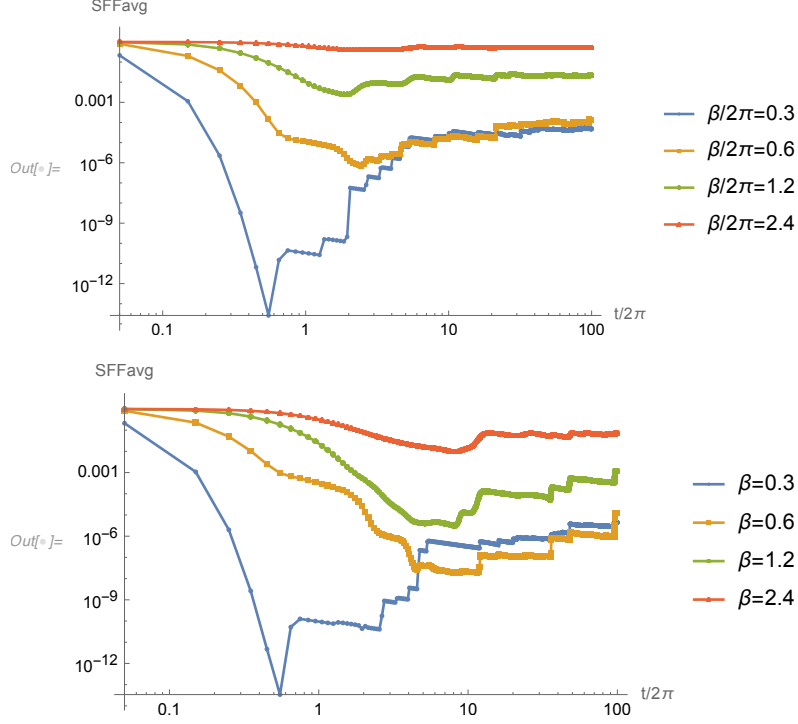


Figure 4.6.: Time-averaged SFFs of compact boson symmetric orbifolds with $N = 16$ at various temperatures. The upper panel shows the result for fermion radius, while the lower panel shows $R = 3$. Inverse temperature is taken to be $\beta/2\pi = 0.3, 0.6, 1.2, 2.4$.

(4.2) intermittently captures these quasi-periodic peaks, producing the apparent “one-step-up” evolution of $\overline{\text{SFF}}(\beta, t)$.

From the viewpoint of holographic CFTs dual to semiclassical gravity, it is widely expected that after a suitable averaging procedure the SFF approaches the smooth ramp and plateau characteristic of random-matrix universality [78, 83, 79, 90]. In our case, however, pronounced seed-sensitive quasi-periodic contributions persist even after the progressive time average, and the resulting ramp remains step-like rather than smooth. This strongly suggests that, at the symmetric orbifold point, the time-averaged SFF is still dominated by structured light-sector interference, instead of universal heavy-state correlations associated with random-matrix universality.

Recall that symmetric product orbifolds with an exactly marginal twisted-sector deformation (such as the D1-D5 CFT) are widely believed to flow, upon turning on the deformation, toward a regime admitting a semiclassical gravity (supergravity) dual [17, 18, 19, 20]. In particular, marginal deformations are expected to substantially reorganize the non-BPS light spectrum, giving large anomalous dimensions to many operators that are light at the orbifold point and lifting stringy/higher-spin degrees of freedom. This expectation is consistent with our findings: the late-time ramp of the time-averaged SFF is strongly obstructed by coherent contributions from the light

sector at the orbifold point, so that removing or lifting a significant fraction of this light structure under marginal deformation is a natural mechanism by which a smoother, more universal ramp could emerge.

4.2.3. Outlook

It is worth emphasizing a general caveat: in two-dimensional CFTs, even highly structured theories (including rational CFTs) can exhibit an apparent dip-ramp-plateau pattern after appropriate time averaging. Accordingly, when one attempts to diagnose random-matrix universality from a *single* 2d CFT using time-averaged observables, some care is required [84, 85]. We will not pursue this broader interpretational issue further here.

For the present work, the main takeaway is more concrete: at the symmetric-orbifold point, the progressively time-averaged SFF displays a robust ramp-like growth even though it remains manifestly non-smooth and step-like. This suggests that time averaging can extract a degree of self-averaging-like late-time growth while still retaining sensitivity to light-sector structure. It would therefore be interesting to investigate, *within symmetric orbifolds*, whether similar time-averaging prescriptions can help reveal holography-mimicking behavior in other observables, such as real-time correlation functions and entanglement-related quantities, that are otherwise dominated by recurrences and structured oscillations [5].

Finally, a straightforward next step on the computational side is to push the numerics further so that the plateau onset time can be read off more reliably. Since the step-like ramp obscures a clean saturation, extending the accessible time range and improving numerical control (e.g. larger N where appropriate) may allow a sharper determination of when $\overline{\text{SFF}}(\beta, t)$ truly reaches its plateau.

Part III.

Open Twists and Interfaces

5. Open Twists and Interfaces in Symmetric Orbifolds

In this chapter, we introduce, in the setting of symmetric orbifolds, a new class of objects that we call *interfaces*, together with a class of excitations which we refer to as *open twists*. An interface is a one-dimensional object that connects symmetric orbifolds with different copy numbers $N_{\pm}$, and can be viewed as a generalization of boundary states and conformal defects. See Fig. 5.1 for a schematic depiction.

An interface comes with several pieces of discrete data. Among these, a particularly important parameter is the *degree of transmissivity* P . Intuitively, P measures “how many of the $N_{\pm}$ copies of the seed theory are actually connected across the interface”. In that sense, P satisfies $0 \leq P \leq \min(N_{\pm})$. In the presence of the interface, the permutation gauge symmetry on the $\text{Sym}^{N_-}(\mathcal{M})$ side is broken according to

$$S_{N_-} \longrightarrow S_{N_- - P} \times S_P,$$

while on the $\text{Sym}^{N_+}(\mathcal{M})$ side, it is broken as

$$S_{N_+} \longrightarrow S_{N_+ - P} \times S_P.$$

The common factor S_P is naturally interpreted as the gauge group acting on the block

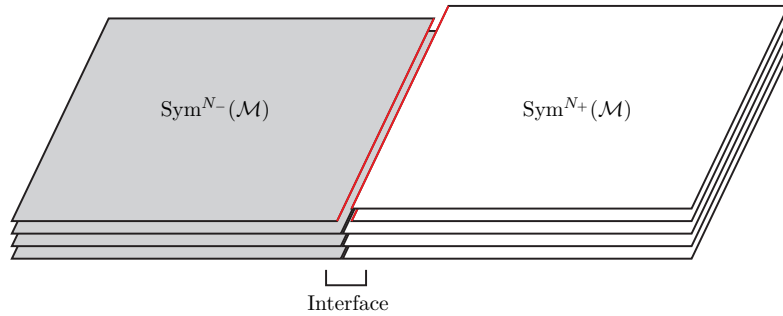


Figure 5.1.: Interface between symmetric orbifolds $\text{Sym}^{N_-}\mathcal{M}$ and $\text{Sym}^{N_+}\mathcal{M}$. In the figure we set $N_- = 4$, $N_+ = 5$ and $P = 3$. Red lines indicate boundaries in seed theories. This figure is reproduced from Fig. 1.1.

of copies that are transmitted through the interface.

The presence of an interface allows us to consider ramification points that lie on top of the interface and are attached to the boundary of the seed theory. Ordinary ramification points, namely those associated with closed twists, correspond to branch points whose neighborhood has an opening angle $2\pi w$ ($w \in \mathbb{Z}_{>0}$). By contrast, a ramification point attached to the boundary is allowed to carry an opening angle that is a half-integer multiple of 2π , namely, $2\pi w$ with $w \in \frac{1}{2}\mathbb{Z}_{>0}$. We refer to the excitations on the interface that are associated with such boundary ramification points as *open twists*. When w happens to be an integer, the corresponding insertion induces a definite local change in the degree of transmissivity P in the neighborhood of the insertion.

These constructions are motivated by holography, and in particular by the description of tensionless strings in AdS_3 . The interpretation of the interfaces introduced here as AdS_2 branes in a pure NS–NS background will be discussed in detail in chapter 6.

In Section 5.1, we give an explicit definition of the interface as a state in the Hilbert space of the symmetric orbifold. We then compute the vacuum expectation value of the interface operator and verify that the effective gauge symmetry of the full system is broken to

$$S_{N_-} \times S_{N_+} \longrightarrow S_{N_- - P} \times S_P \times S_{N_+ - P}. \quad (5.1)$$

Additionally, we apply the folding trick, in close analogy with the earlier discussion of defects in Sec. 2.4. The interface can be realized as a boundary state of the folded theory.

In Section 5.2, we study the cylinder amplitude of boundary states of interfaces. From this cylinder amplitude, we extract the spectrum of boundary condition changing operators between the boundary state realizations of two interfaces: that is, the spectrum of *interface-changing operators*. This spectrum includes twist operators localized on top of the interface: in particular, makes the spectrum of open twists fully explicit.

Finally, in Section 5.3, we introduce a general method for computing correlation functions in the presence of an interface.

5.1. Interfaces in symmetric orbifold

In this section, we introduce *interfaces*, which generalize the boundary states and topological defects reviewed in Secs. 2.3 and 2.4 and connect symmetric orbifolds with different copy numbers N but a common seed CFT $\mathcal{M}$. We label these different symmetric orbifolds as $\text{Sym}^{N_{\pm}}\mathcal{M}$, as shown in Fig. 5.1.

To keep track of the discrete data labeling an interface $\mathcal{I}$ in a compact way, we introduce a convenient multi-index ϖ :

$$\varpi = \left((P, R_I), (\vec{n}_-, \vec{r}_-; \vec{n}_+, \vec{r}_+; \vec{a}) \right), \quad (5.2)$$

where P is the degree of transmissivity that determines the size of the subsector of the interface that behaves as a topological defect between the two symmetric orbifolds: P seed copies out of $N_{\pm}$ are geometrically connected along the interface. P satisfies $0 \leq P \leq \min(N_{\pm})$. R_I indicates an irreducible representation of the permutation group S_P associated with the transmissive subsector. The collection of vectors $(\vec{n}_-, \vec{r}_-; \vec{n}_+, \vec{r}_+; \vec{a})$ determines the boundary condition of $N_{\pm} - P$ copies of both sides:

$$(\vec{n}_-, \vec{r}_-; \vec{n}_+, \vec{r}_+; \vec{a}) = \left\{ (n_{-,i}, r_{-,i}; n_{+,i}, r_{+,i}; a_i) \right\}_{i=1}^{n_b}, \quad (5.3)$$

where

- n_b is the number of distinct seed Cardy boundary states,
- a_i denotes the i -th seed Cardy boundary condition out of these n_b ,
- $n_{\pm,i}$ specifies the size of the subset of $\{P, (P+1), \dots, N_{\pm}\}$, i.e. the number of seed copies associated with a_i on the $N_{\pm}$ side,
- $r_{\pm,i}$ is an irreducible representation of the permutation group $S_{n_{\pm,i}}$ acting on that subset.

In particular, the integers $n_{\pm,i}$ obey

$$\sum_{i=1}^{n_b} n_{-,i} = N_- - P, \quad \sum_{i=1}^{n_b} n_{+,i} = N_+ - P. \quad (5.4)$$

Generic interface $\mathcal{I}_{\varpi}$ is then defined by averaging over the permutation gauge group $S_{N_-} \times S_{N_+}$ acting on the boundary states and topological defects:

$$\begin{aligned} \mathcal{I}_{\varpi} := & \sum_{h_- \in S_{N_-}} \sum_{h_+ \in S_{N_+}} \\ & \prod_{i=1}^{n_b} \left(\sum_{g_{-,i} \in S_{n_{-,i}}} \frac{\chi_{r_{-,i}}(g_{-,i})}{n_{-,i}!} |a_i\rangle_{h_- g_{-,i} h_-^{-1}} \right) \left(\sum_{g_{+,i} \in S_{n_{+,i}}} \frac{\chi_{r_{+,i}}(g_{+,i})}{n_{+,i}!} \langle a_i|_{h_+ g_{+,i} h_+^{-1}} \right) \\ & \left(\sum_{[g] \in \text{Conj}(S_P)} \frac{\chi_{R_I}([g])}{|C(g)|} \sum_{h,n,m} \Pi_{h_- g h_-^{-1}} |h, n\rangle_{h_- g h_-^{-1}} \otimes \overline{|h, m\rangle}_{h_- g h_-^{-1}} \right. \\ & \left. \otimes_{h_+ g h_+^{-1}} \langle h, n| \otimes_{h_+ g h_+^{-1}} \overline{\langle h, m|} \Pi_{h_+ g h_+^{-1}} \right). \end{aligned} \quad (5.5)$$

It is instructive to note that our parametrization interpolates between the ordinary boundary states and the topological defects discussed in Secs. 2.3 and 2.4. First, when $N_+ = 0$ the right-hand side of the interface is absent, and the consistency condition $0 \leq P \leq \min(N_-, N_+)$ forces $P = 0$. In this limit, there is no transmissive subsector and the interface data reduce to

$$\varpi \longrightarrow (\vec{n}_-, \vec{r}_-, \vec{a}), \quad (5.6)$$

so that $\mathcal{I}_\varpi$ reproduces precisely the ordinary boundary state of $\text{Sym}^{N_-} \mathcal{M}$ constructed in Sec. 2.3.

Second, when both sides carry the same number of copies and are fully transmissive $N_- = N_+ = P = N$, there are no remaining reflective copies, $N_\pm - P = 0$, and the boundary-data part of ϖ drops out:

$$\varpi \longrightarrow (P = N, R_I), \quad (5.7)$$

In this case the interface is entirely characterized by the representation R_I of $S_P = S_N$, and $\mathcal{I}_\varpi$ reduces to a purely topological defect between two identical symmetric orbifolds $\text{Sym}^N \mathcal{M}$, in agreement with the construction of topological defects in Sec. 2.4.

In the rest of this thesis, we focus on the case with homogeneous boundary choice;

$$(\vec{n}_-, \vec{r}_-, \vec{a})|_{a_i=a_-} = (N_- - P, R_-; a_-), \quad (\vec{n}_+, \vec{r}_+, \vec{a})|_{a_i=a_+} = (N_+ - P, R_+; a_+), \quad (5.8)$$

where $R_\pm$ is an irreducible representation of $S_{N_\pm - P}$. Also, we often take $R_\pm$ and R_I to be trivial. We denote such an interface by $\mathcal{I}_{a_\pm; R_\pm}^{P; R_I}$ and $\mathcal{I}_{a_\pm}^P$, respectively.

Folding trick and boundary state realization of interface.

Now let us introduce the boundary state realization of interface $\mathcal{I}_{a_\pm; R_\pm}^{P; R_I}$:

$$\begin{aligned} |\mathcal{I}_{a_\pm; R_\pm}^{P; R_I}\rangle\rangle = & \sum_{h_- \in S_{N_- - P}} \sum_{h_+ \in S_{N_+ - P}} \\ & \left(\sum_{g_- \in S_{N_- - P}} \frac{\chi_{R_-}(g_-)}{(N_- - P)!} |a_-\rangle_{h_- g_- h_-^{-1}}^{(-)} \right) \left(\sum_{g_+ \in S_{N_+ - P}} \frac{\chi_{R_+}(g_+)}{(N_+ - P)!} |a_+\rangle_{h_+ g_+ h_+^{-1}}^{(+)} \right) \\ & \left(\sum_{g \in S_P} \frac{\chi_{R_I}(g)}{P!} \sum_{h, n, m} \Pi_{h_- g h_-^{-1}}^{(-)} |h, n\rangle_{h_- g h_-^{-1}}^{(-)} \otimes \overline{|h, m\rangle_{h_- g h_-^{-1}}^{(-)}} \right. \\ & \left. \otimes \Pi_{h_+ g h_+^{-1}}^{(+)} \overline{|h, n\rangle_{h_+ g h_+^{-1}}^{(+)}} \otimes |h, m\rangle_{h_+ g h_+^{-1}}^{(+)} \right). \end{aligned} \quad (5.9)$$

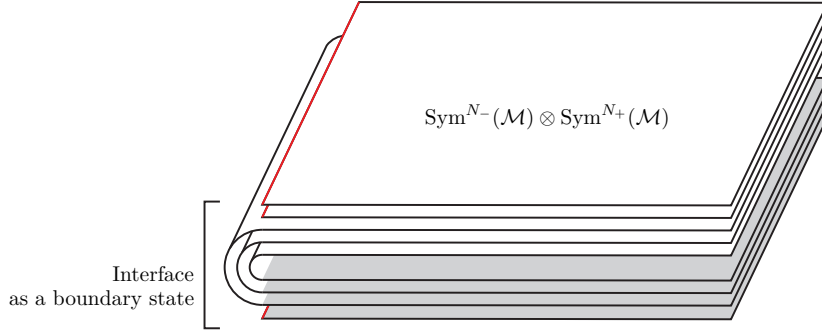


Figure 5.2.: Interface realized as a boundary state in folded theory $\text{Sym}^{N_-} \mathcal{M} \otimes \text{Sym}^{N_+} \mathcal{M}$. In the figure we set $N_- = 4$, $N_+ = 5$ and $P = 3$. Red lines indicate boundaries in seed theories.

See Fig. 5.2.

Vacuum expectation value of the interface

Let us compute the vacuum expectation value of $\mathcal{I}_{a_{\pm}; R_{\pm}}^{P; R_I}$, or equivalently the g -function associated with the corresponding boundary state $\|\mathcal{I}_{a_{\pm}; R_{\pm}}^{P; R_I}\rangle\rangle$. In the overlap with the vacuum, only the contribution from the identity elements $g_{\pm} = g = e$ in eq. (5.9) survives, and we obtain

$$\langle 0 | \mathcal{I}_{a_{\pm}; R_{\pm}}^{P; R_I} \rangle\rangle_{\text{phys}} = \frac{\dim R_- \dim R_+ \dim R_I}{(N_- - P)! (N_+ - P)! P!} g_-^{N_- - P} g_+^{N_+ - P} z^P, \quad (5.10)$$

where $g_{\pm} = \langle 0 | a_{\pm} \rangle_{\text{seed}}$ are the g -functions in the seed theory, and $z = \langle 0 | 0 \rangle_{\text{seed}}$ is the sphere partition function of the seed theory.

In particular, when the representations $R_{\pm}$ and R_I are taken to be trivial, this vacuum expectation value reduces to the product of disk and sphere partition functions of the seed theory, divided by the order of the effectively unbroken permutation group $S_{N_- - P} \times S_P \times S_{N_+ - P}$:

$$\langle 0 | \mathcal{I}_{a_{\pm}}^P \rangle\rangle_{\text{phys}} = \frac{g_-^{N_- - P} z^P g_+^{N_+ - P}}{|S_{N_- - P} \times S_P \times S_{N_+ - P}|}. \quad (5.11)$$

This provides one piece of evidence that the effective permutation gauge symmetry of the full configuration is broken as

$$S_{N_-} \times S_{N_+} \longrightarrow S_{N_- - P} \times S_P \times S_{N_+ - P}. \quad (5.12)$$

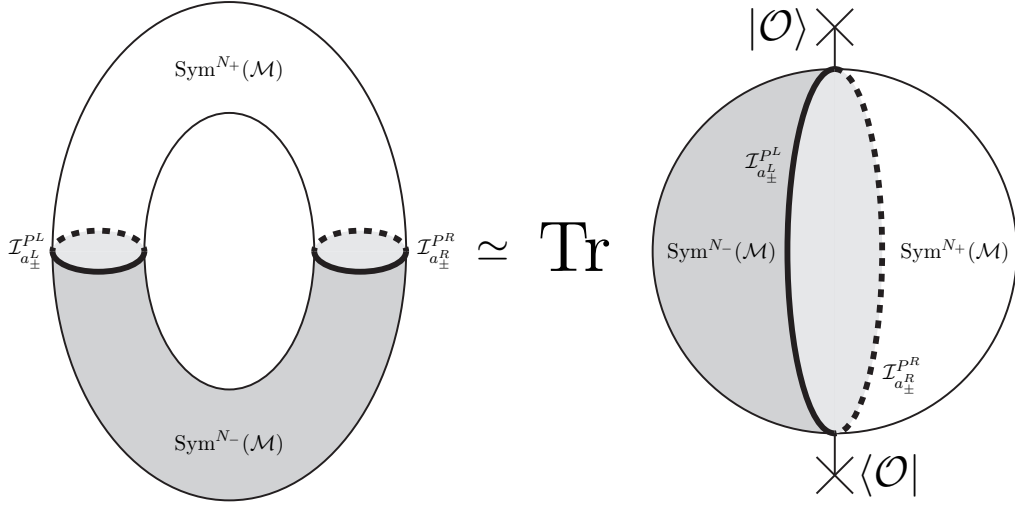


Figure 5.3.: Two interfaces connecting the theories $\text{Sym}^{N\pm}\mathcal{M}$ on the cylinder. The partition function can be regarded as a trace over interface changing operators. On the right-hand side of the figure, the interface is changed when one passes through the insertion point of the interface changing operator $\mathcal{O}$.

5.2. Interface changing operators / open twists

Now we consider the cylinder amplitude of interfaces.

$$Z_{N_{\pm}, PL/R}^{\mathbb{T}^2} := \langle \mathcal{I}_{a_{\pm}^L}^{PL} | \sqrt{\tilde{q}}^{L_0^{(-)} + L_0^{(+)} - \frac{(N_- + N_+)c}{12}} \sqrt{\tilde{q}}^{\bar{L}_0^{(-)} + \bar{L}_0^{(+)} - \frac{(N_- + N_+)c}{12}} | \mathcal{I}_{a_{\pm}^R}^{PR} \rangle_{\text{phys}} \quad (5.13)$$

In the unfolded theory picture it is like the left-hand side of Fig. 5.3. We can read off the spectrum of interface changing operators between $\mathcal{I}_{a_{\pm}^L}^{PL}$ and $\mathcal{I}_{a_{\pm}^R}^{PR}$.

In the result we have these five types of the contributions

$$Z_{\mathcal{M}}^{\mathbb{T}^2} \left(\frac{k\tau + b}{l} \right), \quad Z_{a_{\pm}^L a_{\pm}^R} \left(\frac{k\tau}{2l - 1} \right), \quad Z_{a_-^{L/R} a_+^{L/R}} \left(\frac{k\tau}{2l} \right), \quad (5.14)$$

where $k, l \in \mathbb{Z}_{>0}$. Note that $Z_{\mathcal{M}}^{\mathbb{T}^2} \left(\frac{k\tau + b}{l} \right)$ always appears in the combination

$$Z_{\text{closed}} \left(\frac{k\tau}{l} \right) := \frac{1}{l} \sum_{b=0}^{l-1} Z_{\mathcal{M}}^{\mathbb{T}^2} \left(\frac{k\tau + b}{l} \right). \quad (5.15)$$

Let us define the set of subscripts to fix the feature of partition functions:

$$f \in F := \left\{ \text{closed}, (a_-^L a_-^R), (a_+^L a_+^R), (a_-^L a_+^L), (a_-^R a_+^R) \right\}. \quad (5.16)$$

By using this f , each term in (5.13) is identified by $m_f^{(k,l)}$ as follows, up to coefficient:

$$\prod_{f \in F} \prod_{k,l} Z_f(k,l)^{m_f^{(k,l)}}. \quad (5.17)$$

where non-negative integers $m_f^{(k,l)}$ satisfy the following conditions:

$$N_- - P^L = \sum_{k,l} k \left(m_{(a_-^L a_-^R)}^{(k,l)} + m_{(a_-^L a_+^L)}^{(k,l)} \right), \quad (5.18a)$$

$$N_+ - P^L = \sum_{k,l} k \left(m_{(a_+^L a_+^R)}^{(k,l)} + m_{(a_-^L a_+^L)}^{(k,l)} \right), \quad (5.18b)$$

$$N_- - P^R = \sum_{k,l} k \left(m_{(a_-^L a_-^R)}^{(k,l)} + m_{(a_-^R a_+^R)}^{(k,l)} \right), \quad (5.18c)$$

$$N_+ - P^R = \sum_{k,l} k \left(m_{(a_+^L a_+^R)}^{(k,l)} + m_{(a_-^R a_+^R)}^{(k,l)} \right), \quad (5.18d)$$

as a match of boundary seed numbers, and

$$P^L = \sum_{k,l} kl \left(m_{\text{closed}}^{(k,l)} + m_{(a_-^R a_+^R)}^{(k,l)} \right) + \sum_{k,l} k(l-1) \left(m_{(a_-^L a_+^L)}^{(k,l)} + m_{(a_-^L a_-^R)}^{(k,l)} + m_{(a_+^L a_+^R)}^{(k,l)} \right), \quad (5.18e)$$

$$P^R = \sum_{k,l} kl \left(m_{\text{closed}}^{(k,l)} + m_{(a_-^L a_+^L)}^{(k,l)} \right) + \sum_{k,l} k(l-1) \left(m_{(a_-^R a_+^R)}^{(k,l)} + m_{(a_-^L a_-^R)}^{(k,l)} + m_{(a_+^L a_+^R)}^{(k,l)} \right), \quad (5.18f)$$

as a match of transmissive part. The multiplier can be determined as

$$\prod_{f \in F} \prod_{k,l} \frac{\left(\frac{1}{k} Z_f(k\tau/l) \right)^{m_f^{(k,l)}}}{m_f^{(k,l)}!}, \quad (5.19)$$

by looking at the number of combinations that realize $\prod_{f \in F} \prod_{k,l} Z_f(k,l)^{m_f^{(k,l)}}$ divided by the gauge volume. We obtain

$$Z_{N_{\pm}, P^L/R}^{\mathbb{T}^2} = \sum_{m_f^{(k,l)}} \prod_{f \in F} \prod_{k,l} \frac{\left(\frac{1}{k} Z_f(k,l) \right)^{m_f^{(k,l)}}}{m_f^{(k,l)}!}, \quad (5.20)$$

where $m_f^{(k,l)}$ satisfies the constraint (5.18).

5.2.1. Grand-canonical uplift of overlap

It is natural to introduce chemical potentials $\mu^{L/R}$ and $\nu_{\pm}^{L/R}$ in order to construct the grand-canonical uplift of eq. (5.20). We define the corresponding grand-canonical partition function by

$$\mathcal{Z}_{\mu^{L/R}, \nu_{\pm}^{L/R}}^{\mathbb{T}^2} := \sum_{N_{\pm} \geq 0} \sum_{P^{L/R}=0}^{\min N_{\pm}} \prod_{A \in \{L, R\}} (\nu_{-}^A)^{N_{-}-P^A} (\mu^A)^{P^A} (\nu_{+}^A)^{N_{+}-P^A} Z_{N_{\pm}, P^{L/R}}^{\mathbb{T}^2}. \quad (5.21)$$

For later convenience, we also use the shorthand $\mu := \mu^L \mu^R$. We find the following DMVV-like formula

$$\mathcal{Z}_{\mu^{L/R}, \nu_{\pm}^{L/R}}^{\mathbb{T}^2}(\tau) = \prod_{k \geq 1} \prod_{l \geq 1} \exp \left[(\mu^L \mu^R)^{kl} \frac{1}{kl} \sum_{b=0}^{l-1} Z_{\mathcal{M}}^{\mathbb{T}^2} \left(\frac{k\tau + b}{l} \right) \right] \quad (5.22a)$$

$$\times \exp \left[\left(\nu_{-}^L \nu_{-}^R (\mu^L \mu^R)^{l-1} \right)^k \frac{1}{k} Z_{(a_{-}^L a_{-}^R)} \left(\frac{k\tau}{2l-1} \right) \right] \quad (5.22b)$$

$$\times \exp \left[\left(\nu_{+}^L \nu_{+}^R (\mu^L \mu^R)^{l-1} \right)^k \frac{1}{k} Z_{(a_{+}^L a_{+}^R)} \left(\frac{k\tau}{2l-1} \right) \right] \quad (5.22c)$$

$$\times \exp \left[\left(\nu_{-}^L \nu_{+}^L (\mu^L)^{l-1} (\mu^R)^l \right)^k \frac{1}{k} Z_{(a_{-}^L a_{+}^L)} \left(\frac{k\tau}{2l} \right) \right] \quad (5.22d)$$

$$\times \exp \left[\left(\nu_{-}^R \nu_{+}^R (\mu^L)^l (\mu^R)^{l-1} \right)^k \frac{1}{k} Z_{(a_{-}^R a_{+}^R)} \left(\frac{k\tau}{2l} \right) \right]. \quad (5.22e)$$

Recall that we have the following q -expansions:

$$Z_{\mathcal{M}}^{\mathbb{T}^2}(\tau) = \sum_{(m, \bar{m}) \in I} d(m, \bar{m}) q^m \bar{q}^{\bar{m}} \quad \text{and} \quad Z_{ab}^{(\text{open})}(\tau) = \sum_{m \in I_{ab}} d_{ab}(m) q^m.$$

Again, using the identities (2.31), we can rewrite the grand-canonical partition function in the following product form:

$$\begin{aligned} \mathcal{Z}_{\mu^{L/R}, \nu_{\pm}^{L/R}}^{\mathbb{T}^2}(\tau) &= \prod_{w \in \mathbb{Z}_{>0}} \prod_{\substack{(m, \bar{m}) \in I \\ m - \bar{m} \equiv 0 \pmod{w}}} \left(\frac{1}{1 - \mu^w q^{\frac{m}{w}} \bar{q}^{\frac{\bar{m}}{w}}} \right)^{d(m, \bar{m})} \\ &\times \prod_{\sigma = \pm} \prod_{w \in \mathbb{Z}_{>0} + \frac{1}{2}} \prod_{m \in I_{a_{\sigma}^L a_{\sigma}^R}} \left(\frac{1}{1 - \nu_{\sigma}^L \nu_{\sigma}^R (\mu)^{w - \frac{1}{2}} q^{\frac{m}{2w}}} \right)^{d_{a_{\sigma}^L a_{\sigma}^R}(m)} \\ &\times \prod_{AB=LR, RL} \prod_{w \in \mathbb{Z}_{>0}} \prod_{m \in I_{a_{-}^A a_{+}^A}} \left(\frac{1}{1 - \nu_{-}^A \nu_{+}^A \mu^B (\mu)^{w-1} q^{\frac{m}{2w}}} \right)^{d_{a_{-}^A a_{+}^A}(m)}. \end{aligned} \quad (5.23)$$

5.2.2. Spectrum of interface changing operators

For fixed grand-canonical charges $(N_{\pm}, P^{L/R})$, we can read off the spectrum of interface changing operators by focusing on the corresponding sector of the generating function. It is natural to work on the vacuum energy given by $-\frac{c}{24}(N_- + N_+)$. For simplicity, in the following we restrict to the case where all seed boundary conditions coincide, namely

$$a_+^L = a_-^L = a_+^R = a_-^R \equiv a. \quad (5.24)$$

In particular, the relevant annulus partition function reduces to

$$Z_{aa}^{(\text{open})}(\tau) = \sum_{m \in I_{aa}} d_{aa}(m) q^m, \quad m = \Delta - \frac{c}{24}, \quad \Delta \geq 0, \quad (5.25)$$

so that the vacuum has $\Delta = 0$ and hence $m_{\text{vac}} = -c/24$.

Reading off the twist ground state from the product.

Restricting attention to the vacuum sector $h = \bar{h} = \Delta = 0$, we extract the spectrum of a particular class of interface changing operators, which we call *interface twist operators*. The spectrum of single-cycle interface twist operators are organized into five series. Three series occur when $P^L = P^R$; among these, one essentially reduces to the familiar twist operator series. The remaining two series appear when the labels differ by one unit, $P^L = P^R \pm 1$. More explicitly:

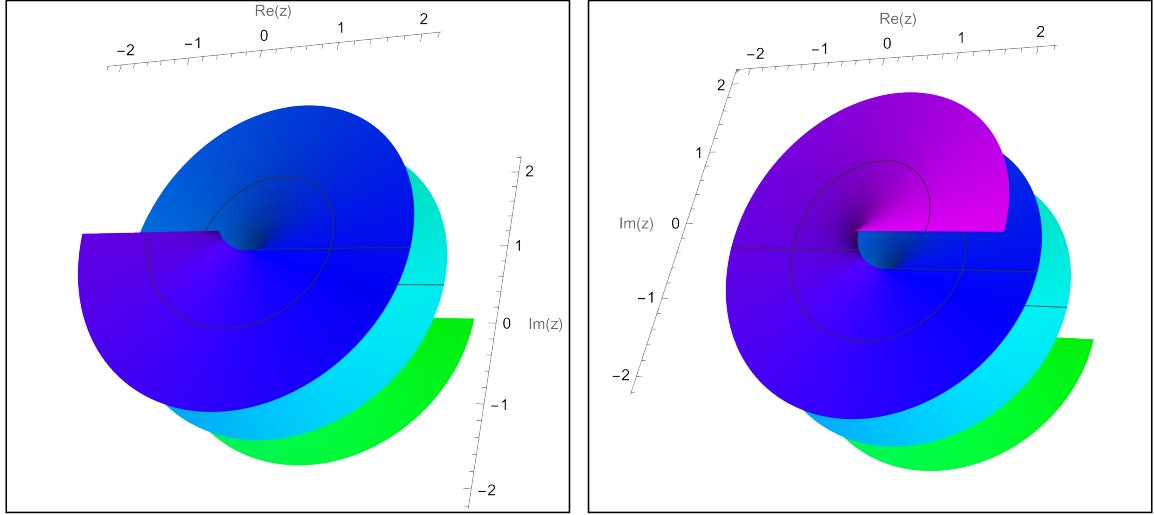
- σ_w^I : $h = \bar{h} = \frac{c_M}{24} \left(w - \frac{1}{w} \right)$ with $P^L = P^R$ and $w \in \mathbb{Z}_{>0}$,
- $\Delta = \frac{c}{24} \left(2w - \frac{1}{2w} \right)$ for

$$\begin{cases} \sigma_w^{I,-}: \text{from } a_-^L a_-^R, & \text{with } P^L = P^R \text{ and } w \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \\ \sigma_w^{I,+}: \text{from } a_+^L a_+^R, & \text{with } P^L = P^R \text{ and } w \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \\ \sigma_w^{I,>}: \text{from } a_-^L a_+^R, & \text{with } P^L + 1 = P^R \text{ and } w \in \mathbb{Z}_{>0}, \\ \sigma_w^{I,<}: \text{from } a_+^L a_-^R, & \text{with } P^L = P^R + 1 \text{ and } w \in \mathbb{Z}_{>0}. \end{cases}$$

The first series is called *closed interface twists*, while others are *open interface twists* or simply *open twists*. To make this distinction explicit, we sometimes refer to the usual twist operators specifically as *bulk twists* or *bulk closed twists*.

Open twists as local operators

Open twists are identified as the ramification points with boundaries in the covering surface. Note that this type of excitation was first introduced in an intuitive way in [91]. See Figure 5.4 for examples.



(a) $w = \frac{5}{2}$ ramification point of a $N_- = 3$, $N_+ = 2$ covering created by $\sigma^{I,-}$. The degree of transmissivity is constantly 2.

(b) $w = \frac{6}{2}$ ramification point of a $N_{\pm} = 3$ covering created by $\sigma^{I,<}$. The degree of transmissivity jumps from 3 to 2 at the ramification point.

Figure 5.4.: Visualization of coverings created by open twists. The covering map is the vertical projection down the symmetry axis of the spiral. The preimage of the interface is marked by the real axis $\text{Im } z = 0$.

5.3. Computation of interface correlation functions

Even in the presence of interfaces, the computation of correlation functions can be reduced, as in the bulk case, to calculations in the seed theory on a suitable covering surface obtained by pulling back along a covering map. However, there are several respects in which the interface case differs from the purely bulk case:

- An interface generically involves a boundary of the seed theory. As a consequence, the covering surface Σ is a Riemann surface with boundary. In particular, both the evaluation of the seed theory on Σ and the evaluation of the Liouville factor must be performed on a surface with boundary.
- The degree of the covering map depends on which symmetric orbifold the map takes values in. For example, consider a single interface $\mathcal{I}$ that connects a symmetric orbifold with N_- copies on X_- to a symmetric orbifold with N_+ copies on X_+ along their common one-dimensional locus $X_- \cap X_+ = S^1$. In this situation, the degrees of the covering maps to $X_{\pm}$ are

$$\deg(\Sigma \rightarrow X_{\pm}) = N_{\pm}. \quad (5.26)$$

- There exist boundary ramification points on Σ corresponding to open twists on X . More precisely, for each open twist operator $\mathcal{O}_w^I$ on X there appears a

boundary branch point on Σ with opening angle $2\pi w$, where w is a positive half-integer or integer as introduced in Sec. 5.2.

In this chapter, we will organize the discussion so as to proceed in a way that is as parallel as possible to the bulk symmetric orbifold case. As a first step, we extend the definition of the covering map to the situation with interfaces. In what follows we will specialize to the case where the base is the Riemann sphere $X = S^2$, and a single interface circle S^1 divides X into two regions

$$X_{\pm} = \mathbb{D}_{\pm}. \quad (5.27)$$

5.3.1. Definition of interface branched covering map.

Let X and Σ be compact Riemann surfaces, possibly with boundary.

Assume that there exist finitely many disjoint simple closed curves

$$S_1^1, \dots, S_s^1 \subset X \quad (5.28)$$

such that the complement

$$X \setminus \bigcup_{a=1}^s S_a^1 \quad (5.29)$$

has r connected components $U_1, \dots, U_r$. We denote by

$$X_i := \overline{U_i} \quad (i = 1, \dots, r) \quad (5.30)$$

the corresponding closed regions. Then X is covered by the X_i , and different X_i intersect only along subsets of the interface curves S_a^1 .

Let

$$\Gamma : \Sigma \rightarrow X \quad (5.31)$$

be a map that is holomorphic on the interior of Σ and continuous up to the boundary. We define

$$\Sigma_i := \Gamma^{-1}(X_i) \quad (5.32)$$

so that Σ is covered by the Σ_i , which meet only along the preimages of the curves S_a^1 .

We say that Γ is an *interface branched covering map of degree* $(N_1, \dots, N_r)$ if the following conditions hold:

- For each $i = 1, \dots, r$ there exists a finite subset $B_i \subset X_i$ (the set of bulk branch points in X_i) such that

$$\Gamma : \Sigma_i \setminus \Gamma^{-1}(B_i) \longrightarrow X_i \setminus B_i$$

is a topological covering of degree N_i . In particular, every point $x \in X_i \setminus B_i$ has exactly N_i preimages in Σ_i .

- For every point $p \in \mathring{\Sigma}_i$ in the interior of Σ_i , there exist local complex coordinates z on Σ near p and x on X near $\Gamma(p)$ such that, up to multiplication by a nonzero constant,

$$x - \Gamma(p) = z^{w_p} \quad (5.33)$$

for some integer $w_p \geq 1$. The integer w_p is called the (bulk) ramification index at p ; points with $w_p > 1$ are bulk ramification points, and their images in X_i are the bulk branch points in B_i .

- For every point

$$p \in \Gamma^{-1}\left(\bigcup_{a=1}^s S_a^1\right) \cap \mathring{\Sigma}$$

in the interior of Σ , there exist local complex coordinates z on Σ near p and x on X near $\Gamma(p)$ such that, up to multiplication by a nonzero constant,

$$x - \Gamma(p) = z^{w_p} \quad (5.34)$$

for some integer $w_p \geq 1$. The integer w_p is called the *interior interface ramification index* at p ; points with $w_p > 1$ are interior interface ramification points, and their images in X are the bulk branch points on the interface curves S_a^1 .

- For every point

$$p \in \Gamma^{-1}\left(\bigcup_{a=1}^s S_a^1\right) \cap \partial\Sigma$$

on the boundary of Σ , there exist local complex coordinates z on Σ near p and x on X near $\Gamma(p)$ such that $\text{Im } z \geq 0$ describe neighborhoods of p as half-disk, with the interface given locally by $\text{Im } z = 0$, and such that, up to multiplication by a nonzero constant,

$$x - \Gamma(p) = z^{2w_p} \quad (5.35)$$

for some

$$w_p \in \mathbb{Z}_{>0}/2. \quad (5.36)$$

The number w_p is called the *boundary ramification index* at p ; points with $w_p > 1/2$ are boundary interface ramification points, and their images in X are the interface branch points on the interface curves S_a^1 .

In applications to symmetric orbifolds with interfaces, each region X_i carries a symmetric product $\text{Sym}^{N_i} \mathcal{M}$. Bulk branch points in B_i coincide with the insertion points

of bulk twist operators on X_i , with (bulk) ramification indices $w_p \in \mathbb{Z}_{\geq 1}$ matching the corresponding cycle lengths. Interior interface ramification points, i.e. points $p \in \Gamma^{-1}(\bigcup_a S_a^1) \cap \mathring{\Sigma}$, correspond to twist operators localized on the interface curves S_a^1 in the interior of X , again with integer ramification indices $w_p \in \mathbb{Z}_{\geq 1}$ identified with the relevant cycle lengths.

Boundary ramification points on the curves S_a^1 , i.e. points $p \in \Gamma^{-1}(\bigcup_a S_a^1) \cap \partial\Sigma$, correspond to open twist operators $\mathcal{O}_w^I$ supported on the interfaces. Their boundary ramification indices $w_p \in \mathbb{Z}_{>0}/2$ reproduce the opening angle $2\pi w$ associated with such open twists; in particular, $w_p = \frac{1}{2}$ corresponds to the trivial case with no change of opening angle, while $w_p > \frac{1}{2}$ gives a genuine boundary interface ramification point.

Degrees of transmissivity.

For each interface component S_i we can further define a *degree of transmissivity* P_i . Let $t_i \in S_i$ be a point that is not a nontrivial branch point. We define the local degree of transmissivity at t_i by

$$P_i(t_i) := \# \left(\Gamma^{-1}(t_i) \cap \mathring{\Sigma} \right), \quad (5.37)$$

i.e. as the number of preimages of t_i that lie in the interior of Σ , and hence correspond to sheets that pass through the interface rather than ending on it.

Away from interface branch points and insertions of interface changing operators, this quantity is locally constant along S_i . However, in the presence of interface excitations (such as open twist operators or more general interface changing operators inserted on S_i), the local transmissivity $P_i(t)$ can jump when crossing the insertion points along the interface. In order to have a definite label for each interface component, we choose for every S_i a reference point $t_i^* \in S_i$ which does not coincide with any interface branch point or interface changing operator insertion, and define

$$P_i := P_i(t_i^*). \quad (5.38)$$

We refer to P_i as the (reference) degree of transmissivity of the interface S_i , and will often use P_i as a label for S_i itself while keeping in mind that the local quantity $P_i(t)$ may take different values on segments of S_i separated by interface insertions.

5.3.2. Normalization of interface correlation functions

Let us discuss the normalization of the interface correlation function. Generically, we can insert multiple interfaces. In each intermediate symmetric orbifold, one can insert

bulk twists, and on each interfaces, one can insert interface twists. At any case,

$$Z_{\text{Sym}^N, \dots, \mathcal{M}}^X [\prod \mathcal{I} \prod O_{w_i} \prod O_{w_j}^I] = \sum_{\Gamma} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma} [\prod O_i \prod O_j^I]}{|\text{Aut}(\Gamma)|}, \quad (5.39)$$

where $\hat{Z}_{\mathcal{M}}^{\Sigma}$ denotes the correlation function of the seed CFT on the covering surface (possibly with boundary) including the Liouville factor. Note that this does not depend on the choice of reference point t^* for each interface curve S_i^1 .

To obtain the *normalized* interface correlation function on the sphere, we need to divide once by the vacuum partition function with interface. In this normalized convention, the contribution of Γ becomes

$$\begin{aligned} \langle \mathcal{I}_{a_{\pm}}^P \prod O_{w_i} \prod O_{w_j}^I \rangle_{N_{\pm}} &= \frac{Z_{\text{Sym}^{N_{\pm}}, \mathcal{M}}^{S^2} [\mathcal{I}_{a_{\pm}}^P \prod O_{w_i} \prod O_{w_j}^I]}{Z_{\text{Sym}^{N_{\pm}}, \mathcal{M}}^{S^2} [\mathcal{I}_{a_{\pm}}^P]} \\ &= \sum_{\Gamma} \frac{(N_- - P)! (N_+ - P)! P!}{|\text{Aut}(\Gamma)|} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma} [\prod O_i \prod O_j^I]}{g_-^{N_- - P} g_+^{N_+ - P} z^P}, \end{aligned} \quad (5.40)$$

where $Z_{\text{Sym}^{N_{\pm}}, \mathcal{M}}^{S^2} [\mathcal{I}_{a_{\pm}}^P] = \langle 0 | \mathcal{I}_{a_{\pm}}^P \rangle_{\text{phys}}$ is calculated in eqs. (5.10) and (5.11). Thus, in our normalization, the coefficient of each covering is given by

$$\frac{(N_- - P)! (N_+ - P)! P!}{|\text{Aut}(\Gamma)|}, \quad (5.41)$$

where $\text{Aut}(\Gamma)$ is the remaining gauge group associated with Γ .

It is important to stress that, while the unnormalized quantity (5.39) does not depend on the choice of the reference points t_i^* on the interface curves S_i^1 , the normalized correlator (5.40) *does*. As explained above, once the local degrees of transmissivity $P_i(t)$ along each interface curve S_i^1 are specified, we choose reference points $t_i^* \in S_i^1$ away from interface branch points and interface changing operator insertions and define $P_i := P_i(t_i^*)$. The vacuum partition function with interface that appears in the denominator of the normalized correlator (5.40) is then computed with this choice of P and t^* , and hence depends on it. When it is essential to make this dependence explicit, we denote the corresponding interface insertion as $\mathcal{I}_{a_{\pm}}^P(t^*)$.

5.3.3. Interface diagrammatics

As we did in Secs. 2.5.2 and 2.5.3, we can consider generalized PRR diagrammatics compatible with the interface and excitations on it. Let us draw on $X = S^2$ a double-line graph, made of solid and dashed lines, that passes through the insertion points of all twist excitations. Here, we take its path mostly along the interface; see Fig. 5.5 for

an example.

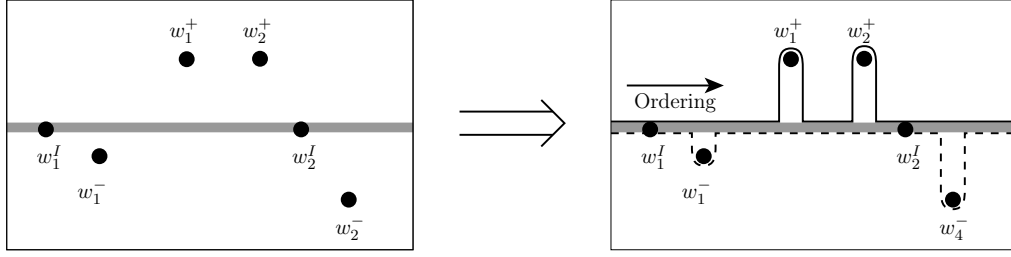


Figure 5.5.: Double-line graph made of solid and dashed lines on base manifold S^2 in the presence of an interface. The gray line indicates the location of interface. The solid (dashed) line is deformed so that it passes through bulk twist insertion points, while the dashed (solid) line stays on the reference line. Ordering of twists is naturally induced.

Given an interface covering map $\Gamma : \Sigma \rightarrow X = S^2$, we can again consider the inverse image of this double-line graph under Γ and thus obtain a double-line diagram drawn on the covering surface Σ . As in the bulk case, the diagrams with appropriate properties are in one-to-one correspondence with the branched coverings of fixed degree $N_{\pm}$ that contribute to interface correlators of bulk/interface twists. We refer to such diagrams as a *generalized PRR diagram*.

Generalized PRR rules.

To describe coverings in the presence of interfaces we introduce a generalized version of the PRR diagrams. They are built from a larger set of local ingredients: in addition to the bulk twist vertices and redundant vertices of the original PRR rules shown in Fig. 2.8, we use the splitting vertices shown in Fig. 5.6, together with the interface twist vertices in Fig. 5.7. The interface twist vertices, in particular the open interface twists, are genuinely new elements of the construction, but their representation in terms of alternating solid and dashed strands is chosen so that their combinatorics remains as transparent as in the bulk case.

We now summarize the generalized PRR rules for fixed degree $N_{\pm}$:

- A w -twist insertion at the upper (lower) half-plane is represented by a vertex surrounded by w solid (dashed) strands, respectively. See the two leftmost figures in Fig. 2.8 for example.
- Redundant vertices are introduced. A redundant vertex is surrounded by two solid (dashed) strands and one dashed (solid) strand, respectively. See the two rightmost figures in Fig. 2.8 for example.

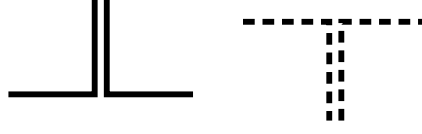


Figure 5.6.: Splitting vertices for generalized PRR diagrams. The vertex on the left (right) represents a splitting of a solid (dashed) double line, which must be attached to a bulk twist vertex in the upper (lower) half-plane, respectively. In each case the outgoing single line corresponds to a segment of the boundary of the covering surface Σ .

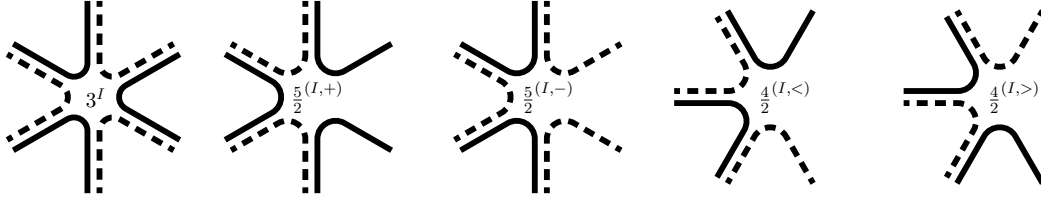


Figure 5.7.: Vertices for interface twist insertions. The leftmost vertex represents a closed interface twist of length $w = 3$. The remaining four vertices represent open interface twists with $w = \frac{5}{2}$ of type $(I, +)$ and $(I, -)$ and $w = \frac{4}{2}$ of type $(I, <)$ and $(I, >)$, respectively. In each case the vertex is surrounded by $2w$ strands with alternating solid and dashed segments, with the choice of the first (and last) strand fixed by the type $\pm, \gtrless$ of the open twist.

- Splitting vertices are introduced. A splitting vertex is surrounded by two solid (dashed) strands, where two outgoing lines form a double line. See Fig. 5.6 for example.
- A closed w -twist insertion on the interface is represented by a vertex surrounded by w solid and w dashed strands. See the leftmost figure in Fig. 5.7 for example.
- An open w -twist insertion on the interface is represented by a vertex surrounded by $2w$ strands which, when one goes around the vertex, appear in an alternating pattern of solid and dashed lines. The precise choice of whether the sequence starts (and hence ends) on a solid or on a dashed strand, and which bulk side the first strand is attached to, is fixed by the type of the open twist $(\pm, \gtrless)$. See the four rightmost figures in Fig. 5.7 for example.
- A propagator inside Σ is represented by a double line, while a propagator along the boundary of Σ is a single line.
- Each connected component of the solid (respectively dashed) line is a contractible closed curve on Σ .

- Traversing a solid line counter-clockwise, one encounters the twist insertions and the redundant vertices in the partial cyclic order as on the base sphere X .
- Traversing a dashed line clockwise, one again encounters the twist insertions and the redundant vertices in the partial cyclic order as on X .
- The numbers of solid and dashed closed curves are each equal to the degree of the covering map, N_+ and N_- , respectively.

In the following we will illustrate the above rules using explicit examples of generalized PRR diagrams. All such diagrams will be shown in the simplified form obtained by contracting redundant vertices, in the sense explained around Fig. 2.8.

With this convention, the graphical rules introduced here reduce exactly to the original PRR rules in the special case $N_{\pm} = P$ with no open twist insertions, so that our generalized PRR diagrams indeed extend the original PRR diagrams.

Fixing transmissivity by t^* on the interface.

The generalized PRR rules described above determine the covering $\Gamma : \Sigma \rightarrow X = S^2$ and its degrees (N_+, N_-) , but they do not yet fix how many sheets are actually transmitted across the interface, i.e. the degree of transmissivity P .

To distinguish these possibilities and to implement the choice of transmissivity P in the diagrammatics, it is useful to introduce a reference point t^* on the double line of $X = S^2$. We choose t^* such that it does not coincide with any interface changing operator. On the covering surface Σ , or equivalently on the generalized PRR diagram drawn on Σ , we then mark all preimages of t^* under Γ by black dots.

As in the definition of the local degree of transmissivity, we regard only those preimages that lie in the interior of Σ as transmitted sheets. We therefore define

$$P(\Gamma) := \left| \Gamma^{-1}(t^*) \cap \overset{\circ}{\Sigma} \right| \quad (5.42)$$

to be the number of black dots in the interior of Σ .

For a fixed choice of transmissivity P we then restrict the sum over coverings (or over generalized PRR diagrams) to those Γ for which

$$P(\Gamma) = P. \quad (5.43)$$

Equivalently, at the level of the generalized PRR rules, this amounts to augmenting the list of local rules above by the following two items:

- On each closed solid or dashed circle in the generalized PRR diagram, place black dots along the propagators so that, when one goes once around the circle,

the black dots appear in the same cyclic order as the vertices on the base sphere and each circle is crossed exactly once by a black dot.

- Among all generalized PRR diagrams compatible with the local rules above, only those in which the total number of black dots on the double line in the interior of Σ equals P are kept in the interface correlator with degree of transmissivity P .

Disconnected covering surface and remaining gauge group.

In the presence of an interface, the completed generalized PRR diagram is again realized on a (possibly) disconnected Riemann surface Σ that is identified as the covering surface. For a given interface covering of degrees (N_+, N_-) and the degree of transmissivity P at t^* , the covering surface naturally decomposes into trivial and non-trivial components.

Let $\Sigma_{\text{nontrivial}}$ denote the union of all connected components that carry at least one twist or interface insertion. We denote by

$$\Gamma_{\pm}|_{\Sigma_{\text{nontrivial}}} : \Sigma_{\text{nontrivial}} \rightarrow X_{\pm} = \mathbb{D}_{\pm}, \quad (5.44)$$

the restrictions of the covering map to the two sides of the interface, with degrees $n_{\pm} := \deg(\Gamma_{\pm}|_{\Sigma_{\text{nontrivial}}})$. Among these $n_{\pm}$ sheets, we assume that the restriction $\Gamma|_{\Sigma_{\text{nontrivial}}}$ have the degree of transmissivity p . It is convenient to define

$$B_{\pm} := N_{\pm} - P, \quad b_{\pm} := n_{\pm} - p, \quad (5.45)$$

so that $B_{\pm}$ measures the total number of sheets that are *not* transmitting on side $\pm$, while $b_{\pm}$ counts those among them that belong to $\Sigma_{\text{nontrivial}}$. The remaining

$$B_{\pm} - b_{\pm} = (N_{\pm} - P) - (n_{\pm} - p) \quad (5.46)$$

sheets give rise to trivial disk components on side $\pm$.

In addition, there may be sheets that are transmitting but do not carry any insertions at all; these contribute trivial spherical components whose interface is entirely unexcited. Collecting all pieces, we can write the covering surface as

$$\Sigma = \left(S_{\text{trivial}}^2\right)^{\sqcup(P-p)} \sqcup \left(\mathbb{D}_{-}^{\text{trivial}}\right)^{\sqcup(B_{-}-b_{-})} \sqcup \left(\mathbb{D}_{+}^{\text{trivial}}\right)^{\sqcup(B_{+}-b_{+})} \sqcup \Sigma_{\text{nontrivial}}. \quad (5.47)$$

On the trivial components, the generalized PRR diagram reduces to a single double-line circle (for S_{trivial}^2) or a single-line circle on a disk (for $\mathbb{D}_{\pm}^{\text{trivial}}$), and these components

do not carry any non-trivial dynamical information. Thus, these trivial components in the generalized PRR diagram will be systematically omitted in what follows.

Nevertheless, they contribute to the remaining gauge group of the covering, i.e. to the automorphism group

$$\text{Aut}(\Gamma) \equiv \left\{ h : \Sigma \rightarrow \Sigma \mid \Gamma \circ h = \Gamma \right\}. \quad (5.48)$$

Since the $(P - p)$ trivial spheres S_{trivial}^2 and the $(B_{\pm} - b_{\pm})$ trivial disks $\mathbb{D}_{\pm}^{\text{trivial}}$ are indistinguishable among themselves, any permutation of these components defines an automorphism of the covering. Therefore, the automorphism group factorizes as

$$\text{Aut}(\Gamma) = S_{P-p} \times S_{B_- - b_-} \times S_{B_+ - b_+} \times \text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}}), \quad (5.49)$$

and the symmetry factor can be written as

$$|\text{Aut}(\Gamma)| = (P - p)! (B_- - b_-)! (B_+ - b_+)! \times |\text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}})|. \quad (5.50)$$

The remaining factor $\text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}})$ encodes genuine symmetries of the non-trivial part of the covering: cyclic permutations of sheets around identical twist insertions, permutations of identical connected components, and similar discrete symmetries. These are of the same type as in the bulk case discussed in Sec. 2.5.2. In practice, all such factors can be read off directly from the generalized PRR diagram.

Using this decomposition, let us see how eq. (5.40) can be written explicitly. It takes the following form:

$$\begin{aligned} & \left\langle \mathcal{I}_{a_{\pm}}^P \prod O_{w_i} \prod O_{w_j}^I \right\rangle_{N_{\pm}} \\ &= \sum_{\Gamma} \frac{P! B_-! B_+!}{(P - p)! (B_- - b_-)! (B_+ - b_+)!} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma_{\text{nontrivial}}} [\prod O_i \prod O_j^I]}{|\text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}})| g_-^{b_-} g_+^{b_+} z^p}. \end{aligned} \quad (5.51)$$

Here, all of the $N_{\pm}$ - and P -dependence is contained in the prefactor

$$\frac{P! B_-! B_+!}{(P - p)! (B_- - b_-)! (B_+ - b_+)!},$$

while the remaining factor is independent of $N_{\pm}$ and P .

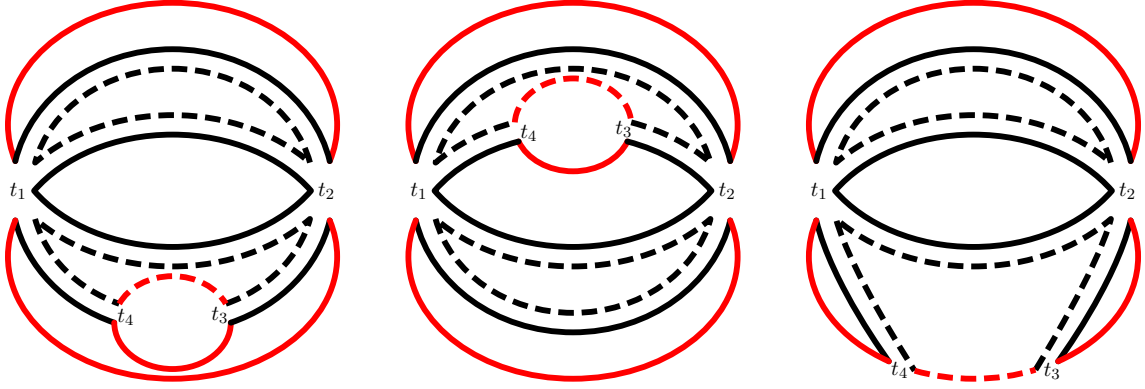


Figure 5.8.: Generalized PRR diagrams on the covering surface Σ for the interface four-point function $\langle \sigma_{\frac{5}{2}}^{I,+}(t_1) \sigma_{\frac{5}{2}}^{I,+}(t_2) \sigma_{\frac{2}{2}}^{I,<}(t_3) \sigma_{\frac{2}{2}}^{I,>}(t_4) \rangle$. The three panels show the three inequivalent connected generalized PRR diagrams compatible with the local rules. For visual clarity, single-line propagators (which represent boundary segments of Σ arising from splitting vertices) are drawn in red lines. The rightmost covering has degrees $(N_+, N_-) = (4, 2)$, while others have $(N_+, N_-) = (3, 2)$

Exapmle: $\langle \sigma_{\frac{5}{2}}^{I,+} \sigma_{\frac{5}{2}}^{I,+} \sigma_{\frac{2}{2}}^{I,<} \sigma_{\frac{2}{2}}^{I,>} \rangle$.

As an explicit example of the generalized PRR rules with an interface, let us consider the four-point function

$$\langle \sigma_{\frac{5}{2}}^{I,+}(t_1) \sigma_{\frac{5}{2}}^{I,+}(t_2) \sigma_{\frac{2}{2}}^{I,<}(t_3) \sigma_{\frac{2}{2}}^{I,>}(t_4) \rangle, \quad (5.52)$$

with the insertions ordered as t_1, t_2, t_3, t_4 along the interface. Figure 5.8 shows the corresponding generalized PRR diagram on the covering surface Σ . Figure 5.9 displays the same diagram after placing black dots in the order t^*, t_1, t_2, t_3, t_4 , so that the degree of transmissivity P can be read off directly from the diagram.

Gauge fixing of twists and interface.

On the sphere, it is often useful to separate explicitly the overall gauge volume in order to count inequivalent gauge fixings. For a given covering Γ , the contribution to the unnormalized correlator can be written schematically as

$$Z_{\text{Sym}^{N_{\pm}\mathcal{M}}}^{S^2} [\mathcal{I}_{a_{\pm}}^P(t^*) \prod O_{w_i} \prod O_{w_j}^I] = \frac{1}{N_-!N_+!} \sum_{\Gamma} \frac{N_-!N_+!}{|\text{Aut}(\Gamma)|} \hat{Z}_{\mathcal{M}}^{\Sigma} [\prod O_i \prod O_j^I], \quad (5.53)$$

where the factor $N_-!N_+!/|\text{Aut}(\Gamma)|$ can be interpreted as the number of essentially different gauge fixings of the objects $\mathcal{I}_{a_{\pm}}^P, O_w, O_w^I$ that give rise to the same covering Γ .

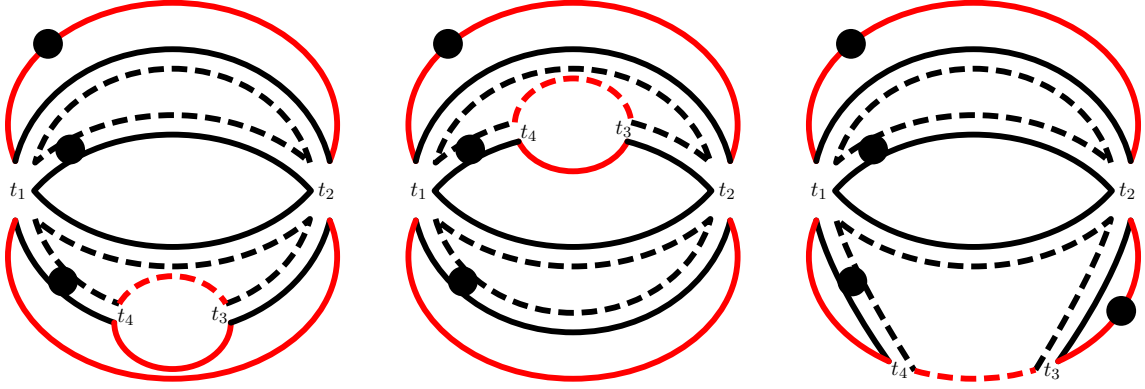


Figure 5.9.: The same generalized PRR diagram as in Fig. 5.8, with black dots implementing the choice of a reference point t^* on the interface. All contributions have $P = 2$.

First, to see the gauge-fixing through the generalized PRR diagrammatics, we bijectively assign integers in $\underline{N}_+ := \{1, \dots, N_+\}$ to solid circles, and $\underline{N}_- := \{1, \dots, N_-\}$ to dashed circles, in the diagram. We call a generalized PRR diagram whose circles are labeled in such a way *gauge fixed*. To give a more detailed description, it is useful to introduce different variables a and j to denote elements of $\underline{N}_+$ and $\underline{N}_-$, respectively: $a \in \underline{N}_+$, $j \in \underline{N}_-$. Note that, in order to see the gauge-fixing of the interface itself, we also need to fix the reference point t^* on it. For each object $\mathcal{I}_{a\pm}^P(t^*)$, O_w , O_w^I , one can assign gauge-fixed configuration in the following way:

- For each vertex corresponding to the bulk twist $\sigma_w^\pm$, we assign the permutation $\mathbf{g} = (a_1 \dots a_w) \in S_{N_+}$ or $\mathbf{g} = (j_1 \dots j_w) \in S_{N_-}$ by circumnavigating the associated vertex of the diagram anti-clockwise and reading off the encountered $\underline{N}_\pm$ labels: $(a_i)_{i=1}^w$ of the traversed solid lines for σ^+ , or $(j_i)_{i=1}^w$ of the traversed dashed lines for σ^- , respectively.
- For each vertex corresponding to the closed interface twist σ_w^I , we assign the permutation $\mathbf{g} = (a_1 j_1 \dots a_w j_w) \in S_{N_++N_-}$ by circumnavigating the associated vertex of the diagram anti-clockwise and reading off the encountered labels of $\underline{N}_+ \sqcup \underline{N}_-$ in the order they appear.
- For the interface $\mathcal{I}_{a\pm}^P(t^*)$, we assign the permutation $\boldsymbol{\iota} = (a_1 j_1) \dots (a_P j_P) \in S_{N_++N_-}$ by reading off the labels of P double lines on which the preimage of t^* locates. These P transpositions encode which sheets are transmitted across the interface.
- For each vertex corresponding to the open interface twist $\sigma_w^{I,*}$, we assign the sequence $\boldsymbol{\gamma}^*$ by circumnavigating the associated vertex of the diagram anti-

clockwise and reading off the encountered labels. More precisely,

$$\gamma^+ = (a_1 j_1 \dots j_{w-1/2} a_{w+1/2}), \quad (5.54a)$$

$$\gamma^- = (j_1 a_1 \dots a_{w-1/2} j_{w+1/2}), \quad (5.54b)$$

$$\gamma^< = (a_1 j_1 \dots a_w j_w), \quad (5.54c)$$

$$\gamma^> = (j_1 a_1 \dots j_w a_w). \quad (5.54d)$$

Here the superscript $*, = +, -, <, >$ labels the type of the open interface twist and fixes the alternating pattern of a 's and j 's as respecting the type of strands shown in Fig. 5.7.

From the orbit–stabilizer theorem, the prefactor in (5.53) can be interpreted as the number of inequivalent gauge fixings of the corresponding generalized PRR diagram:

$$\frac{N_+! N_-!}{|\text{Aut}(\Gamma)|} = \left| [\mathbf{g}_1, \dots, \mathbf{g}_{n_c}, \boldsymbol{\iota}, \boldsymbol{\gamma}_1^*, \dots, \boldsymbol{\gamma}_{n_o}^*] \right|, \quad (5.55)$$

where the right-hand side is the size of the orbit

$$\begin{aligned} & [\mathbf{g}_1, \dots, \mathbf{g}_{n_c}, \boldsymbol{\iota}, \boldsymbol{\gamma}_1^*, \dots, \boldsymbol{\gamma}_{n_o}^*] \\ & := \left\{ (g \mathbf{g}_1 g^{-1}, \dots, g \mathbf{g}_{n_c} g^{-1}, g \boldsymbol{\iota} g^{-1}, g \cdot \boldsymbol{\gamma}_1^*, \dots, g \cdot \boldsymbol{\gamma}_{n_o}^*) \mid g \in S_{N_+} \times S_{N_-} \right\}. \end{aligned} \quad (5.56)$$

Here $S_{N_+} \times S_{N_-}$ acts on $\mathbf{g}_i$ and $\boldsymbol{\iota}$ by conjugation (via the natural embedding into the permutation group of all sheet labels), and acts on $\boldsymbol{\gamma}_j^*$ by permuting the labels appearing in the sequences $\boldsymbol{\gamma}_j^*$. Diagrammatically, this group action corresponds to the relabeling of solid and dashed circles in the gauge fixed PRR diagram.

In analogy with the bulk case, one may therefore regard the interface correlator as a sum over gauge-fixed data. Schematically, we can write

$$Z_{\text{Sym}^{N_{\pm} \mathcal{M}}}^{S^2} \left[\mathcal{I}_{a_{\pm}}^P(t^*) \prod_{i=1}^{n_c} O_{w_i} \prod_{j=1}^{n_o} O_{w_j}^{I, *j} \right] = \sum_{\mathbf{g}_i, \boldsymbol{\iota}, \boldsymbol{\gamma}_j^*} Z_{\text{Sym}^{N_{\pm} \mathcal{M}}}^{S^2} \left[\mathcal{I}_{a_{\pm}}^{(\boldsymbol{\iota})}(t^*) \prod_{i=1}^{n_c} O_{\mathbf{g}_i} \prod_{j=1}^{n_o} O_{\boldsymbol{\gamma}_j^*}^{I, *j} \right], \quad (5.57)$$

where we understand that incompatible choices of $\mathbf{g}_i, \boldsymbol{\iota}, \boldsymbol{\gamma}_j^*$ give vanishing contributions, as in the bulk case.

A slightly nontrivial consequence of this description appears for the *normalized* sphere correlator (5.40). In the normalized convention, the vacuum partition function with interface in the denominator is computed with the same choice of transmissivity as in the numerator. As a result, global relabelings that change the interface permutation $\boldsymbol{\iota}$ contribute equally to numerator and denominator and cancel out. Equivalently, one may fix a reference interface permutation $\boldsymbol{\iota}_0$ and rewrite the normalized correlator as

a sum over gauge-fixed data with ι held fixed:

$$\left\langle \mathcal{I}_{a_{\pm}}^P \prod_{i=1}^{n_c} O_{w_i} \prod_{j=1}^{n_o} O_{w_j}^{I, *j} \right\rangle_{N_{\pm}} = \sum_{\mathbf{g}_i, \boldsymbol{\gamma}_j^*} \left\langle \mathcal{I}_{a_{\pm}}^{(\iota_0)}(t^*) \prod_{i=1}^{n_c} O_{\mathbf{g}_i} \prod_{j=1}^{n_o} O_{\boldsymbol{\gamma}_j^*}^{I, *j} \right\rangle_{N_{\pm}}. \quad (5.58)$$

The summation over $\mathbf{g}_i$ and $\boldsymbol{\gamma}_j^*$ completely reproduces the factor in the normalized correlator (5.41) i.e.

$$\frac{(N_- - P)! (N_+ - P)! P!}{|\text{Aut}(\Gamma)|}.$$

5.3.4. Covering-space prescription with interface

We now state the interface/boundary extension of the Lunin–Mathur covering-space prescription, following Section 4 of [4]. Since the bulk ingredients have been reviewed in Section 2.5.4, we only highlight the new structural element.

For an interface correlator on a base surface X with a boundary segment supporting the interface, the covering-surface expansion again reduces the computation to the seed theory on a covering surface $\Gamma : \Sigma \rightarrow X$. The only conceptual change is that Σ is generically a surface with boundary $\partial\Sigma = \Gamma^{-1}(\partial X)$, and therefore the Weyl-anomaly factor relating the physical pullback metric Γ^*g to a smooth reference metric $\hat{g}$ acquires an additional boundary contribution. Concretely, writing $\Gamma^*g = e^{\phi} \hat{g}$, the seed-theory correlator satisfies

$$Z_{\mathcal{M}}^{\Sigma}[\Gamma^*g; \dots] = \exp\left(S_L^{(\text{bdy})}[\phi; \hat{g}]\right) Z_{\mathcal{M}}^{\Sigma}[\hat{g}; \dots], \quad (5.59)$$

where $S_L^{(\text{bdy})}$ is the Liouville functional appropriate to a surface with boundary,

$$S_L^{(\text{bdy})}[\phi; \hat{g}] = \frac{c}{96\pi} \left[\int_{\Sigma} d^2z \sqrt{\hat{g}} \left(\hat{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + 2\hat{R} \phi \right) + 4 \int_{\partial\Sigma} ds \sqrt{\hat{h}} \hat{K} \phi \right]. \quad (5.60)$$

Here $\hat{h}$ is the induced metric on $\partial\Sigma$ and $\hat{K}$ its extrinsic curvature (both computed in the reference geometry). Equation (5.60) is precisely the modification emphasized in [4]: relative to the bulk discussion of Section 2.5.4, one simply adds the boundary term to the Liouville action.

With a fixed UV/IR regularization scheme and fixed local counterterms/normalizations (chosen uniformly for the seed theory), the dependence of a given interface covering Γ again localizes on the regulated boundary components and small circles/semi-circles around ramification points, and can be expressed in terms of the corresponding local covering data. In the following we therefore implement the interface covering-space prescription by using the same definition of $\hat{Z}_{\mathcal{M}}^{\Sigma}$ as in (2.131), but replacing S_L by $S_L^{(\text{bdy})}$ as in (5.60). See [4] for the explicit evaluation and normalization choices in

the interface setup.

5.3.5. Normalization of twist operators

We now fix the normalization factors $\mathcal{N}$ of interface twist operators for a fixed choice of $N_{\pm}$ and P :

$$\hat{\sigma}_w^c := \mathcal{N}_w^c \sigma_w^c, \quad \hat{\sigma}_w^{I,*} := \mathcal{N}_w^{I,*} \sigma_w^{I,*}, \quad (5.61a)$$

where $c \in \{+, -, I\}$ labels the type of closed twists, while $* \in \{\pm, \gtrless\}$ labels the type of open interface twists. We first make explicit the standard normalization of (closed) twists. Subsequently, we discuss the normalization of open interface twists. We choose a normalization in which two-point functions are normalized to one. Finally, we comment on caveats that arise when introducing the normalization of interface twists in actual correlation functions; the normalization must be evaluated with the appropriate local value of P .

Bulk twists and closed interface twists.

For bulk twists, the normalization factor is determined by the local gauge group $S_{N_{\pm}}$ and takes the form

$$\hat{\sigma}_w^{\pm} = \mathcal{N}_w^{\pm} \sigma_w^{\pm} = \sqrt{\frac{w(N_{\pm} - w)!}{N_{\pm}!}} \sigma_w^{\pm}. \quad (5.62)$$

For closed interface twists, we similarly have

$$\hat{\sigma}_w^I(t) = \mathcal{N}_w^I \sigma_w^I(t) = \sqrt{\frac{w(P - w)!}{P!}} \sigma_w^I(t). \quad (5.63)$$

The normalization factors shown above have a similar form, but their dependence on the operator location differs.

Open interface twists.

In the case of open interface twists $\hat{\sigma}_w^{I,*}$, the normalization constant is fixed so that the coefficient of corresponding two-point function is equal to one. When $* = \pm$, the two-point function $\langle \mathcal{I}^P \hat{\sigma}_w^{I,\pm} \hat{\sigma}_w^{I,\pm} \rangle$ is easily determined: the covering map Γ is unique, and corresponding nontrivial component $\Gamma_{\text{nontrivial}}$ satisfies $|\text{Aut}[\Gamma_{\text{nontrivial}}]| = 1$. We

find

$$\hat{\sigma}_w^{I,+} = \mathcal{N}_w^{I,+} \sigma_w^{I,+} = \sqrt{\frac{(P - (w - \frac{1}{2}))!}{(N_+ - P) P!}} \sigma_w^{I,+}, \quad (5.64)$$

$$\hat{\sigma}_w^{I,-} = \mathcal{N}_w^{I,-} \sigma_w^{I,-} = \sqrt{\frac{(P - (w - \frac{1}{2}))!}{(N_- - P) P!}} \sigma_w^{I,-}, \quad (5.65)$$

where w is supposed to be a half-integer $w \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$.

In the case $* = \gtrless$, there is an ambiguity in fixing the normalization. The nonzero two-point functions are

$$\langle \mathcal{I}^P(t^*) \sigma_w^{I,<}(t_1) \sigma_w^{I,>}(t_2) \rangle, \quad (5.66a)$$

$$\langle \mathcal{I}^P(t^*) \sigma_w^{I,>}(t_1) \sigma_w^{I,<}(t_2) \rangle, \quad (5.66b)$$

and, for generic values of $N_{\pm}$ and P , they cannot be normalized simultaneously¹. Let us normalize the former two-point function; this choice results in

$$\hat{\sigma}_w^{I,\gtrless} = \mathcal{N}_w^{I,\gtrless} \sigma_w^{I,\gtrless} = \sqrt{\frac{(P - w)!}{P!}} \sigma_w^{I,\gtrless}, \quad (5.67)$$

with w as a positive integer. With this normalization, the latter two-point function is related to the former as

$$\langle \mathcal{I}^P(t^*) \hat{\sigma}_w^{I,>}(t_1) \hat{\sigma}_w^{I,<}(t_2) \rangle = \frac{(N_+ - P)(N_- - P)}{P + 1} \langle \mathcal{I}^P(t^*) \hat{\sigma}_w^{I,<}(t_1) \hat{\sigma}_w^{I,>}(t_2) \rangle. \quad (5.68)$$

Interface twists and the local value of P .

It is important to stress that the parameter P entering the normalization factors has to be understood as a *local* quantity along the interface. In particular, whenever an open interface twist $\sigma_w^{I,\gtrless}(t)$ is inserted, the value of $P(t)$ jumps across t , as shown in the right panel of Fig. 5.4. In all of the above formulas, the symbol P should therefore be interpreted as this local multiplicity evaluated at the insertion point.

At points where $P(t)$ is discontinuous, such as the insertion points of $\sigma_w^{I,\gtrless}$ itself, we adopt the convention

$$P(t) = \max(P(t - \epsilon), P(t + \epsilon)). \quad (5.69)$$

¹In the first correlator, the operator $\sigma_w^{I,<}$ locally changes the interface from $\mathcal{I}^P$ to $\mathcal{I}^{P-1}$, while $\sigma_w^{I,>}$ brings it back from $\mathcal{I}^{P-1}$ to $\mathcal{I}^P$. In contrast, in the second correlator, the intermediate interface is $\mathcal{I}^{P+1}$. This difference in the intermediate interface, $\mathcal{I}^{P-1}$ versus $\mathcal{I}^{P+1}$, is exactly reflected in the combinatorial factors.

That is, $P(t)$ is taken to be the larger of the two values immediately to the left and right of the insertion. This prescription ensures that the normalization of $\sigma_w^{I,*}(t)$ always uses the correct local value of P appropriate to the interface configuration at that point.

5.3.6. Large- N limit and genus expansion with interface

In this subsection we analyze the large- N behaviour of correlation functions of *normalized* twist operators in the presence of an interface, and show that their N -dependence organizes itself into a standard string-theoretic genus expansion. Using the decomposition of the covering surface and the remaining gauge group in the previous subsection, the normalized correlator in the presence of an interface can be written as

$$\left\langle \mathcal{I}_{a_{\pm}}^P \prod_{i=1}^{n_c} \hat{\sigma}_{w_i}^{c_i}(x_i) \prod_{j=1}^{n_o} \hat{\sigma}_{w_j}^{I,*j}(t_j) \right\rangle_{N_{\pm}} = \sum_{\Gamma} C_{\Gamma}(N_{\pm}, P) \frac{\hat{Z}_{\mathcal{M}}^{\Sigma_{\text{nontrivial}}}}{|\text{Aut}(\Gamma|_{\Sigma_{\text{nontrivial}}})| g_-^{b_-} g_+^{b_+} z^p}, \quad (5.70)$$

where:

- $c_i \in \{+, -, I\}$ and $*_j \in \{\pm, \geq\}$,
- $n_{\pm}$ is the degree of $\Gamma|_{\Sigma_{\text{nontrivial}}}$,
- p is the degree of transmissivity of the nontrivial covering $\Sigma_{\text{nontrivial}}$,
- $B_{\pm} := N_{\pm} - P$ and $b_{\pm} := n_{\pm} - p$ are as in the previous subsection,
- $C_{\Gamma}(N_{\pm}, P)$ collects all the $N_{\pm}$ - and P -dependence of its contribution:

$$C_{\Gamma}(N_{\pm}, P) = \frac{P! B_-! B_+!}{(P-p)!(B_- - b_-)!(B_+ - b_+)!} \prod_{i=1}^{n_c} \mathcal{N}_{w_i}^{c_i} \prod_{j=1}^{n_o} \mathcal{N}_{w_j}^{I,*j}. \quad (5.71)$$

Large- N scalings.

We now adopt the following large- N scaling:

$$N_{\pm} \sim N, \quad P \sim N, \quad B_{\pm} := N_{\pm} - P \sim \sqrt{N}, \quad (5.72)$$

so that the number $B_{\pm}$ of non-transmitting copies on each side is of order $\sqrt{N}$, while the number P of transmitting copies is of order N . This scaling is naturally singled out by requiring that normalized two-point functions of open twists (or, more generally, of open multi-particle excitations) remain of order $O(1)$ and do not depend on the interface insertion point t^* ; see eq. (5.68).

Under these assumptions, the explicit formulas in Sec. 5.3.5 show that, for any twist (bulk, closed interface, or open interface), the normalization factor scales universally

as

$$\mathcal{N}_w \sim N^{-\frac{w}{2}}. \quad (5.73a)$$

We denote the sizes of closed twists by w_i ($i = 1, \dots, n_c$) and those of open interface twists by w_j ($j = 1, \dots, n_o$). The product of normalization factors then behaves as

$$\prod_{i=1}^{n_c} \mathcal{N}_{w_i}^{c_i} \prod_{j=1}^{n_o} \mathcal{N}_{w_j}^{I,*j} \sim N^{-\frac{1}{2} \sum_{i=1}^{n_c} w_i - \frac{1}{2} \sum_{j=1}^{n_o} w_j}. \quad (5.73b)$$

The remaining combinatorial factor contributes an additional power of N :

$$\frac{P! B_-! B_+!}{(P-p)!(B_- - b_-)!(B_+ - b_+)!} \sim N^p (\sqrt{N})^{b_- + b_+} = N^{\frac{n_- + n_+}{2}}. \quad (5.74)$$

Thus, the N -dependence of the contribution from a fixed covering Γ is

$$C_\Gamma(N_\pm, P) \sim N^{\frac{1}{2} \left(n_- + n_+ - \sum_{i=1}^{n_c} w_i - \sum_{j=1}^{n_o} w_j \right)}. \quad (5.75)$$

Riemann-Hurwitz formula and genus expansion.

We now relate the quantities $n_\pm$, the twist lengths w_i, w_j , and the numbers of insertions (n_c, n_o) to the topology of the covering surface. Let the nontrivial part of the covering $\Sigma_{\text{nontrivial}}$ have genus g and b boundary components. Following the mirroring trick, we take the mirror image $\bar{\Gamma}$ of Γ and glue Γ and $\bar{\Gamma}$ along every boundary component. The resulting doubled surface has the following properties:

- its degree is $n_{\text{dbl}} = 2p + b_- + b_+ = n_- + n_+$,
- its genus is $g_{\text{dbl}} = 2g + b - 1$,
- it carries $n_{c,\text{dbl}} = 2n_c + n_o$ closed twist insertions in total;
- each closed twist $\sigma_{w_i}^{c_i}$ of the original covering appears twice with size $w = w_i$, while each open interface twist $\sigma_{w_j}^{I,*j}$ gives rise to a single closed twist of size $2w_j$ on the doubled surface.

Hence, on the doubled covering surface the sum of ramification indices is

$$\sum_{\text{dbl}} w = 2 \sum_{i=1}^{n_c} w_i + \sum_{j=1}^{n_o} 2w_j. \quad (5.76)$$

The Riemann-Hurwitz formula for this doubled surface $n_{\text{dbl}} - \frac{1}{2} \sum_{\text{dbl}} w = 1 - g_{\text{dbl}} - \frac{n_{c,\text{dbl}}}{2}$ gives

$$\begin{aligned} (n_- + n_+) - \frac{1}{2} \left(2 \sum_{i=1}^{n_c} w_i + \sum_{j=1}^{n_o} 2w_j \right) &= 1 - (2g + b - 1) - \frac{2n_c + n_o}{2} \\ \implies \frac{1}{2} \left(n_- + n_+ - \sum_{i=1}^{n_c} w_i - \sum_{j=1}^{n_o} w_j \right) &= \frac{1}{2} \left(2 - 2g - b - n_c - \frac{n_o}{2} \right). \end{aligned} \quad (5.77)$$

Here we are implicitly focusing on a single connected nontrivial component of the covering surface. For coverings whose nontrivial part decomposes into several connected components, the Riemann-Hurwitz formula and the large- N counting apply to each component separately, and the total contribution is obtained by taking the product over components.

Combining (5.75) with the topological identity (5.77), we obtain

$$C_{\Gamma}(N_{\pm}, P) \sim N^{\frac{1}{2}(2-2g-b-n_c-\frac{n_o}{2})}. \quad (5.78)$$

Introducing the string coupling via $g_s^2 \sim N^{-1}$, this scaling can be rewritten as

$$C_{\Gamma}(N_{\pm}, P) \sim g_s^{-2+2g+b+n_c+\frac{n_o}{2}}, \quad (5.79)$$

In particular, the dependence on the degree of the covering map, the twist lengths, and related data has reorganized itself into the simple topological data (g, b, n_c, n_o) through the combination $-2 + 2g + b + n_c + \frac{n_o}{2}$. It is reassuring (and somewhat non-trivial) that the interface generalization of the symmetric product orbifold still exhibits precisely the string-theoretic genus counting expected from this topology. Note that this simplification relies on the scaling assumption (5.72); conversely, requiring such a genus expansion constrains the allowed large- N behavior of $(N_{\pm}, P)$. In particular, configurations such as $N_+ \sim N$ with $N_- = P = 0$ are incompatible with (5.79) and therefore do not belong to this stringy large- N regime.

6. Interfaces in Tensionless Holography

6.1. Short review of tensionless holography on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$

In this section we briefly review the current understanding of tensionless holography, namely type IIB superstring theory on

$$\text{AdS}_3 \times S^3 \times \mathbb{T}^4 \tag{6.1}$$

with pure, one unit $k = 1$ of NSNS three-form flux. We will always restrict attention to the $\mathbb{T}^4$ case and will not discuss the closely related backgrounds with $M_4 = K3$ or $M_4 = S^3 \times S^1$ in any detail.

The main message of this section is that, at the tensionless point $k = 1$, the bulk string theory and the boundary CFT match very precisely: their spectra and symmetry algebras agree, and this agreement extends to torus partition functions and, to a large extent, to correlation functions. In particular, string theory on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ with minimal NSNS flux admits an exact description in terms of the grand-canonical ensemble of free symmetric product orbifolds $\text{Sym}^N(\mathbb{T}^4)$, and many observables can be computed equivalently on either side of the duality.

This section is organized as follows: In the first subsection, we briefly review the generic pure-NSNS backgrounds at integer $k > 1$, emphasizing the role of the long-string continuum and the appearance of a symmetric product of an interacting $\mathcal{N} = 4$ Liouville-like theory tensored with $\mathbb{T}^4$.

In the second subsection, we then specialize to the minimal value $k = 1$, the “tensionless point”, and give a more detailed overview of the hybrid worldsheet description and its symmetric orbifold dual $\text{Sym}^N(\mathbb{T}^4)$. We explain how the matching of spectra, partition functions and correlation functions is realized in detail, and how the duality works for non-perturbative objects.

6.1.1. $\text{AdS}_3/\text{CFT}_2$ duality for pure NSNS strings at $k > 1$

We first recall the worldsheet description of string theory on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ with pure NSNS flux at generic integer level $k > 1$. In the conventional RNS formalism, the $\text{AdS}_3 \times S^3$ part is described by an $\mathcal{N} = 1$ super WZW model based on

$$\mathfrak{sl}(2, \mathbb{R})_k \oplus \mathfrak{su}(2)_k, \quad (6.2)$$

coupled to an $\mathcal{N} = 1$ supersymmetric sigma model on $\mathbb{T}^4$ and the usual superconformal ghosts [92, 93]. The level k is proportional to the NSNS three-form flux through S^3 and controls the radius of AdS_3 in string units $l_{\text{AdS}} = \sqrt{k}l_s$.

The central charge of the matter sector can be written as

$$c_{\text{matter}} = c_{\mathfrak{sl}(2, \mathbb{R})_k}^{\mathcal{N}=1} + c_{\mathfrak{su}(2)_k}^{\mathcal{N}=1} + c_{\mathbb{T}^4}^{\mathcal{N}=1}. \quad (6.3)$$

For the supersymmetric WZW models one finds

$$c_{\mathfrak{sl}(2, \mathbb{R})_k}^{\mathcal{N}=1} = \frac{3(k+2)}{k} + \frac{3}{2} = 3 + \frac{6}{k} + \frac{3}{2}, \quad (6.4)$$

$$c_{\mathfrak{su}(2)_k}^{\mathcal{N}=1} = \frac{3(k-2)}{k} + \frac{3}{2} = 3 - \frac{6}{k} + \frac{3}{2}, \quad (6.5)$$

where the first term comes from the bosonic currents and the second from the adjoint fermions, while the supersymmetric sigma model on $\mathbb{T}^4$ contributes $c_{\mathbb{T}^4}^{\mathcal{N}=1} = 6$. Altogether this gives $c_{\text{matter}} = 15$, as required for a critical type II superstring background.

Current understanding of correspondence.

For generic integer level $k > 1$, the $\text{AdS}_3/\text{CFT}_2$ duality with pure NSNS flux can be organized schematically as follows:

$$\begin{array}{c} \text{Type IIB on } \text{AdS}_3 \times S^3 \times \mathbb{T}^4 \text{ with } k > 1 \text{ NSNS flux} \\ \Updownarrow \\ \text{GCE of } \text{Sym}^N \left(\mathcal{N} = 4, c = 6(k-1) \text{ LD} \times \mathbb{T}^4 \right) + (\text{marginal deformation}). \end{array}$$

The evidence for this duality can be summarized as follows. First, at the level of the spectrum and symmetry algebra, the long-string states constructed from worldsheet DDF operators reproduce the single-particle spectrum and chiral algebra of the symmetric product orbifold of $\mathcal{N} = 4$ Liouville $\times \mathbb{T}^4$, i.e. of an $\mathcal{N} = 4$ linear dilaton $\times \mathbb{T}^4$ with a marginal deformation by the untwisted Liouville screening charge, as proposed in [94]. Additional evidence for the Liouville factor in the seed theory comes from [95].

Second, there is strong evidence from correlation functions: various classes of three- and four-point functions have been shown to agree with the CFT proposal [96, 97, 98], and an even broader class of correlators has been matched in a purely bosonic setup in [99]. These works focus primarily on the long-string sector, but to some extent, they also manage to probe the short-string sector. In these analyses, once one includes on the CFT side the twist-two marginal deformation, which is the symmetric-product cousin of the Liouville screening charge, the short-string sector appears as a family of bound states: they appear as discrete poles in the dependence of correlators on the linear-dilaton momentum. On the $\mathfrak{sl}(2, \mathbb{R})$ WZW side, this twist-two marginal deformation is realized, in the Wakimoto free-field representation, by spectrally-flowed screening (interaction) terms [99].

6.1.2. $\text{AdS}_3/\text{CFT}_2$ duality for the tensionless point $k = 1$

At $k = 1$, string theory on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ is widely viewed as a particularly clean realization of an $\text{AdS}_3/\text{CFT}_2$ duality, with the bulk described by the symmetric product orbifold $\text{Sym}^N(\mathbb{T}^4)$:

$$\begin{array}{c} \text{Type IIB on } \text{AdS}_3 \times S^3 \times \mathbb{T}^4 \text{ with } k = 1 \text{ NSNS flux} \\ \Downarrow \\ \text{GCE of } \text{Sym}^N \mathbb{T}^4. \end{array}$$

In this regime, the usual RNS treatment becomes technically unstable because the $\mathfrak{sl}(2, \mathbb{R})_k$ and $\mathfrak{su}(2)_k$ sectors are tightly coupled through their fermions and ghosts, and in the standard “bosonic WZW + fermions” bookkeeping, the bosonic $\mathfrak{su}(2)$ level is shifted to $k - 2$, i.e. $\mathfrak{su}(2)_{-1}$ at $k = 1$. This makes the standard representation-theoretic organization of states and the control of spectral-flow/long-string sectors in RNS unreliable in practice. Therefore, the *hybrid formalism* is used in the literature [100, 101, 102]: it packages the $\text{AdS}_3 \times S^3$ degrees of freedom into a $\text{PSU}(1, 1|2)_1$ WZW model as the worldsheet theory and replaces the RNS superghost machinery by a cohomological (BRST-like) definition of physical states. In practice, this is the framework in which many of computations and duality checks at $k = 1$ are carried out cleanly. See also [103, 104].

In the hybrid description at $k = 1$, the worldsheet CFT naturally splits into two parts: (i) a $\text{PSU}(1, 1|2)_k$ supergroup WZW model (with $k = 1$ here), capturing the $\text{AdS}_3 \times S^3$ geometry with manifest supersymmetry, and (ii) an internal $\mathbb{T}^4$ sector, which is most efficiently treated after a topological twist. Concretely, one selects an $\mathcal{N} = 2$ subalgebra inside the worldsheet supersymmetry (embedded in the small

$\mathcal{N} = 4$ structure) and modifies the stress tensor by a derivative of the associated $U(1)$ current. After this topological twist, one of the worldsheet supercurrents becomes a dimension-one BRST-like current. Physical vertex operators are then defined as cohomology classes of the corresponding BRST-like charge, and the remaining generators act naturally on the cohomology. This twist is a technical device: it aligns the internal $\mathbb{T}^4$ degrees of freedom with the cohomological physical-state condition and prepares a direct comparison with the organization of states and correlators on the symmetric orbifold CFT side.

A key simplification at $k = 1$ is that the relevant representation theory of the $\text{PSU}(1, 1|2)_1$ current algebra effectively excludes the long sectors that, at $k > 1$, are tied to continuous families of representations and to the long-string continuum in AdS_3 . The structural reason is that $k = 1$ is a highly degenerate point of the affine supergroup theory: null states proliferate, and the typical (long) modules that would normally generate continuous labels become incompatible with the hybrid physical-state conditions. Equivalently, once one imposes the hybrid BRST-like constraints, the would-be long/continuous contributions do not survive in the physical cohomology. The practical outcome is that the $k = 1$ spectrum in the $\text{PSU}(1, 1|2)$ sector organizes into discrete, short (atypical) representations, matching the expected tensionless string / symmetric orbifold structure.

Evidence for the $k = 1$ correspondence.

The claim that the pure-NSNS background at $k = 1$ is dual to the symmetric product orbifold $\text{Sym}^N(\mathbb{T}^4)$ is supported by several complementary tests described below.

Spectrum and symmetry. On the bulk side, the hybrid description in terms of the $\text{PSU}(1, 1|2)_1$ WZW model provides a controlled, cohomological definition of physical states. A detailed comparison shows that the resulting single-particle spectrum and symmetry algebra match those of the single-cycle sector of $\text{Sym}^N(\mathbb{T}^4)$, including the expected organization into short multiplets and the characteristic structure of twisted sectors. In particular, the absence (in the physical cohomology) of the long-string continuum at $k = 1$ aligns naturally with the purely discrete spectrum of the symmetric orbifold at the orbifold point [105, 106, 107].

Partition functions. At the level of one-loop/thermal data, the string partition function extracted from the hybrid worldsheet reproduces the partition function of the symmetric product theory, once the comparison is phrased in terms of the appropriate single-particle building blocks and their second-quantized lift [106, 108]. See further interpretations in [109].

Correlation functions. On the string side, a striking (and highly constraining) feature of the $k = 1$ hybrid worldsheet theory is that genus-zero n -point amplitudes do not involve a generic integral over the full moduli space: instead, the worldsheet correlators become *delta-function localized* onto those configurations that admit a holomorphic branched covering map from the worldsheet to the S^2 boundary of AdS_3 , with prescribed branching near the insertion points. In this way the moduli-space integral collapses to a finite sum over covering maps (with the appropriate Jacobian/measure factors determined on the worldsheet), making the tensionless theory unusually close to an exactly solvable point [110, 111]. This localization can be exhibited concretely using free-field realizations of the $\mathfrak{psu}(1,1|2)_1$ WZW model, which allow one to compute and constrain worldsheet correlators directly and to derive the branched-cover support of the amplitudes in a controlled way [111]. Moreover, the same mechanism extends beyond tree level: at higher genus, the correlators again localize onto worldsheets that cover the AdS_3 boundary holomorphically, reducing the genus- g moduli integral to a finite sum and providing a worldsheet route to organizing perturbative corrections order-by-order in genus [112, 113]. Finally, a complementary path-integral treatment makes the emergence of localization particularly transparent and enables explicit tree-level correlator computations (alongside explicit spectrally-flowed vertex operators and DDF operators) within a unified $k = 1$ framework [107].

Taken together, these results establish the $k = 1$ correspondence as a highly constrained and extensively tested example of $\text{AdS}_3/\text{CFT}_2$, even though a completely general proof at the level of arbitrary correlators is not yet available.

Defects as a complementary non-perturbative test.

Even when closed-string correlators are under good control, correlators alone do not exhaust the non-perturbative content of the theory. In AdS_3 , defects and D-branes provide especially sharp probes because they encode global, nonlocal constraints: they must satisfy consistency requirements (including factorization and modular properties of cylinder amplitudes) and must be compatible with the full operator algebra. On the CFT side, their holographic avatars are naturally formulated in terms of boundary states and defect data.

At $k = 1$, some class of D-branes i.e. spherical branes have been constructed and matched to candidate CFT duals with detailed amplitude-level checks in [50], indicating that branes are not an auxiliary decoration but an integral component of the same mechanism underlying the matching of closed-string observables. This motivates using branes as a complementary test of the duality in a genuinely non-perturbative sector.

The remainder of this chapter focuses on the AdS_2 brane, following [3, 4]. Our

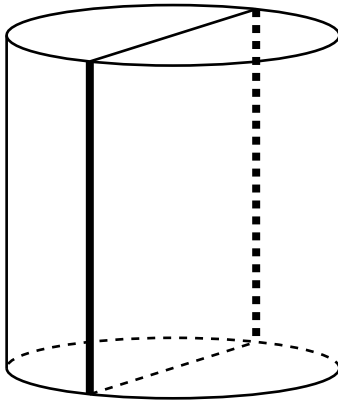


Figure 6.1.: AdS_2 brane in AdS_3 background. Endpoints of AdS_2 brane are located at the antipodal points.

goals are twofold. First, we review and sharpen the evidence that the AdS_2 brane is correctly captured by the symmetric orbifold description, in the same spirit as other $k = 1$ branes/defects whose matching is established in the literature. Second, we qualitatively test the duality at the level of open-string degrees of freedom on the CFT side. This provides a complementary line of evidence: the matching of a non-perturbative defect and its open-string sector imposes strong constraints and further supports the conclusion that the $k = 1$ tensionless duality is essentially correct.

6.2. Quantitative matching of interface partition functions

In this section we give a quantitative test of tensionless holography with D-branes by comparing the spacetime partition function of an AdS_2 brane in thermal $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ with the torus partition function of two interface insertions in the symmetric product orbifold. On the bulk side, the relevant quantity is obtained by computing the worldsheet annulus amplitude for open strings ending on the AdS_2 brane, following [3, 50]. Schematic figure is shown in Fig. 6.1. See the literature on AdS_2 branes in AdS_3 in [114, 115, 116, 117, 118, 119]. See also [120]. We will consider the partition function in thermal AdS_3 , in spirit of [108].

On the CFT side, we consider the grand-canonical ensemble of symmetric product orbifolds $\text{Sym}^{N_{\pm}}(\mathbb{T}^4)$ on the torus, with two interface insertions. The resulting torus partition function (5.22), constructed in Sec. 5.2.1, will be shown to agree exactly with the bulk AdS_2 brane partition function once we specialize the seed to $\mathbb{T}^4$ and choose matching boundary conditions.

Throughout this section we follow exactly the conventions of [50] for the worldsheet theory on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ and organize our computation in parallel to their analysis

of spherical D-branes. The only substantive difference is that we use the AdS_2 branes, as we done in [3].

6.2.1. Partition function for AdS_2 branes in AdS_3

In this subsection we compute the spacetime partition function associated with an AdS_2 brane in thermal $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ at the tensionless point $k = 1$.

The spectrum of $\text{PSU}(1,1|2)_1$ WZW model is built from *short multiplets* and its spectral flow. We denote the corresponding representation by $\mathcal{F}_\lambda$ and $\sigma^w(\mathcal{F}_\lambda)$, respectively. The Hilbert space can be schematically decomposed as

$$\mathcal{H}_{\text{WZW}} = \bigoplus_{w \in \mathbb{Z}} \int_{\lambda \in [0,1)} d\lambda \, \sigma^w(\mathcal{F}_\lambda) \otimes \overline{\sigma^w(\mathcal{F}_\lambda)}. \quad (6.6)$$

For each representation, its supercharacter takes the following form:

$$\begin{aligned} \tilde{\text{ch}}[\sigma^w(\mathcal{F}_\lambda)] &:= \text{Tr}_{\sigma^w(\mathcal{F}_\lambda)} \left[(-1)^F q^{L_0 - \frac{c}{24}} x^{J_0^3} y^{K_0^3} \right] \\ &= (-1)^w q^{\frac{w^2}{2}} \sum_{r \in \mathbb{Z} + \lambda} x^r q^{-rw} \frac{\theta_1(\frac{t+z}{2}|\tau) \theta_1(\frac{t-z}{2}|\tau)}{\eta(\tau)^4}, \end{aligned} \quad (6.7)$$

where $q = e^{2\pi i \tau}$ is the modular parameter, $x = e^{2\pi i t}$ is the $\mathfrak{sl}(2, \mathbb{R})$ fugacity, and $y = e^{2\pi i z}$ is the $\mathfrak{su}(2)$ fugacity.

Boundary states and closed-string channel.

We now consider Ishibashi states $|w, \lambda, \varepsilon\rangle\rangle$ associated with the representations $\mathcal{F}_\lambda$ and their spectral flows $\sigma^w(\mathcal{F}_\lambda)$. Here ε labels the gluing condition of fermionic modes. Following [50], the consistent boundary state for the AdS_2 brane is given by the following linear combination

$$|\varepsilon\rangle\rangle_A = \frac{1}{\sqrt{2}} \sum_{\lambda=0,1/2} e^{2\pi i(\lambda-1/2)\delta_A^L} |0, \lambda, \varepsilon\rangle\rangle. \quad (6.8)$$

Here, the index A labels which asymptotic end of the AdS_2 brane the boundary state lives on: $A \in \{L, R\}$. As in [50], the AdS_2 brane couples only to the unflowed ($w = 0$) short representations, so the boundary state involves $w = 0$ Ishibashi states only. Combining with a $\mathbb{T}^4$ sector boundary state and a ghost sector boundary state, the worldsheet boundary states take the form

$$|u, \varepsilon\rangle\rangle_A = |\varepsilon\rangle\rangle_A |u, R, \varepsilon\rangle\rangle_{\mathbb{T}^4} |\text{ghost}, \varepsilon\rangle\rangle. \quad (6.9)$$

We now consider the cylinder amplitude

$$\hat{Z}_{A,u|B,v}^{\text{WS}}(\hat{t}, \hat{z}; \hat{\tau}) := {}_A\langle\langle u, \mp \| \hat{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} \hat{x}^{\frac{1}{2}(J_0^3 - \bar{J}_0^3)} \hat{y}^{\frac{1}{2}(K_0^3 - \bar{K}_0^3)} \| v, \pm \rangle\rangle_B. \quad (6.10)$$

Each factor can be computed separately as

$$\hat{Z}_{A,u|B,v}^{\text{WS}}(\hat{t}, \hat{z}; \hat{\tau}) = \hat{Z}_{A|B}^{\text{PSU}}(\hat{t}, \hat{z}; \hat{\tau}) \hat{Z}_{u|v}^{\mathbb{T}^4}(\hat{\tau}) \hat{Z}^{\text{ghost}}(\hat{\tau}), \quad (6.11a)$$

$$\hat{Z}_{u|v}^{\mathbb{T}^4}(\hat{\tau}) = \frac{\theta_1(0|\hat{\tau})\theta_1(0|\hat{\tau})}{\eta(\hat{\tau})^6} \hat{\Theta}_{u|v}^{\mathbb{T}^4}(\hat{\tau}), \quad (6.11b)$$

$$\hat{Z}^{\text{ghost}}(\hat{\tau}) = \frac{\eta(\hat{\tau})^4}{\theta_1(0|\hat{\tau})\theta_1(0|\hat{\tau})}. \quad (6.11c)$$

The PSU(1, 1|2) part is given by

$$\begin{aligned} \hat{Z}_{A|B}^{\text{PSU}}(\hat{t}, \hat{z}; \hat{\tau}) &= \frac{1}{2} \sum_{\lambda=0,1/2} e^{2\pi i(1/2-\lambda)(\delta_A^L - \delta_B^L)} \text{ch}[\mathcal{F}_\lambda](\hat{t}, \hat{z}; \hat{\tau}) \\ &= \frac{1}{2} \sum_{\lambda=0,1/2} e^{2\pi i(1/2-\lambda)(\delta_A^L - \delta_B^L)} \hat{x}^\lambda \sum_{l \in \mathbb{Z}} \frac{\delta(\hat{t} - l)}{\eta(\hat{\tau})^4} \theta_1\left(\frac{\hat{t} + \hat{z}}{2} | \hat{\tau}\right) \theta_1\left(\frac{\hat{t} - \hat{z}}{2} | \hat{\tau}\right) \\ &= \begin{cases} + \sum_{l \in 2\mathbb{Z}_{>0}} \frac{\delta(\hat{t} - l)}{\eta(\hat{\tau})^4} \theta_1\left(+\frac{\hat{z}}{2} | \hat{\tau}\right) \theta_1\left(-\frac{\hat{z}}{2} | \hat{\tau}\right), & A = B, \\ - \sum_{l \in 2\mathbb{Z}_{>0}+1} \frac{\delta(\hat{t} - l)}{\eta(\hat{\tau})^4} \theta_2\left(+\frac{\hat{z}}{2} | \hat{\tau}\right) \theta_2\left(-\frac{\hat{z}}{2} | \hat{\tau}\right), & A \neq B. \end{cases} \end{aligned} \quad (6.12)$$

Combining all contributions, we obtain

$$\hat{Z}_{A,u|B,v}^{\text{WS}}(\hat{t}, \hat{z}; \hat{\tau}) = \begin{cases} + \sum_{l \in 2\mathbb{Z}_{>0}} \delta(\hat{t} - l) \hat{Z}_{u|v, \tilde{R}}^{\mathbb{T}^4}(\hat{z} | \hat{\tau}), & A = B, \\ - \sum_{l \in 2\mathbb{Z}_{>0}+1} \delta(\hat{t} - l) \hat{Z}_{u|v, R}^{\mathbb{T}^4}(\hat{z} | \hat{\tau}), & A \neq B, \end{cases} \quad (6.13)$$

where we defined $\hat{Z}_{u|v,*}^{\mathbb{T}^4}$ as

$$\hat{Z}_{u|v, \tilde{R}}^{\mathbb{T}^4}(\hat{z} | \hat{\tau}) = \frac{\theta_1\left(+\frac{\hat{z}}{2} | \hat{\tau}\right) \theta_1\left(-\frac{\hat{z}}{2} | \hat{\tau}\right)}{\eta(\hat{\tau})^6} \hat{\Theta}_{u|v}^{\mathbb{T}^4}(\hat{\tau}), \quad (6.14a)$$

$$\hat{Z}_{u|v, R}^{\mathbb{T}^4}(\hat{z} | \hat{\tau}) = \frac{\theta_2\left(+\frac{\hat{z}}{2} | \hat{\tau}\right) \theta_2\left(-\frac{\hat{z}}{2} | \hat{\tau}\right)}{\eta(\hat{\tau})^6} \hat{\Theta}_{u|v}^{\mathbb{T}^4}(\hat{\tau}), \quad (6.14b)$$

$$\hat{Z}_{u|v, \text{NS}}^{\mathbb{T}^4}(\hat{z} | \hat{\tau}) = \frac{\theta_3\left(+\frac{\hat{z}}{2} | \hat{\tau}\right) \theta_3\left(-\frac{\hat{z}}{2} | \hat{\tau}\right)}{\eta(\hat{\tau})^6} \hat{\Theta}_{u|v}^{\mathbb{T}^4}(\hat{\tau}), \quad (6.14c)$$

$$\hat{Z}_{u|v, \tilde{\text{NS}}}^{\mathbb{T}^4}(\hat{z} | \hat{\tau}) = \frac{\theta_4\left(+\frac{\hat{z}}{2} | \hat{\tau}\right) \theta_4\left(-\frac{\hat{z}}{2} | \hat{\tau}\right)}{\eta(\hat{\tau})^6} \hat{\Theta}_{u|v}^{\mathbb{T}^4}(\hat{\tau}). \quad (6.14d)$$

Open-string channel and partition functions.

Now we perform the modular transformation to obtain the open string annulus amplitude:

$$\hat{\tau} = -\frac{1}{\tau}, \quad \hat{t} = \frac{t}{\tau}, \quad \hat{z} = \frac{z}{\tau}. \quad (6.15)$$

The result is

$$Z_{A,u|B,v}^{\text{WS}}(t, z; \tau) = \begin{cases} - \sum_{l \in 2\mathbb{Z}_{>0}} \frac{\tau^2}{t} \delta\left(\tau - \frac{t}{l}\right) e^{\frac{\pi i z^2}{2\tau}} Z_{u|v, \tilde{\text{R}}}^{\mathbb{T}^4}(z|\tau), & A = B, \\ - \sum_{l \in 2\mathbb{Z}_{>0}+1} \frac{\tau^2}{t} \delta\left(\tau - \frac{t}{l}\right) e^{\frac{\pi i z^2}{2\tau}} Z_{u|v, \tilde{\text{NS}}}^{\mathbb{T}^4}(z|\tau), & A \neq B. \end{cases} \quad (6.16)$$

Here, even l correspond to open strings that begin and end on the same asymptotic end of the AdS_2 brane; they contribute with the $\tilde{\text{R}}$ spin structure. The minus sign comes from the modular properties of θ_1 . Odd l describe open strings stretching between the two ends and contribute with the $\tilde{\text{NS}}$ spin structure. Integrating over the worldsheet modulus τ yields the following expression for the open-string partition function:

$$\int \frac{d\tau}{\tau^2} Z_{A,u|B,v}^{\text{WS}}(t, z; \tau) = \begin{cases} -\frac{1}{t} \sum_{l \in 2\mathbb{Z}_{>0}} e^{\frac{\pi i l z^2}{2t}} Z_{u|v, \tilde{\text{R}}}^{\mathbb{T}^4}\left(z\left|\frac{t}{l}\right.\right), & A = B, \\ -\frac{1}{t} \sum_{l \in 2\mathbb{Z}_{>0}+1} e^{\frac{\pi i l z^2}{2t}} Z_{u|v, \tilde{\text{NS}}}^{\mathbb{T}^4}\left(z\left|\frac{t}{l}\right.\right), & A \neq B. \end{cases} \quad (6.17)$$

Thermal AdS_3 partition function from winding sums.

Following the general strategy of [108] and its open-string adaptation in [3], thermal AdS_3 is obtained by a discrete identification along the Euclidean time direction. On the worldsheet this amounts, in the open-string channel, to summing over images of the cylinder amplitude under winding of the open string around the thermal circle. In the present setup, where we focus only on the open-string contribution, this is implemented simply by replacing

$$t \longrightarrow kt, \quad z \longrightarrow kz, \quad (6.18)$$

in the global AdS_3 annulus amplitude (6.16) and summing over all windings $k \in \mathbb{Z}_{>0}$. Introducing an overall normalization factor $\mathcal{N}$, the thermal worldsheet amplitude is thus

$$Z_{A,u|B,v}^{\text{WS,th}}(t, z; \tau) := \mathcal{N} \sum_{k=1}^{\infty} Z_{A,u|B,v}^{\text{WS}}(kt, kz; \tau). \quad (6.19)$$

Using the explicit expression for the global amplitude, we obtain

$$Z_{A,u|B,v}^{\text{WS,th}}(t, z; \tau) = \begin{cases} -\sum_{k=1}^{\infty} \sum_{l \in 2\mathbb{Z}_{>0}} \tau^2 \frac{\mathcal{N}}{t} \frac{1}{k} \delta\left(\tau - \frac{kt}{l}\right) \exp\left(\frac{\pi i k^2 z^2}{2\tau}\right) Z_{u|v, \tilde{R}}^{\mathbb{T}^4}(kz|\tau), & A = B, \\ -\sum_{k=1}^{\infty} \sum_{l \in 2\mathbb{Z}_{>0}+1} \tau^2 \frac{\mathcal{N}}{t} \frac{1}{k} \delta\left(\tau - \frac{kt}{l}\right) \exp\left(\frac{\pi i k^2 z^2}{2\tau}\right) Z_{u|v, \tilde{NS}}^{\mathbb{T}^4}(kz|\tau), & A \neq B. \end{cases} \quad (6.20)$$

This is the thermal worldsheet annulus amplitude for open strings ending on the AdS_2 branes. Integrating over the worldsheet modulus τ now yields the thermal open-string partition function. We obtain, by setting $\mathcal{N} = -t$,

$$\begin{aligned} Z_{A,u|B,v}^{\text{open}}(t, z) &:= \int \frac{d\tau}{\tau^2} Z_{A,u|B,v}^{\text{WS,th}}(t, z; \tau) \\ &= \begin{cases} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l \in 2\mathbb{Z}_{>0}} \exp\left(\frac{\pi i (kz)^2}{2(kt/l)}\right) Z_{u|v, \tilde{R}}^{\mathbb{T}^4}\left(kz \left| \frac{kt}{l} \right.\right), & A = B, \\ \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l \in 2\mathbb{Z}_{>0}+1} \exp\left(\frac{\pi i (kz)^2}{2(kt/l)}\right) Z_{u|v, \tilde{NS}}^{\mathbb{T}^4}\left(kz \left| \frac{kt}{l} \right.\right), & A \neq B. \end{cases} \end{aligned} \quad (6.21)$$

This is the thermal AdS_3 open-string partition function for AdS_2 branes in the tensionless background, expressed as a double sum over the open-string winding number l and the thermal winding number k .

6.2.2. Interface partition function for the symmetric orbifold

On the bulk side, the AdS_2 brane has two asymptotic ends on the boundary cylinder of thermal AdS_3 , which we label by the index $A \in \{L, R\}$. To model a holographic interface between two copies of the symmetric orbifold with boundary conditions u and v in the internal $\mathbb{T}^4$ sector, we place for example the boundary state

$$\|u, +\rangle\rangle_L \quad \text{on the } L \text{ end}, \quad \|v, -\rangle\rangle_R \quad \text{on the } R \text{ end}, \quad (6.22)$$

and sum (6.21) over all contributions $(A, B) = (L, L), (R, R), (L, R), (R, L)$. Open strings can begin and end either on the same end (“ LL ” or “ RR ” sectors) or stretch between the two ends (“ LR ” and “ RL ” sectors), with the $\mathbb{T}^4$ boundary labels chosen appropriately as $(u|u)$, $(v|v)$, $(u|v)$ and $(v|u)$, respectively. The full thermal open-string partition function associated with the AdS_2 brane is then obtained by summing over all four sectors,

$$Z^{\text{open}}(z|\tau) := Z_{L,u|L,u}^{\text{open}}(z|\tau) + Z_{R,v|R,v}^{\text{open}}(z|\tau) + Z_{L,u|R,v}^{\text{open}}(z|\tau) + Z_{R,v|L,u}^{\text{open}}(z|\tau), \quad (6.23)$$

where each $Z_{A,u|B,v}^{\text{open}}(t, z)$ is given by the thermal annulus partition function (6.21) derived in the previous subsection, where t is replaced by τ .

On the symmetric-orbifold side, the grand-canonical torus partition function with two interface insertions was shown in Sec. 5.2.1 to take the DMVV-like form (5.22). For our purposes it is convenient to take the logarithm and focus on the interface (“open-string”) contribution. Just ignoring all for simplicity¹, the interface part of $\log \mathcal{Z}$ reads

$$\log \mathcal{Z}_{\mu^{L/R}, \nu_{\pm}^{L/R}}^{\mathbb{T}^2}(\tau) \Big|_{\text{interface}} = \sum_{k \geq 1} \sum_{l \geq 1} \frac{1}{k} \left[Z_{(a_-^L a_-^R)} \left(\frac{k\tau}{2l-1} \right) + Z_{(a_+^L a_+^R)} \left(\frac{k\tau}{2l-1} \right) + Z_{(a_-^L a_+^L)} \left(\frac{k\tau}{2l} \right) + Z_{(a_-^R a_+^R)} \left(\frac{k\tau}{2l} \right) \right]. \quad (6.24)$$

Here $Z_{(ab)}(\tau)$ denotes the seed torus partition function on $\mathbb{T}^2$ with two interface insertions labeled by boundary conditions a and b , as defined in Sec. 5.2.1. We now specialize to the $\mathbb{T}^4$ seed and identify the interface labels with the boundary conditions u and v of the open-string computation. Concretely, we choose

$$a_-^L = a_+^L = \|u, +\rangle_{\mathbb{T}^4}, \quad a_-^R = a_+^R = \|v, -\rangle_{\mathbb{T}^4}, \quad (6.25)$$

so that the four seed interface partition functions appearing in (6.24) are identified as

$$Z_{(a_-^L a_+^L)}(\tau) \longleftrightarrow e^{\frac{\pi i z^2}{2\tau}} Z_{u|\tilde{R}}^{\text{open}}(z|\tau), \quad Z_{(a_-^R a_+^R)}(\tau) \longleftrightarrow e^{\frac{\pi i z^2}{2\tau}} Z_{v|\tilde{R}}^{\text{open}}(z|\tau), \quad (6.26a)$$

$$Z_{(a_-^L a_-^R)}(\tau) \longleftrightarrow e^{\frac{\pi i z^2}{2\tau}} Z_{u|v, \tilde{\text{NS}}}^{\text{open}}(z|\tau), \quad Z_{(a_+^L a_+^R)}(\tau) \longleftrightarrow e^{\frac{\pi i z^2}{2\tau}} Z_{v|u, \tilde{\text{NS}}}^{\text{open}}(z|\tau), \quad (6.26b)$$

where the $\tilde{R}/\tilde{\text{NS}}$ labels match the spin structures appearing in (6.21). With these identifications, the double sum over (k, l) in (6.24) reproduces term by term the thermal open-string partition function (6.23) built from the four sectors LL , RR , LR and RL .

To summarize, setting

$$a_{\pm}^{L/R} \longleftrightarrow \text{boundary states } \|u, \varepsilon\rangle_{\mathbb{T}^4}, \|v, \varepsilon\rangle_{\mathbb{T}^4}, \quad (6.27)$$

the interface contribution to the grand-canonical symmetric-orbifold partition function (5.22) reproduces exactly the thermal AdS_2 brane partition function obtained from the worldsheet open-string computation. This provides a quantitative check of tensionless

¹This operation isolates the combinatorial structure of the covering contributions; the dependence on the chemical potentials $\mu^{L/R}, \nu_{\pm}^{L/R}$ can be reinstated straightforwardly and does not affect the matching with the open-string result.

holography with AdS_2 branes: the open-string degrees of freedom in the bulk encode precisely the interface degrees of freedom of the symmetric product orbifold.

6.3. Qualitative matching of interface correlation functions

The quantitative agreement of partition functions in Sec. 6.2 provides a stringent check of tensionless holography with AdS_2 branes at the level of a thermal one-loop amplitude. In this section we take a complementary viewpoint and focus on *correlation functions*: rather than matching a single protected quantity, we ask how the interface sector reorganizes symmetric-orbifold observables into the qualitative structures familiar from open/closed string perturbation theory.

The key input is that an interface between two symmetric orbifolds supports two natural classes of twist-sector insertions. Besides the usual *bulk* twists, which behave as closed-string insertions, one can also insert the *interface* twists introduced in Sec. 5.2, which may admit an interpretation as open-string vertex operators. To make the open/closed structure manifest, we therefore introduce in Sec. 6.3.1 a grand-canonical uplift of the fixed- $(N_+, N_-; P)$ correlators of Sec. 5.3. Summing over $(N_+, N_-; P)$ with fugacities conjugate to the transmitting and non-transmitting sectors lifts the global constraints and exponentiates the connected covering contributions. With an appropriate g_s -dependence of the fugacities, the resulting expansion is controlled purely by the topology of the connected covering surface Σ , reproducing the standard open/closed coupling counting. Section 6.3.2 summarizes the resulting dictionary between connected interface coverings and open/closed string worldsheets, including the characteristic weight $g_s^{-2+2g(\Sigma)+b(\Sigma)+n_c+n_o/2}$.

6.3.1. Grand-canonical uplift of interface correlators

In this section, we construct the grand-canonical uplift of interface correlators in symmetric orbifold of an arbitrary seed theory, which provides a convenient framework for identifying which degrees of freedom correspond to open strings in the dual string theory. As a byproduct, we relate this grand-canonical description to the canonical large- N analysis of Sec. 5.3.6 via a saddle-point argument. This section proceeds in parallel with the discussion in Sec. 2.5.6.

Interface grand-canonical generating functional.

We consider a single interface $\mathcal{I}_{a\pm}^P$ inserted between two copies of the seed theory living on Riemann surfaces $X_{\pm}$ of genus $G_{\pm}$ that are glued along their common boundary

$$\partial X_- \simeq \partial X_+ \simeq S^1, \quad (6.28)$$

to form a closed Riemann surface $X = X_+ \cup X_-$ of genus G . On a fixed background (X_-, X_+) and for fixed data $(N_+, N_-; P)$, we define an interface partition function with sources by

$$\begin{aligned} & Z_{N_{\pm}; P}^{X_{\pm}; S^1}[J] \\ &:= Z_{\text{Sym}^{N_{\pm}} \mathcal{M}}^{X_{\pm}; S^1} \left[\mathcal{I}_{a\pm}^P(t^*) \exp \left(\sum_{w=1}^{\infty} \int_X d^2x g_s^w J_w(x) \sigma_w(x) + \sum_{w,*} \int_{S^1} dt g_s^w J_w^{I,*}(t) \sigma_w^{I,*}(t) \right) \right] \end{aligned} \quad (6.29)$$

where:

- $J_w(x)$ are sources for single-cycle *closed* twists of length w on X (bulk twists in either X_+ , X_- or closed interface twists on the equator),
- $J_w^{I,*}(t)$ are sources for single *open* interface twists of length w inserted along the interface, with $*$ $\in \{\pm, \gtrless\}$,
- we have introduced a formal parameter g_s by assigning a factor g_s^w to each insertion of a single-cycle twist of length w .

The precise ranges of w in the second term depend on the type of open interface twist (integer or half-integer), as spelled out in Sec. 5.3.5; we keep the notation schematic here for brevity.

The grand-canonical interface generating functional is then defined by summing over the integers $(N_+, N_-; P)$ with three fugacities:

$$\mathcal{Z}_{\mu, \nu_{\pm}}^{X_{\pm}; S^1}[J] := \sum_{N_{\pm}=0}^{\infty} \sum_{P=0}^{\min(N_{\pm})} \mu^P \nu_-^{N_- - P} \nu_+^{N_+ - P} Z_{N_{\pm}; P}^{X_{\pm}; S^1}[J]. \quad (6.30)$$

Here μ is the fugacity conjugate to the degree P of the interface, while $\nu_{\pm}$ weigh the numbers $B_{\pm} = N_{\pm} - P$ of non-transmitting copies on each side.

The normalized grand-canonical correlators in the presence of an interface are obtained from $\mathcal{Z}_{g_s}^{X_{\pm}; S^1}[J]$ by functional differentiation with respect to the sources and

dividing by the vacuum partition function,

$$\left\langle \mathcal{I}_{a_{\pm}} \prod_{i=1}^{n_c} \sigma_{w_i}(x_i) \prod_{j=1}^{n_o} \sigma_{w_j}^{I,*j}(t_j) \right\rangle_{g_s}^{X_{\pm}} := \left[\prod_{i=1}^{n_c} \frac{\delta}{\delta J_{w_i}(x_i)} \prod_{j=1}^{n_o} \frac{\delta}{\delta J_{w_j}^{I,*j}(t_j)} \right] \frac{\mathcal{Z}_{g_s}^{X_{\pm};S^1}[J]}{\mathcal{Z}_{g_s}^{X_{\pm};S^1}[0]} \Big|_{J=0}. \quad (6.31)$$

Factorization into connected coverings.

In the covering-space representation reviewed in Sec. 2.5.2 and Sec. 5.3.3, each fixed- $(N_+, N_-; P)$ correlator, and hence each $Z_{N_{\pm};P}^{X_{\pm};S^1}[J]$, can be written as a sum over branched coverings $\Gamma : \Sigma \rightarrow X$ with prescribed ramification data. Expanding the exponential in (6.29) and reorganizing the result in terms of covering maps gives a schematic expression of the form

$$Z_{N_{\pm};P}^{X_{\pm};S^1}[J] = \exp \left(\sum_{w=1}^{\infty} \int_X d^2x J_w(x) + \sum_{w,*} \int_{S^1} dt J_w^{I,*}(t) \right) \sum_{\Gamma} g_s^{w[\Gamma]} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma}}{|\text{Aut}(\Gamma)|}, \quad (6.32)$$

where the sum runs over all coverings Γ with appropriate bulk and boundary ramification points and $w[\Gamma]$ is the sum of ramification indices.

We now make the decomposition of interface coverings into connected components explicit, in close analogy with the bulk case reviewed in Sec. 2.5.6. A generic covering surface Σ decomposes into several connected components

$$\Sigma = \bigsqcup_{\alpha} \Sigma_{\alpha}^{\sqcup k_{\alpha}}. \quad (6.33)$$

On each Σ_{α} , we can in turn restrict to the two sides of the interface,

$$\Gamma_{\pm}|_{\Sigma_{\alpha}} : \Sigma_{\alpha} \longrightarrow X_{\pm}, \quad (6.34)$$

and define

- $n_{\pm,\alpha} := \deg(\Gamma_{\pm}|_{\Sigma_{\alpha}})$, the degrees of the covering on the $X_{\pm}$ sides,
- p_{α} , the degree of transmissivity of $\Gamma|_{\Sigma_{\alpha}}$ at the reference point t^* .
- $b_{\pm,\alpha} := n_{\pm,\alpha} - p_{\alpha}$, the number of non-transmitting sheets at the reference point t^* on side $\pm$, belonging to the connected component Σ_{α} .

Since the sheets of the full covering are partitioned among the connected components, these satisfy the constraints

$$\sum_{\alpha} k_{\alpha} n_{\pm,\alpha} = N_{\pm}, \quad \sum_{\alpha} k_{\alpha} p_{\alpha} = P. \quad (6.35)$$

Using this decomposition, the fixed- $(N_+, N_-; P)$ correlator can be reorganized as a sum over isomorphism classes of connected interface coverings Γ_α and their multiplicities k_α . For example, for a correlator with insertions $\{\sigma_{w_i}\}$ in the bulk and $\{\sigma_{w_j}^{I,*j}\}$ on the interface, the covering-space expression takes the schematic form

$$Z_{N_\pm;P}^{X_\pm;S^1}[J] = \exp \left(\sum_{w=1}^{\infty} \int_X d^2x J_w(x) + \sum_{w,*} \int_{S^1} dt J_w^{I,*}(t) \right) \sum_{\Gamma} \prod_{\alpha} \frac{1}{k_{\alpha}!} \left(g_s^{w[\Gamma]_{\Sigma_{\alpha}}} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma_{\alpha}}}{|\text{Aut}(\Gamma|_{\Sigma_{\alpha}})|} \right)^{k_{\alpha}}, \quad (6.36)$$

Once we insert this into the grand-canonical partition function (6.30) and perform the sums over $(N_+, N_-; P)$, all constraints tying together the degrees and transmissivities of different connected components are lifted. The resulting grand-canonical partition function takes the exponential form

$$\log \mathcal{Z}_{\mu,\nu_{\pm}}^{X_{\pm};S^1}[J] = \sum_{\{w_i, (*_j, w_j)\}} \exp \left(\sum_{w=1}^{\infty} \int_X d^2x J_w(x) + \sum_{w,*} \int_{S^1} dt J_w^{I,*}(t) \right) \Big|_{\{w_i, (*_j, w_j)\}} \sum_{\substack{\Gamma: \Sigma \rightarrow X \\ \text{connected}}} \mu^{p(\Gamma)} \nu_-^{b_-(\Gamma)} \nu_+^{b_+(\Gamma)} g_s^{w[\Gamma]} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma}}{|\text{Aut}(\Gamma)|}, \quad (6.37)$$

where:

- The sum $\{w_i, (*_j, w_j)\}$ runs over finite multisets specifying the closed and open insertions.
- For a fixed choice of $\{w_i, (*_j, w_j)\}$, the notation

$$\exp \left(\sum_{w=1}^{\infty} \int_X d^2x J_w(x) + \sum_{w,*} \int_{S^1} dt J_w^{I,*}(t) \right) \Big|_{\{w_i, (*_j, w_j)\}}$$

denotes the coefficient of the monomial built from the corresponding integrals of sources.

- In the second sum, $\Gamma : \Sigma \rightarrow X$ runs over connected interface coverings whose ramification profile is encoded by the multiset selected in the first sum and the coordinates in the integrals.
- $p(\Gamma)$ is the degree of transmissivity of the connected covering Γ at t^* . $b_{\pm}(\Gamma)$ count the non-transmitting sheets on side $\pm$ at t^* .

- $w(\Gamma)$ is the total twist length carried by the covering,

$$w(\Gamma) = \sum_{i=1}^{n_c} w_i + \sum_{j=1}^{n_o} w_j.$$

Interface genus expansion.

To identify the g_s -dependence of a given connected covering, let us look at the weight $\mu^{p(\Gamma)} \nu_-^{b_-(\Gamma)} \nu_+^{b_+(\Gamma)} g_s^{w(\Gamma)}$ that appears in (6.37). Requiring that this weight can be written purely in terms of the topological data (g, b, n_c, n_o) through an open/closed string genus expansion fixes how the fugacities must depend on the coupling parameter g_s . In direct analogy with the bulk case, we choose

$$\mu = g_s^{2G-2}, \quad \nu_{\pm} = g_s^{2G_{\pm}-1}, \quad (6.38)$$

where G and $G_{\pm}$ are the genera of X and $X_{\pm}$, respectively. With this choice, the total power of g_s associated with Γ is

$$\begin{aligned} \mathcal{E}(\Gamma) &= p(\Gamma)(2G-2) + b_-(\Gamma)(2G_- - 1) + b_+(\Gamma)(2G_+ - 1) + w(\Gamma) \\ &= n_-(\Gamma)(2G_- - 1) + n_+(\Gamma)(2G_+ - 1) + \sum_{i=1}^{n_c} w_i + \sum_{j=1}^{n_o} w_j. \end{aligned} \quad (6.39)$$

The topology of a connected interface covering is constrained by an interface generalization of the Riemann-Hurwitz formula. For a covering with n_c bulk insertions of lengths w_i and n_o interface insertions of lengths w_j , the following holds:

$$n_-(2G_- - 1) + n_+(2G_+ - 1) + \sum_{i=1}^{n_c} w_i + \sum_{j=1}^{n_o} w_j = -2 + 2g(\Sigma) + b(\Sigma) + n_c + \frac{n_o}{2}. \quad (6.40)$$

This is the natural interface generalization of the usual Riemann-Hurwitz formula, and reduces to (5.77) when $X = S^2$. A derivation of this formula is given in [4].

Comparing (6.39) with (6.40), we see that $\mathcal{E}(\Gamma)$ can be expressed purely in terms of the topological data,

$$\mathcal{E}(\Gamma) = -2 + 2g(\Sigma) + b(\Sigma) + n_c + \frac{n_o}{2}. \quad (6.41)$$

Thus the contribution of a connected covering Γ to the exponent in (6.37) is weighted

by

$$g_s^{\mathcal{E}(\Gamma)} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma}}{|\text{Aut}(\Gamma)|} = g_s^{-2+2g(\Sigma)+b(\Sigma)+n_c+\frac{n_o}{2}} \frac{\hat{Z}_{\mathcal{M}}^{\Sigma}}{|\text{Aut}(\Gamma)|}. \quad (6.42)$$

In other words, after the grand-canonical uplift and the identification (6.38), the dependence on $(N_{\pm}; P)$ and the detailed ramification data reorganizes itself into the simple topological combination $-2 + 2g(\Sigma) + b(\Sigma) + n_c + \frac{n_o}{2}$. This is precisely the genus counting expected for an open/closed string background of worldsheet (Σ) with genus g with b boundaries, n_c closed-string insertions, and n_o open-string insertions on the boundaries.

The case $X_{\pm} = \mathbb{D}_{\pm}$ and saddle-point approximation.

Let us now specialize to the case relevant for our interface setup, namely $X_{\pm} = \mathbb{D}_{\pm}$ so that $G_+ = G_- = 0$ and hence $G = 0$. The natural choice (6.38) then reduces to

$$\mu = g_s^{-2}, \quad \nu_{\pm} = g_s^{-1}, \quad (6.43)$$

which assigns μ to spheres and $\nu_{\pm}$ to disks in the covering-space description.

In the absence of any insertions ($J = 0$), the vacuum grand-canonical partition function for the interface on $(\mathbb{D}_{\pm})$ takes the form

$$\mathcal{Z}_{g_s}^{\mathbb{D}_-, \mathbb{D}_+}[0] = \exp\left(\mu z + \nu_- g_- + \nu_+ g_+\right), \quad (6.44)$$

where $z = Z_{\mathcal{M}}^{S^2}$ is the sphere partition function of the seed theory, and $g_{\pm} = \langle 0|a_{\pm} \rangle$ are the boundary g -functions of the boundary states $|a_{\pm}\rangle$ in the seed theory. The expectation values of the number P of transmitting copies and of the numbers $B_{\pm} = N_{\pm} - P$ of non-transmitting copies on each side are then

$$\overline{P} = \mu \partial_{\mu} \log \mathcal{Z}_{g_s}^{\mathbb{D}_{\pm}}[0] = \mu z, \quad (6.45)$$

$$\overline{B}_{\pm} = \nu_{\pm} \partial_{\nu_{\pm}} \log \mathcal{Z}_{g_s}^{\mathbb{D}_{\pm}}[0] = \nu_{\pm} g_{\pm}. \quad (6.46)$$

With the identifications (6.43), we obtain

$$\overline{P} \sim g_s^{-2}, \quad \overline{B}_{\pm} \sim g_s^{-1}, \quad (6.47)$$

so that $\overline{B}_{\pm}^2 \propto \overline{P}$. In particular, in the weak-coupling regime $g_s \rightarrow 0$ we have

$$\overline{P} \sim \overline{N}_{\pm} \sim g_s^{-2}, \quad \overline{B}_{\pm} \sim \sqrt{\overline{P}}, \quad (6.48)$$

in precise agreement with the large- N scaling assumption (5.72). In particular, with $g_s^2 \sim \bar{P}^{-1}$, this identification also reproduces the large- N scaling of the interface twist normalizations in eq. (5.73). Moreover, the ratio of non-transmitting copies on the two sides is controlled by the ratio of boundary entropies,

$$\frac{\bar{B}_+}{\bar{B}_-} = \frac{g_+}{g_-}. \quad (6.49)$$

Let us now relate the grand-canonical correlators to the canonical large- N correlators in Sec. 5.3.6 by a saddle-point approximation, in direct analogy with the bulk discussion on S^2 . Consider normalized interface correlators of the form

$$\left\langle \mathcal{I}_{a_{\pm}} \prod_{i=1}^{n_c} \sigma_{w_i}(x_i) \prod_{j=1}^{n_o} \sigma_{w_j}^{I,*j}(t_j) \right\rangle_{g_s}^{\mathbb{D}_{\pm}} = \left[\prod_{i=1}^{n_c} \frac{\delta}{\delta J_{w_i}(x_i)} \prod_{j=1}^{n_o} \frac{\delta}{\delta J_{w_j}^{I,*j}(t_j)} \right] \frac{\mathcal{Z}_{g_s}^{\mathbb{D}_-, \mathbb{D}_+}[J]}{\mathcal{Z}_{g_s}^{\mathbb{D}_-, \mathbb{D}_+}[0]} \Big|_{J=0}. \quad (6.50)$$

Both the numerator and the denominator are triple sums over (P, B_-, B_+) with weights $\mu^P \nu_-^{B_-} \nu_+^{B_+}$. More precisely,

$$\mathcal{Z}_{g_s}^{\mathbb{D}_-, \mathbb{D}_+}[J] = \sum_{P, B_-, B_+} \mu^P \nu_-^{B_-} \nu_+^{B_+} Z_{P, B_-, B_+}^{\mathbb{D}_{\pm}}[J], \quad (6.51)$$

$$\mathcal{Z}_{g_s}^{\mathbb{D}_-, \mathbb{D}_+}[0] = \sum_{P, B_-, B_+} \mu^P \nu_-^{B_-} \nu_+^{B_+} Z_{P, B_-, B_+}^{\mathbb{D}_{\pm}}[0], \quad (6.52)$$

with

$$Z_{P, B_-, B_+}^{\mathbb{D}_{\pm}}[0] = \frac{z^P}{P!} \frac{g_-^{B_-}}{B_-!} \frac{g_+^{B_+}}{B_+!}. \quad (6.53)$$

Each of the three sums is a Poisson-type sum, sharply peaked at

$$P \simeq \bar{P} = \mu z, \quad B_{\pm} \simeq \bar{B}_{\pm} = \nu_{\pm} g_{\pm},$$

with Gaussian widths

$$\Delta P \sim \sqrt{\bar{P}}, \quad \Delta B_{\pm} \sim \sqrt{\bar{B}_{\pm}}.$$

Approximating the triple sum in the denominator by its saddle point gives

$$\begin{aligned} \mathcal{Z}_{g_s}^{\mathbb{D}_-, \mathbb{D}_+}[0] &\sim \mu^{\bar{P}} \nu_-^{\bar{B}_-} \nu_+^{\bar{B}_+} Z_{\bar{P}, \bar{B}_-, \bar{B}_+}^{\mathbb{D}_{\pm}}[0] \times (2\pi)^{3/2} \sqrt{\bar{P} \bar{B}_- \bar{B}_+} \\ &= \mu^{\bar{P}} \nu_-^{\bar{B}_-} \nu_+^{\bar{B}_+} \frac{z^{\bar{P}}}{\bar{P}!} \frac{g_-^{\bar{B}_-}}{\bar{B}_-!} \frac{g_+^{\bar{B}_+}}{\bar{B}_+!} \times (2\pi)^{3/2} \sqrt{\bar{P} \bar{B}_- \bar{B}_+}. \end{aligned} \quad (6.54)$$

Similarly, differentiating $\mathcal{Z}_{g_s}^{\mathbb{D}_-, \mathbb{D}_+}[J]$ with respect to the sources and evaluating at $J = 0$

yields a sum of the same form but with $Z_{P,B_-,B_+}^{\mathbb{D}_\pm}[J]$ replaced by the corresponding fixed- (P, B_-, B_+) correlator. To leading order in g_s this sum is again dominated by the same saddle point $(\bar{P}, \bar{B}_-, \bar{B}_+)$, so that

$$\begin{aligned} & \left[\prod_{i=1}^{n_c} \frac{\delta}{\delta J_{w_i}(x_i)} \prod_{j=1}^{n_o} \frac{\delta}{\delta J_{w_j}^{I,*j}(t_j)} \right] \mathcal{Z}_{g_s}^{\mathbb{D}_-, \mathbb{D}_+}[J] \Big|_{J=0} \\ & \sim \mu^{\bar{P}} \nu_-^{\bar{B}_-} \nu_+^{\bar{B}_+} Z_{\text{Sym}^{\bar{N}_\pm} \mathcal{M}}^{\mathbb{D}_\pm} \left[\mathcal{I}_{a_\pm}^{\bar{P}} \prod_{i=1}^{n_c} \hat{\sigma}_{w_i}(x_i) \prod_{j=1}^{n_o} \hat{\sigma}_{w_j}^{I,*j}(t_j) \right] \times (2\pi)^{3/2} \sqrt{\bar{P} \bar{B}_- \bar{B}_+}. \end{aligned} \quad (6.55)$$

In the normalized ratio (6.50), the Gaussian prefactors $(2\pi)^{3/2} \sqrt{\bar{P} \bar{B}_- \bar{B}_+}$ as well as the overall powers $\mu^{\bar{P}} \nu_-^{\bar{B}_-} \nu_+^{\bar{B}_+}$ cancel between numerator and denominator. We therefore obtain, to leading order in g_s ,

$$\left\langle \mathcal{I}_{a_\pm} \prod_{i=1}^{n_c} \sigma_{w_i}(x_i) \prod_{j=1}^{n_o} \sigma_{w_j}^{I,*j}(t_j) \right\rangle_{g_s}^{\mathbb{D}_\pm} \simeq \left\langle \mathcal{I}_{a_\pm}^{\bar{P}} \prod_{i=1}^{n_c} \hat{\sigma}_{w_i}^{c_i}(x_i) \prod_{j=1}^{n_o} \hat{\sigma}_{w_j}^{I,*j}(t_j) \right\rangle_{\bar{N}_\pm}^{\mathbb{D}_\pm}, \quad (6.56)$$

where c_i s are determined by the location of x_i . In particular, the normalized grand-canonical interface correlators reproduce, to leading order in $g_s \sim N^{-1/2}$, the canonical large- N behavior of normalized twist correlators in the presence of an interface, with the same genus expansion (6.42) as derived directly in Sec. 5.3.6.

6.3.2. Comparison with open/closed string perturbation theory

Let us now spell out more explicitly in which sense the grand-canonical interface ensemble constructed above reproduces the qualitative structure of open/closed string perturbation theory in tensionless holography.

Our analysis in this subsection makes this structural matching manifest in the interface context. The dictionary may be summarized as follows:

- connected interface coverings $\Gamma : \Sigma \rightarrow X \leftrightarrow$ connected open/closed string world-sheets with genus $g(\Sigma)$ and $b(\Sigma)$ boundaries;
- bulk twist insertions $\sigma_{w_i}(x_i) \leftrightarrow$ closed-string vertex operators in the AdS_3 background;
- interface twist insertions $\sigma_{w_j}^{I,*j}(t_j) \leftrightarrow$ open-string vertex operators on AdS_2 branes;
- the power of g_s associated with Γ , $g_s^{-2+2g(\Sigma)+b(\Sigma)+n_c+n_o/2}$, $\leftrightarrow$ the usual open/closed string coupling factor;

- the seed-theory factor $\hat{Z}_{\mathcal{M}}^{\Sigma}/|\text{Aut}(\Gamma)| \leftrightarrow$ the worldsheet CFT correlator on Σ divided by the symmetry factor of the corresponding string diagram.

In this sense, the grand-canonical interface ensemble provides a qualitative, and in fact structural, realization of open/closed string perturbation theory in the dual tensionless AdS_3 background. A detailed comparison of individual amplitudes, including their kinematical dependence and vertex operator normalizations, would require a more refined matching to the worldsheet description of strings on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ with AdS_2 branes, which goes beyond the scope of this work. For our purposes, it suffices that the interface correlators reproduce the correct topological structure and coupling dependence of string scattering amplitudes to all orders in the string coupling.

Part IV.

Conclusions and Outlook

7. Conclusions and Outlook

7.1. Conclusions

This thesis uses symmetric product orbifold CFTs as a controllable laboratory for $\text{AdS}_3/\text{CFT}_2$, with the aim of sharpening what it means for the theory to exhibit “holographic” behavior. In this setting, the word “holographic” should be interpreted in two senses: one is the semiclassical notion tied to an effective gravity description, and the other is a more stringy notion appropriate to the tensionless regime. The former provides a useful benchmark but is not automatically realized at the symmetric orbifold point. Motivated by this, we adopted the latter viewpoint and asked how a stringy AdS_3 dual can be made concrete on the CFT side, in particular by identifying and computing open-string-like sectors and comparing them with expectations from the stringy bulk dual.

Chapters 3 and 4 develop complementary tests of what can and cannot be cleanly extracted at the orbifold point. Chapter 3 provides a controlled analytic setup for single-interval entanglement on the torus, deriving closed-form Rényi entropies at integer $n > 1$ in a Dirac-fermion symmetric-orbifold toy model. This provides a concrete starting point for future comparisons, including the $n \rightarrow 1$ limit and direct tests against holographic Rényi data. Chapter 4, by contrast, shows that even after applying a progressive time average the spectral form factor does not approach random-matrix universality: the late-time growth is dominated by a robust step-like structure whose characteristic times are largely insensitive to β and, at sufficiently large N , to N , while remaining seed-dependent. This points to persistent light-sector coherence at the orbifold point, and thereby strongly supports the usual picture that twist-two marginal deformations must lift and reorganize a large part of the light spectrum before random-matrix-like chaotic diagnostics can become visible.

The constructive core of the thesis is developed in Chapters 5 and 6. Since, at the symmetric orbifold point, semiclassical-holographic benchmarks are not automatically realized, we instead work in the tensionless setting. To access a broader class of excitations in the tensionless regime, we made open-string-like sectors concrete on the CFT side via interfaces connecting theories with different N . This framework yields new computable sectors (“open twists”) and practical tools for partition functions and

correlators; after lifting to the grand-canonical ensemble natural for tensionless holography, it reproduces characteristic AdS_2 -brane structures, including the organization of correlators into a g_s expansion. Taken together, these results provide a quantitative route to comparisons with the stringy bulk dual.

7.2. Outlook

Finally, we collect the main open problems and extensions suggested by this work, and present them as an outlook for future studies.

Exact entanglement computations in symmetric orbifolds.

In Chapter 3, we derived closed-form expressions for the Rényi entropy of subsystems in thermal states in a fermionic toy model of a symmetric orbifold, for integer replica index $n > 1$. A natural next step is to extract the entanglement entropy by taking the analytic continuation $n \rightarrow 1$, either by numerical evaluation of the replica expressions or by developing new analytic techniques to control the continuation. Even prior to the $n \rightarrow 1$ limit, it would already be meaningful to compare the $n > 1$ data with holographic predictions for Rényi entropies, in order to test whether the leading contribution agrees and how subleading terms deviate. Such a comparison could help identify which pieces of the entanglement observables receive genuinely stringy corrections and at what order they first appear.

A complementary direction is to analyze entanglement in global AdS rather than in thermal AdS/BTZ kinematics. From the viewpoint advocated by Ohmori and Tachikawa, defining the reduced density matrix implicitly introduces a small physical boundary at the entangling cut, i.e. an entanglement-cut boundary condition [121]. A concrete open problem is then to determine *which* boundary states in symmetric orbifolds can realize (or approximate) such entanglement-cut boundary conditions, and how their data control subleading contributions to entanglement.

Shortest next step: implement a robust numerical pipeline for the $n > 1$ formulas and test the stability of extrapolations toward $n \rightarrow 1$ on controlled kinematic slices.

High-risk/high-payoff bet: classify (a subclass of) boundary states compatible with entanglement-cut boundary conditions in symmetric orbifolds, and extract universal subleading data from that classification.

Generalizations and applications of the computational method.

Recently, thermal two-point functions of untwisted states (and related probes) in symmetric product orbifolds were computed and interpreted, revealing large- N universality

patterns that can mimic BTZ correlators despite the absence of a semiclassical gravity dual [122]. It would be interesting to apply the “decomposition to cyclic orbifolds” technology developed in this thesis to extend such thermal correlator computations to broader observables e.g. correlators involving light twisted operators (such as “light-twist” two-point functions), or mixed twist/untwist probes, thereby providing independent support for, or sharper tests of, the proposed universality by making the twist-structure dependence explicit.

Shortest next step: reproduce a benchmark untwisted thermal correlator from [122] with the cyclic-orbifold decomposition as a cross-check of conventions and numerics.

High-risk/high-payoff bet: compute a genuinely new class (light-twist thermal correlators) where the twist data *could* spoil universality, and isolate the precise criterion under which universality survives.

Further study of “coarse-graining” in twisted sectors.

Once one turns on a twisted-sector marginal deformation (notably the twist-2 deformation relevant to the D1-D5 moduli space), non-BPS states at the orbifold point are not expected to remain eigenstates of the deformed Hamiltonian. A widespread expectation is that such light states undergo strong mixing and flow into heavier states above the BTZ threshold, but verifying this beyond perturbation theory is notoriously difficult.

We propose a *phenomenological* model designed to capture only the combinatorial twisted-sector mixing aspect, rather than the full energetic dynamics of the marginally deformed CFT. Since the combinatorics of twisted sectors is governed by S_N , one may model sector-mixing using a two-dimensional finite-group gauge theory / Dijkgraaf-Witten-type TQFT based on S_N , which captures the topological data of permutation monodromies [123]. See recent works focusing on holographic perspectives of such theories in [124, 125]. One can then introduce, as a phenomenological proxy for twist-2 interactions, a stochastic “random-transposition” process on S_N and ask for the timescale at which the distribution over conjugacy classes becomes close to uniform. Mathematically, the random-transposition walk exhibits cutoff at time $\frac{1}{2}N \log N$ (up to an $O(N)$ window), suggesting a characteristic coarse-graining scale $\sim N \log N$. See original work [126] and related works [127, 128, 129]. A key future goal is to relate this abstract mixing time to physical timescales and deformation parameters on the CFT side, thereby sharpening our understanding of how (and how fast) a symmetric orbifold operator could flow toward more chaotic behavior under marginal deformations.

Shortest next step: formulate an explicit map from “twist-2 interaction events” to random transpositions and identify which coarse-grained observable in the CFT it is meant to approximate.

High-risk/high-payoff bet: establish a quantitative dictionary between the Markov-chain mixing time and CFT time evolution under the marginal deformation (including the scaling with the deformation coupling), producing a predictive estimate for when chaotic diagnostics should begin to emerge.

Physical meaning of nontrivial “character” data for boundary states, defects, and interfaces.

Boundary states, topological defects, and interfaces in symmetric orbifold CFTs can carry nontrivial character data, i.e. labels of internal degrees of freedom of these objects, as shown in Secs. 2.3, 2.4, 5.1, respectively. At present, a direct tensionless-holographic interpretation of this character data is not fully understood. Our computations indicate that, already at fixed N , collisions of multiple interfaces can reproduce certain nontrivial character degrees of freedom. This motivates three concrete problems: (i) to systematize the algebra of interface collisions, (ii) to provide a holographic interpretation of the resulting character data, and (iii) to study compatibility with the twist-2 marginal deformation.

In particular, recent work on generalized symmetries in symmetric product orbifolds indicates that twisted-sector marginal deformations strongly constrain which topological symmetries can survive: except for special cases, only invertible and maximally symmetric structures persist [41]. Incorporating the effect of marginal deformations into the interface/defect framework may therefore open a path toward applications to $k > 1$ holography and to the D1-D5 CFT away from the orbifold point.

Shortest next step: write down and test associativity/consistency constraints for the interface-collision algebra in a set of computable examples at fixed N , and provide a holographic interpretation from the results.

High-risk/high-payoff bet: track explicitly how the twist-2 marginal deformation breaks (or preserves) the character-labeled defect/interface structures, and identify the surviving sector with a clean bulk interpretation.

Matching open-string scattering data.

As discussed in Chapter 6, we established a CFT-side framework to compute correlation functions that correspond to open-string sectors in the tensionless bulk interpretation. The next step is to perform an explicit matching with string-theory scattering amplitudes (or their worldsheet counterparts) and verify equality of the relevant kinematic dependence and coupling expansions, thereby turning the qualitative identification of “open-twist” degrees of freedom into a quantitative duality check.

Shortest next step: pick the simplest nontrivial correlator with an unambiguous bulk

channel (minimal number of insertions) and match its kinematic structure and normalization.

High-risk/high-payoff bet: match a multi-point amplitude (or a higher-order term in the g_s expansion) where contact terms / operator mixing are unavoidable, thereby stress-testing the dictionary.

Resurgence in higher-genus grand-canonical partition functions.

In Chapter 2, we carefully tracked the contribution of the finite gauge-group volume when evaluating orbifold quantities. For permutation orbifolds, higher-genus partition functions can be expressed systematically in terms of covering-space data [44]. From the path-integral viewpoint, the orbifold partition function takes the schematic form $|G|^{-1} \sum_{\text{holonomies}}(\cdots)$, and in factorization limits (e.g. pinching degenerations) one expects the vacuum-block contributions to organize themselves compatibly with this normalization. In the conventions adopted in this thesis, we therefore summarize the net gauge-group-volume effect at genus g by an overall scaling $|G|^{g-1}$.

In a grand-canonical formulation of symmetric orbifolds relevant for tensionless holography, this factor becomes $(N!)^{g-1}$, suggesting factorial growth and the need for non-perturbative completion. This motivates a systematic analysis using Borel resummation in defining higher-genus grand-canonical partition functions. We have observed indications of Stokes phenomena in naive resummations, and a detailed analysis along these lines is currently in progress.

Shortest next step: isolate a clean sector (e.g. fixed spin structure / restricted class of covers) where the large- N asymptotics can be extracted reliably and compared across genera.

High-risk/high-payoff bet: construct a fully resurgent transseries completion of the grand-canonical higher-genus partition function and interpret its nonperturbative sectors in bulk terms.

Appendices

A. Calculational Details of Dirac Fermion Entropy

In this appendix, we present the details of the calculations carried out in Chapter 3. Section A.1 stands for identities for theta functions, including definitions to clarify the notation and conventions used throughout, mostly referring to [1].

A.1. Theta functions and identities

We use standard convention $q \equiv e^{2\pi i\tau}$ and $y \equiv e^{2\pi iz}$. The Dedekind eta function is defined to be

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m). \quad (\text{A.1})$$

The Jacobi theta functions have two equivalent definitions. In the sum representation,

$$\begin{aligned} \theta_1(z \mid \tau) &= \sum_{r \in \mathbb{Z} + 1/2} q^{\frac{r^2}{2}} y^r e^{-\pi i r}, \\ \theta_2(z \mid \tau) &= \sum_{r \in \mathbb{Z} + 1/2} q^{\frac{r^2}{2}} y^r, \\ \theta_3(z \mid \tau) &= \sum_{r \in \mathbb{Z}} q^{\frac{r^2}{2}} y^r, \\ \theta_4(z \mid \tau) &= \sum_{r \in \mathbb{Z}} q^{\frac{r^2}{2}} y^r e^{-\pi i r}. \end{aligned} \quad (\text{A.2})$$

In the product representation,

$$\begin{aligned}
 \theta_1(z | \tau) &= 2 \sin(\pi z) q^{\frac{1}{8}} \prod_{m=1}^{\infty} (1 - q^m) (1 - y q^m) (1 - y^{-1} q^m), \\
 \theta_2(z | \tau) &= 2 \cos(\pi z) q^{\frac{1}{8}} \prod_{m=1}^{\infty} (1 - q^m) (1 + y q^m) (1 + y^{-1} q^m), \\
 \theta_3(z | \tau) &= \prod_{m=1}^{\infty} (1 - q^m) (1 + y q^{m-\frac{1}{2}}) (1 + y^{-1} q^{m-\frac{1}{2}}), \\
 \theta_4(z | \tau) &= \prod_{m=1}^{\infty} (1 - q^m) (1 - y q^{m-\frac{1}{2}}) (1 - y^{-1} q^{m-\frac{1}{2}}).
 \end{aligned} \tag{A.3}$$

We write $\theta_a(\tau)$ shorthand for $\theta_a(0|\tau)$.

A.1.1. Fundamental formulas

θ_1 satisfies

$$\theta_1(0|\tau) = \theta_1(\tau) = 0, \tag{A.4}$$

$$\partial_z \theta_1(z|\tau)|_{z=0} = 2\pi\eta(\tau)^3. \tag{A.5}$$

η satisfies

$$2\eta(\tau)^3 = \theta_2(\tau)\theta_3(\tau)\theta_4(\tau). \tag{A.6}$$

The famous Jacobi identity is

$$\theta_1(z|\tau)^4 - \theta_2(z|\tau)^4 + \theta_3(z|\tau)^4 - \theta_4(z|\tau)^4 = 0, \tag{A.7}$$

or $z = 0$ version

$$\theta_3(\tau)^4 - \theta_2(\tau)^4 - \theta_4(\tau)^4 = 0. \tag{A.8}$$

The modular- S and modular- T transformation is as follows:

$$\begin{aligned}
 \eta(\tau + 1) &= e^{\frac{i\pi}{12}} \eta(\tau), & \eta\left(-\frac{1}{\tau}\right) &= \sqrt{-i\tau} \eta(\tau), \\
 \theta_1(z|\tau + 1) &= e^{\frac{i\pi}{4}} \theta_1(z|\tau), & \theta_1\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) &= -i\sqrt{-i\tau} e^{\frac{i\pi z^2}{\tau}} \theta_1(z|\tau), \\
 \theta_2(z|\tau + 1) &= e^{\frac{i\pi}{4}} \theta_2(z|\tau), & \theta_2\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) &= \sqrt{-i\tau} e^{\frac{i\pi z^2}{\tau}} \theta_4(z|\tau), \\
 \theta_3(z|\tau + 1) &= \theta_4(z|\tau), & \theta_3\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) &= \sqrt{-i\tau} e^{\frac{i\pi z^2}{\tau}} \theta_3(z|\tau), \\
 \theta_4(z|\tau + 1) &= \theta_3(z|\tau), & \theta_4\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) &= \sqrt{-i\tau} e^{\frac{i\pi z^2}{\tau}} \theta_2(z|\tau).
 \end{aligned} \tag{A.9}$$

Theta functions satisfy the following quasi-double periodicity:

$$\begin{aligned}
 \theta_1(z + m\tau + n|\tau) &= (-1)^{m+n} q^{-\frac{m^2}{2}} y^{-m} \theta_1(z|\tau), \\
 \theta_2(z + m\tau + n|\tau) &= (-1)^n q^{-\frac{m^2}{2}} y^{-m} \theta_2(z|\tau), \\
 \theta_3(z + m\tau + n|\tau) &= q^{-\frac{m^2}{2}} y^{-m} \theta_3(z|\tau), \\
 \theta_4(z + m\tau + n|\tau) &= (-1)^m q^{-\frac{m^2}{2}} y^{-m} \theta_4(z|\tau),
 \end{aligned} \tag{A.10}$$

where m and n are integers. Actually, we can consider the case m and n are half-integers by using the following relations,

$$\begin{aligned}
 \theta_1(z \pm 1/2|\tau) &= \pm \theta_2(z|\tau), \\
 \theta_3(z \pm 1/2|\tau) &= \theta_4(z|\tau),
 \end{aligned} \tag{A.11}$$

$$\begin{aligned}
 \theta_1(z \pm \tau/2|\tau) &= \pm i q^{-\frac{1}{8}} y^{\mp \frac{1}{2}} \theta_4(z|\tau), \\
 \theta_2(z \pm \tau/2|\tau) &= q^{-\frac{1}{8}} y^{\mp \frac{1}{2}} \theta_3(z|\tau), \\
 \theta_3(z \pm \tau/2|\tau) &= q^{-\frac{1}{8}} y^{\mp \frac{1}{2}} \theta_2(z|\tau), \\
 \theta_4(z \pm \tau/2|\tau) &= \pm i q^{-\frac{1}{8}} y^{\mp \frac{1}{2}} \theta_1(z|\tau).
 \end{aligned} \tag{A.12}$$

A.1.2. “Double-angle” formulas

Since we use the “double-angle” formulas $\tau \rightarrow 2\tau$ repeatedly in this paper, we present this special case here. There are specific relations in the “double-angle” case, namely

$$\begin{aligned}\theta_2(2\tau) &= \left(\frac{\theta_3(\tau)^2 - \theta_4(\tau)^2}{2} \right)^{1/2}, \\ \theta_3(2\tau) &= \left(\frac{\theta_3(\tau)^2 + \theta_4(\tau)^2}{2} \right)^{1/2}, \\ \theta_4(2\tau) &= (\theta_3(\tau)\theta_4(\tau))^{1/2}, \\ \eta(2\tau) &= \left(\frac{\theta_2(\tau)\eta(\tau)}{2} \right)^{1/2} = \eta(\tau)^2 (\theta_3(\tau)\theta_4(\tau))^{-1/2}.\end{aligned}\tag{A.13}$$

In addition, we have

$$\begin{aligned}\theta_2\left(\frac{\tau}{2}\right) &= (2\theta_2(\tau)\theta_3(\tau))^{1/2}, \\ \theta_3\left(\frac{\tau}{2}\right) &= (\theta_3(\tau)^2 + \theta_2(\tau)^2)^{1/2}, \\ \theta_4\left(\frac{\tau}{2}\right) &= (\theta_3(\tau)^2 - \theta_2(\tau)^2)^{1/2}, \\ \eta\left(\frac{\tau}{2}\right) &= (\theta_4(\tau)\eta(\tau))^{1/2} = \eta(\tau)^2 \left(\frac{2}{\theta_2(\tau)\theta_3(\tau)} \right)^{1/2},\end{aligned}\tag{A.14}$$

$$\begin{aligned}\theta_2\left(\frac{\tau+1}{2}\right) &= e^{\frac{i\pi}{8}} (2\theta_2(\tau)\theta_4(\tau))^{1/2}, \\ \theta_3\left(\frac{\tau+1}{2}\right) &= (\theta_4(\tau)^2 + i\theta_2(\tau)^2)^{1/2}, \\ \theta_4\left(\frac{\tau+1}{2}\right) &= (\theta_4(\tau)^2 - i\theta_2(\tau)^2)^{1/2}, \\ \eta\left(\frac{\tau+1}{2}\right) &= e^{\frac{i\pi}{24}} (\theta_3(\tau)\eta(\tau))^{1/2} = e^{\frac{i\pi}{24}} \eta(\tau)^2 \left(\frac{2}{\theta_2(\tau)\theta_4(\tau)} \right)^{1/2}.\end{aligned}\tag{A.15}$$

A.1.3. “ n -tuple-angle” formulas

In contrast to the double version, there is no clear relation between $\theta_a(n\tau)$ and $\theta_a(\tau)$ generically. However, as we present in this subsection, we can rewrite “ n -tuple-angle” thetas in terms of $\theta_a(z|\tau)$. In order to derive the “ n -tuple-angle” formulas, we use the

following factorization formulas.

$$1 - q^n = \prod_{k=0}^{n-1} \left(1 - e^{2\pi i \frac{k}{n} q}\right) \quad (\text{A.16})$$

$$1 + q^n = \prod_{k=0}^{n-1} \left(1 - e^{2\pi i \frac{2k+1}{2n} q}\right) = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left(1 + e^{2\pi i \frac{k}{n} q}\right) \quad (\text{A.17})$$

By applying these formulas to the product representations of theta functions, we obtain the following “ n -tuple-angle” formulas:

$$\begin{aligned} \frac{\theta_2(n\tau)}{\eta(n\tau)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\theta_2\left(\frac{k}{n} \mid \tau\right)}{\eta(\tau)}, & \frac{\theta_2\left(\frac{\tau}{n}\right)}{\eta\left(\frac{\tau}{n}\right)} &= \prod_{k=0}^{n-1} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \frac{\theta_2\left(\frac{k}{n} \tau \mid \tau\right)}{\eta(\tau)}, \\ \frac{\theta_3(n\tau)}{\eta(n\tau)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\theta_3\left(\frac{k}{n} \mid \tau\right)}{\eta(\tau)}, & \frac{\theta_3\left(\frac{\tau}{n}\right)}{\eta\left(\frac{\tau}{n}\right)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \frac{\theta_3\left(\frac{k}{n} \tau \mid \tau\right)}{\eta(\tau)}, \\ \frac{\theta_4(n\tau)}{\eta(n\tau)} &= \prod_{k=0}^{n-1} \frac{\theta_4\left(\frac{k}{n} \mid \tau\right)}{\eta(\tau)}, & \frac{\theta_4\left(\frac{\tau}{n}\right)}{\eta\left(\frac{\tau}{n}\right)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \frac{\theta_4\left(\frac{k}{n} \tau \mid \tau\right)}{\eta(\tau)}. \end{aligned} \quad (\text{A.18a})$$

We can extend these formulas straightforwardly to the nonzero z case as

$$\begin{aligned} \frac{\theta_1(\alpha|n\tau)}{\eta(n\tau)} &= \prod_{k=0}^{n-1} \frac{\theta_1\left(\frac{\alpha}{n} + \frac{k}{n} \mid \tau\right)}{\eta(\tau)}, & -i \frac{\theta_1\left(\alpha \mid \frac{\tau}{n}\right)}{\eta\left(\frac{\tau}{n}\right)} &= \prod_{k=0}^{n-1} (-i) e^{2\pi i \frac{k}{n} \alpha} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \frac{\theta_1\left(\alpha + \frac{k}{n} \tau \mid \tau\right)}{\eta(\tau)}, \\ \frac{\theta_2(\alpha|n\tau)}{\eta(n\tau)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\theta_2\left(\frac{\alpha}{n} + \frac{k}{n} \mid \tau\right)}{\eta(\tau)}, & \frac{\theta_2\left(\alpha \mid \frac{\tau}{n}\right)}{\eta\left(\frac{\tau}{n}\right)} &= \prod_{k=0}^{n-1} e^{2\pi i \frac{k}{n} \alpha} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \frac{\theta_2\left(\alpha + \frac{k}{n} \tau \mid \tau\right)}{\eta(\tau)}, \\ \frac{\theta_3(\alpha|n\tau)}{\eta(n\tau)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\theta_3\left(\frac{\alpha}{n} + \frac{k}{n} \mid \tau\right)}{\eta(\tau)}, & \frac{\theta_3\left(\alpha \mid \frac{\tau}{n}\right)}{\eta\left(\frac{\tau}{n}\right)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi i \frac{k}{n} \alpha} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \frac{\theta_3\left(\alpha + \frac{k}{n} \tau \mid \tau\right)}{\eta(\tau)}, \\ \frac{\theta_4(\alpha|n\tau)}{\eta(n\tau)} &= \prod_{k=0}^{n-1} \frac{\theta_4\left(\frac{\alpha}{n} + \frac{k}{n} \mid \tau\right)}{\eta(\tau)}, & \frac{\theta_4\left(\alpha \mid \frac{\tau}{n}\right)}{\eta\left(\frac{\tau}{n}\right)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi i \frac{k}{n} \alpha} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \frac{\theta_4\left(\alpha + \frac{k}{n} \tau \mid \tau\right)}{\eta(\tau)}. \end{aligned} \quad (\text{A.18b})$$

The following enables us to change the range of k .

$$\begin{aligned}
 e^{2\pi i \frac{k}{n} \alpha} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \theta_1 \left(\alpha + \frac{k}{n} \tau \mid \tau \right) &= - e^{2\pi i \frac{k+n}{n} \alpha} e^{\pi i \left(\frac{k+n}{n}\right)^2 \tau} \theta_1 \left(\alpha + \frac{k+n}{n} \tau \mid \tau \right) \\
 e^{2\pi i \frac{k}{n} \alpha} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \theta_2 \left(\alpha + \frac{k}{n} \tau \mid \tau \right) &= e^{2\pi i \frac{k+n}{n} \alpha} e^{\pi i \left(\frac{k+n}{n}\right)^2 \tau} \theta_2 \left(\alpha + \frac{k+n}{n} \tau \mid \tau \right) \\
 e^{2\pi i \frac{k}{n} \alpha} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \theta_3 \left(\alpha + \frac{k}{n} \tau \mid \tau \right) &= e^{2\pi i \frac{k+n}{n} \alpha} e^{\pi i \left(\frac{k+n}{n}\right)^2 \tau} \theta_3 \left(\alpha + \frac{k+n}{n} \tau \mid \tau \right) \\
 e^{2\pi i \frac{k}{n} \alpha} e^{\pi i \left(\frac{k}{n}\right)^2 \tau} \theta_4 \left(\alpha + \frac{k}{n} \tau \mid \tau \right) &= - e^{2\pi i \frac{k+n}{n} \alpha} e^{\pi i \left(\frac{k+n}{n}\right)^2 \tau} \theta_4 \left(\alpha + \frac{k+n}{n} \tau \mid \tau \right)
 \end{aligned} \tag{A.19}$$

We can derive the ordinary n -tuple-angle formulas by substituting the right-hand side of (A.18) to the left-hand side,

$$\begin{aligned}
 \frac{\theta_1(\alpha|\tau)}{\eta(\tau)} &= i^{n-n^2} \prod_{k=0}^{n-1} \prod_{l=0}^{n-1} e^{2\pi i \frac{l}{n} \left(\frac{\alpha}{n} + \frac{k}{n}\right)} e^{\pi i \left(\frac{l}{n}\right)^2 \tau} \frac{\theta_1 \left(\frac{\alpha}{n} + \frac{k}{n} + \frac{l}{n} \tau \mid \tau \right)}{\eta(\tau)}, \\
 \frac{\theta_2(\alpha|\tau)}{\eta(\tau)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \prod_{l=0}^{n-1} e^{2\pi i \frac{l}{n} \left(\frac{\alpha}{n} + \frac{k}{n}\right)} e^{\pi i \left(\frac{l}{n}\right)^2 \tau} \frac{\theta_2 \left(\frac{\alpha}{n} + \frac{k}{n} + \frac{l}{n} \tau \mid \tau \right)}{\eta(\tau)}, \\
 \frac{\theta_3(\alpha|\tau)}{\eta(\tau)} &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \prod_{l=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi i \frac{l}{n} \left(\frac{\alpha}{n} + \frac{k}{n}\right)} e^{\pi i \left(\frac{l}{n}\right)^2 \tau} \frac{\theta_3 \left(\frac{\alpha}{n} + \frac{k}{n} + \frac{l}{n} \tau \mid \tau \right)}{\eta(\tau)}, \\
 \frac{\theta_4(\alpha|\tau)}{\eta(\tau)} &= \prod_{k=0}^{n-1} \prod_{l=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi i \frac{l}{n} \left(\frac{\alpha}{n} + \frac{k}{n}\right)} e^{\pi i \left(\frac{l}{n}\right)^2 \tau} \frac{\theta_4 \left(\frac{\alpha}{n} + \frac{k}{n} + \frac{l}{n} \tau \mid \tau \right)}{\eta(\tau)},
 \end{aligned} \tag{A.20}$$

which are analogous to the following trigonometric function identities,

$$\begin{aligned}
 2 \sin nz &= \prod_{k=0}^{n-1} 2 \sin \left(z + \frac{k\pi}{n} \right), \\
 2 \cos nz &= \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} 2 \cos \left(z + \frac{k\pi}{n} \right).
 \end{aligned} \tag{A.21}$$

A.2. Correlation functions in the compact scalar CFT on the torus

In this section, we present the computation and results for correlation functions in the compact scalar CFT on the torus. Much of the content follows section 12.6 of [88].

Vertex operators

A vertex operator $\mathcal{O}(z, \bar{z})$ can be written in terms of real numbers e and m as

$$\mathcal{O}_{e,m}(z, \bar{z}) = : \exp i \left[\left(\frac{e}{R} + \frac{mR}{\alpha'} \right) X_L(z) + \left(\frac{e}{R} - \frac{mR}{\alpha'} \right) X_R(\bar{z}) \right] : . \quad (\text{A.22})$$

For convenience, we define p_L and p_R by

$$p_L = \frac{e}{R} + \frac{mR}{\alpha'}, \quad p_R = \frac{e}{R} - \frac{mR}{\alpha'}, \quad (\text{A.23})$$

so that the conformal weights can be written as

$$\begin{aligned} (h_{e,m}, \bar{h}_{e,m}) &= \left(\frac{\alpha'}{4} p_L^2, \frac{\alpha'}{4} p_R^2 \right) \\ &= \left(\frac{\alpha'}{4} \left(\frac{e}{R} + \frac{mR}{\alpha'} \right)^2, \frac{\alpha'}{4} \left(\frac{e}{R} - \frac{mR}{\alpha'} \right)^2 \right). \end{aligned} \quad (\text{A.24})$$

A.2.1. Correlation functions

We summarize correlation functions of vertex operators. Two-point functions take the form

$$\begin{aligned} &\langle \mathcal{O}_{e,m}(z_1, \bar{z}_1) \mathcal{O}_{-e,-m}(z_2, \bar{z}_2) \rangle \\ &= \frac{1}{Z(\hat{R}; \tau, \bar{\tau})} \frac{1}{|\eta(\tau)|^2} \left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{12}|\tau)} \right)^{2h_{e,m}} \overline{\left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{12}|\tau)} \right)^{2\bar{h}_{e,m}}} \\ &\quad \times \sum_{n, n'' \in \mathbb{Z}} q^{h_{n'',n}} \bar{q}^{\bar{h}_{n'',n}} \exp [4\pi i (\alpha_{n'',n} \alpha_{e,m} z_{12} - \bar{\alpha}_{n'',n} \bar{\alpha}_{e,m} \bar{z}_{12})]. \end{aligned} \quad (\text{A.25})$$

Introducing the notation $\mathcal{O}_i \equiv \mathcal{O}_{e_i, m_i}(z_i, \bar{z}_i)$ and $\alpha_i \equiv \alpha_{e_i, m_i}$, we can write multi-point functions as

$$\begin{aligned} \left\langle \prod_i \mathcal{O}_i \right\rangle &= \frac{1}{Z(\hat{R}; \tau, \bar{\tau})} \frac{1}{|\eta(\tau)|^2} \prod_{i < j} \left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{ij}|\tau)} \right)^{-2\alpha_i \alpha_j} \overline{\left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{ij}|\tau)} \right)^{-2\bar{\alpha}_i \bar{\alpha}_j}} \\ &\quad \times \sum_{n, n'' \in \mathbb{Z}} q^{h_{n'',n}} \bar{q}^{\bar{h}_{n'',n}} \exp \left[4\pi i \left(\sum_i \alpha_{n'',n} \alpha_i z_i - \bar{\alpha}_{n'',n} \bar{\alpha}_i \bar{z}_i \right) \right]. \end{aligned} \quad (\text{A.26})$$

Here the charges e_i and m_i are assumed to satisfy

$$\sum_i e_i = \sum_i m_i = 0. \quad (\text{A.27})$$

A.2.2. Correlation functions twisted periodic boundary conditions

In this section we compute multi-point functions in the presence of twisted periodic boundary conditions

$$\begin{aligned} X(z+1, \bar{z}+1) &= X(z, \bar{z}) + 2\pi R(n + \eta), \\ X(z + \tau, \bar{z} + \bar{\tau}) &= X(z, \bar{z}) + 2\pi R(n' + \eta'), \end{aligned} \quad (\text{A.28})$$

where η and η' are real numbers. Here η represents a twist in the spatial direction and η' a twist in the (imaginary) time direction. In this case, the partition function is modified as $Z(\hat{R}; \tau, \bar{\tau}) \rightarrow Z(\hat{R}; \tau, \bar{\tau}; \eta, \eta')$. An explicit expression for the partition function will be given below.

For the two-point function, the result can be obtained by replacing the summation part of (A.25) as

$$\begin{aligned} & \sum_{n, n'' \in \mathbb{Z}} q^{h_{n'', n}} \bar{q}^{\bar{h}_{n'', n}} \exp[4\pi i (\alpha_{n'', n} \alpha_{e, m} z_{12} - \bar{\alpha}_{n'', n} \bar{\alpha}_{e, m} \bar{z}_{12})] \\ & \times \exp\left[2\pi i \eta \tau \frac{R}{\sqrt{\alpha'}} \alpha_{n'', n}\right] \exp\left[-2\pi i \eta' \frac{R}{\sqrt{\alpha'}} \alpha_{n'', n}\right] \\ & \times \exp\left[2\pi i \eta \bar{\tau} \frac{R}{\sqrt{\alpha'}} \bar{\alpha}_{n'', n}\right] \exp\left[-2\pi i \eta' \frac{R}{\sqrt{\alpha'}} \bar{\alpha}_{n'', n}\right] \\ & \times \exp\left[-\frac{\pi \tau_2 R^2}{\alpha'} \eta^2\right] \exp\left[2\pi i \eta e \text{Re } z_{12} - 2\pi \eta \frac{m R^2}{\alpha'} \text{Im } z_{12}\right], \end{aligned} \quad (\text{A.29})$$

and the extension to multi-point functions is obtained by replacing the summation part of (A.26) as

$$\sum_{n, n'' \in \mathbb{Z}} q^{h_{n'', n}} \bar{q}^{\bar{h}_{n'', n}} \exp\left[4\pi i \left(\sum_i \alpha_{n'', n} \alpha_i z_i - \bar{\alpha}_{n'', n} \bar{\alpha}_i \bar{z}_i\right)\right] \quad (\text{A.30})$$

$$\begin{aligned} & \times \exp\left[2\pi i \eta \tau \frac{R}{\sqrt{\alpha'}} \alpha_{n'', n}\right] \exp\left[-2\pi i \eta' \frac{R}{\sqrt{\alpha'}} \alpha_{n'', n}\right] \\ & \times \exp\left[2\pi i \eta \bar{\tau} \frac{R}{\sqrt{\alpha'}} \bar{\alpha}_{n'', n}\right] \exp\left[-2\pi i \eta' \frac{R}{\sqrt{\alpha'}} \bar{\alpha}_{n'', n}\right] \\ & \times \exp\left[-\frac{\pi \tau_2 R^2}{\alpha'} \eta^2\right] \exp\left[2\pi i \frac{R}{\sqrt{\alpha'}} \eta \sum_i (\alpha_i z_i + \bar{\alpha}_i \bar{z}_i)\right]. \end{aligned} \quad (\text{A.31})$$

The modified partition function $Z(\hat{R}; \tau, \bar{\tau}; \eta, \eta')$ is obtained by setting $e = m = 0$, or

equivalently $\alpha_i = \bar{\alpha}_i = 0$, in the above, which yields

$$\begin{aligned}
 Z(\hat{R}; \tau, \bar{\tau}; \eta, \eta') = & \exp \left[-\frac{\pi \tau_2 R^2}{\alpha'} \eta^2 \right] \sum_{n, n'' \in \mathbb{Z}} q^{h_{n'', n}} \bar{q}^{\bar{h}_{n'', n}} \\
 & \times \exp \left[2\pi i \eta \tau \frac{R}{\sqrt{\alpha'}} \alpha_{n'', n} \right] \exp \left[-2\pi i \eta' \frac{R}{\sqrt{\alpha'}} \alpha_{n'', n} \right] \\
 & \times \exp \left[2\pi i \eta \bar{\tau} \frac{R}{\sqrt{\alpha'}} \bar{\alpha}_{n'', n} \right] \exp \left[-2\pi i \eta' \frac{R}{\sqrt{\alpha'}} \bar{\alpha}_{n'', n} \right]. \quad (\text{A.32})
 \end{aligned}$$

A.2.3. Example: the fermionic radius $\hat{R} = \sqrt{2}$

As an example, we consider the case of fermionic radius $\hat{R} = \sqrt{2}$, with twisted periodic boundary conditions as in the previous section. We first present the result for the two-point function. Substituting $\hat{R} = R/\alpha' = \sqrt{2}$ into (A.29) and introducing $r = n + n''/2$, $s = n - n''/2$, we obtain

$$\begin{aligned}
 & \sum_{n, n'' \in \mathbb{Z}} q^{r^2/2} \bar{q}^{s^2/2} \exp \left[2\pi i r \left(m + \frac{e}{2} \right) z_{12} \right] \exp \left[-2\pi i s \left(m - \frac{e}{2} \right) \bar{z}_{12} \right] \\
 & \times \exp [2\pi i r \eta \tau] \exp [-2\pi i r \eta'] \exp [-2\pi i s \eta \bar{\tau}] \exp [2\pi i s \eta'] \\
 & \times \exp [-2\pi \tau_2 \eta^2] \exp [2\pi i \eta e \text{Re } z_{12} - 4\pi \eta m \text{Im } z_{12}], \quad (\text{A.33})
 \end{aligned}$$

where the sum over n and n'' can be rewritten in terms of $r = n + n''/2$ and $s = n - n''/2$ as

$$\begin{aligned}
 & \sum_{n, n'' \in \mathbb{Z}} f \left(n + \frac{n''}{2}, n - \frac{n''}{2} \right) \\
 & = \sum_{\substack{n \in \mathbb{Z} \\ n''' \in \mathbb{Z}}} f(n + n''', n - n''') + \sum_{\substack{n \in \mathbb{Z} \\ n''' \in \mathbb{Z} + 1/2}} f(n + n''', n - n''') \\
 & = \sum_{r, s \in \mathbb{Z}} \frac{1 + e^{-\pi i(r-s)}}{2} f(r, s) + \sum_{r, s \in \mathbb{Z} + 1/2} \frac{1 - e^{-\pi i(r-s)}}{2} f(r, s). \quad (\text{A.34})
 \end{aligned}$$

Hence (A.33) can be rewritten as

$$\begin{aligned}
 & \exp [-2\pi \tau_2 \eta^2] \exp [2\pi i \eta e \text{Re } z_{12} - 4\pi \eta m \text{Im } z_{12}] \\
 & \times \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} \theta_a \left(\eta \tau - \eta' + \left(m + \frac{e}{2} \right) z_{12} \middle| \tau \right) \overline{\theta_a \left(\eta \tau - \eta' + \left(m - \frac{e}{2} \right) z_{12} \middle| \tau \right)}, \quad (\text{A.35})
 \end{aligned}$$

which expresses the sum in terms of Jacobi theta functions. Thus the two-point function (for simplicity setting $e = 0$) becomes

$$\begin{aligned} & \langle \mathcal{O}_{0,m}(z_1, \bar{z}_1) \mathcal{O}_{0,-m}(z_2, \bar{z}_2) \rangle_{(\eta, \eta')} \\ &= \frac{1}{Z(\hat{R} = \sqrt{2}; \tau, \bar{\tau}; \eta, \eta')} \left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{12}|\tau)} \right)^{2h_{0,m}} \overline{\left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{12}|\tau)} \right)^{2\bar{h}_{0,m}}} \\ & \times \exp[-4\pi\eta m \text{Im } z_{12}] \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} e^{-2\pi\tau_2\eta^2} \left| \frac{\theta_a(\eta\tau - \eta' + mz_{12}|\tau)}{\eta(\tau)} \right|^2, \end{aligned} \quad (\text{A.36})$$

which can be extended to multi-point functions. First note that (A.31) can be rewritten as

$$\exp[-2\pi\tau_2\eta^2] \exp\left[2\pi i\eta \sum_i \left(\left(\frac{e_i}{2} + m_i \right) z_i + \left(\frac{e_i}{2} - m_i \right) \bar{z}_i \right)\right] \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} \theta_a \bar{\theta}_a, \quad (\text{A.37})$$

where, for brevity, we have written

$$\theta_a \equiv \theta_a \left(\eta\tau - \eta' + \sum_i \left(m_i + \frac{e_i}{2} \right) z_i \middle| \tau \right), \quad (\text{A.38a})$$

$$\bar{\theta}_a \equiv \overline{\theta_a \left(\eta\tau - \eta' + \sum_i \left(m_i - \frac{e_i}{2} \right) z_i \middle| \tau \right)}. \quad (\text{A.38b})$$

Thus, the multi-point function (again setting $e_i = 0$ for simplicity) becomes

$$\begin{aligned} \left\langle \prod_i \mathcal{O}_{0,m_i}(z_i, \bar{z}_i) \right\rangle_{(\eta, \eta')} &= \frac{1}{Z(\hat{R} = \sqrt{2}; \tau, \bar{\tau}; \eta, \eta')} \prod_{i < j} \left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{ij}|\tau)} \right)^{-m_i m_j} \overline{\left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{ij}|\tau)} \right)^{-m_i m_j}} \\ & \times \exp\left[-4\pi\eta \sum_i m_i \text{Im } z_i\right] \\ & \times \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} e^{-2\pi\tau_2\eta^2} \left| \frac{\theta_a(\eta\tau - \eta' + \sum_i m_i z_i|\tau)}{\eta(\tau)} \right|^2, \end{aligned} \quad (\text{A.39})$$

and the partition function $Z(\hat{R} = \sqrt{2}; \tau, \bar{\tau}; \eta, \eta')$ is simply

$$Z(\hat{R} = \sqrt{2}; \tau, \bar{\tau}; \eta, \eta') = \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} e^{-2\pi\tau_2\eta^2} \left| \frac{\theta_a(\eta\tau - \eta'|\tau)}{\eta(\tau)} \right|^2. \quad (\text{A.40})$$

Results in the absence of twists

We now specialize to the case where η and η' are integers, i.e. the twists are trivial. The resulting expressions can also be obtained by substituting the fermionic radius into the formulas of section A.2.1. In this case, the partition function agrees with the partition function of a Dirac fermion,

$$Z_{\mathcal{D}}(\tau, \bar{\tau}) = \frac{1}{2} \left(\left| \frac{\theta_3(\tau)}{\eta(\tau)} \right|^2 + \left| \frac{\theta_2(\tau)}{\eta(\tau)} \right|^2 + \left| \frac{\theta_4(\tau)}{\eta(\tau)} \right|^2 \right). \quad (\text{A.41})$$

The two-point function becomes

$$\begin{aligned} & \langle \mathcal{O}_{0,m}(z_1, \bar{z}_1) \mathcal{O}_{0,-m}(z_2, \bar{z}_2) \rangle \\ &= \frac{1}{Z_{\mathcal{D}}(\tau, \bar{\tau})} \left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{12}|\tau)} \right)^{2h_{0,m}} \overline{\left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{12}|\tau)} \right)^{2\bar{h}_{0,m}}} \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} \left| \frac{\theta_a(mz_{12}|\tau)}{\eta(\tau)} \right|^2, \end{aligned} \quad (\text{A.42})$$

and the multi-point function is

$$\begin{aligned} & \left\langle \prod_i \mathcal{O}_{0,m_i}(z_i, \bar{z}_i) \right\rangle \\ &= \frac{1}{Z_{\mathcal{D}}(\tau, \bar{\tau})} \prod_{i < j} \left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{ij}|\tau)} \right)^{-m_i m_j} \overline{\left(\frac{\partial \theta_1(0|\tau)}{\theta_1(z_{ij}|\tau)} \right)^{-m_i m_j}} \frac{1}{2} \sum_{a=1}^4 (-)^{\delta_{1,a}} e^{-2\pi\tau_2\eta^2} \left| \frac{\theta_a(\sum_i m_i z_i|\tau)}{\eta(\tau)} \right|^2, \end{aligned} \quad (\text{A.43})$$

as desired.

A.3. $N = n = 2$ Rényi entropy calculation

Let us show the details of calculations in Sec. 3.3.3. The discrete Fourier transform over the orbifold-twist index can be performed on each sheet separately. Denoting the $k = 0$ contribution by $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=0}(L)$ and the $k = 1$ contribution by $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L)$, we have

$$Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}}(L) = Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=0}(L) \cdot Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L). \quad (\text{A.44})$$

The $k = 0$ contribution does not depend on l or l' , and thus

$$\begin{aligned} Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=0}(L) &= \prod_{p=-(2-1)/2}^{(2-1)/2} \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{4h_{0,p/2}} \frac{1}{|\eta(\tau)|^2} \left| \theta_3\left(\frac{p}{2}L|\tau\right) \right|^2 \\ &= \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{\frac{1}{4}} \frac{1}{|\eta(\tau)|^4} \left| \theta_3\left(\frac{1}{4}L|\tau\right) \right|^2 \left| \theta_3\left(-\frac{1}{4}L|\tau\right) \right|^2, \end{aligned} \quad (\text{A.45})$$

follows.

Collecting the above results, we obtain

$$\begin{aligned} \left(Z_{\text{Cyc}^2\mathcal{D}}^{\text{NS}} \right)^2 \cdot \text{Tr}_A[\rho_A^2] &= \frac{1}{2} \sum_{l_1, l'_1=0}^1 \frac{1}{2} \sum_{l_2, l'_2=0}^1 Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}} \\ &= \left(\frac{1}{2} \right)^2 \sum_{l_1, l'_1=0}^1 \sum_{l_2, l'_2=0}^1 Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=0}(L) \cdot Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L) \\ &= \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{\frac{1}{4}} \frac{1}{|\eta(\tau)|^4} \left| \theta_3\left(\frac{1}{4}L|\tau\right) \right|^2 \left| \theta_3\left(-\frac{1}{4}L|\tau\right) \right|^2 \\ &\quad \times \left(\frac{1}{2} \right)^2 \sum_{l_1, l'_1=0}^1 \sum_{l_2, l'_2=0}^1 Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L), \end{aligned} \quad (\text{A.46})$$

so it remains to determine $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L)$. This quantity is invariant under the interchange of orbifold boundary conditions:

$$Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS}, k=1}(L) = Z_{(l_2, l'_2)(l_1, l'_1)}^{\text{NS}, k=1}(L). \quad (\text{A.47})$$

This symmetry corresponds to shifting the order of the replica sheets and does not affect the physics.

Diagonal contributions.

We first consider the diagonal contributions, i.e. the case where the orbifold-twist boundary conditions are the same on each replica sheet. Taking $(l_1, l'_1) = (l_2, l'_2) = (l, l')$, we can read off the result from the previous subsection by isolating the $k = 1$ contribution:

$$\begin{aligned} Z_{(l, l')(l, l')}^{\text{NS}, k=1}(L) &= \prod_{p=-(2-1)/2}^{(2-1)/2} \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{4h_{0,p/2}} \frac{e^{-2\pi \frac{l^2}{2^2} \tau_2}}{|\eta(\tau)|^2} \left| \theta_3\left(\frac{l}{2}\tau - \frac{l'}{2} + \frac{p}{2}L|\tau\right) \right|^2 \\ &= \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{\frac{1}{4}} \prod_{p=-\frac{1}{2}, \frac{1}{2}} \frac{e^{-2\pi \frac{l^2}{2^2} \tau_2}}{|\eta(\tau)|^2} \left| \theta_3\left(\frac{l}{2}\tau - \frac{l'}{2} + \frac{p}{2}L|\tau\right) \right|^2, \end{aligned} \quad (\text{A.48})$$

as claimed.

Off-diagonal contributions.

We now turn to the off-diagonal contributions. By the symmetry (A.47), it suffices to consider the following cases:

- $(l_1, l'_1) = (0, 0), (l_2, l'_2) = (1, 0),$
- $(l_1, l'_1) = (0, 0), (l_2, l'_2) = (0, 1),$
- $(l_1, l'_1) = (0, 0), (l_2, l'_2) = (1, 1),$
- $(l_1, l'_1) = (0, 1), (l_2, l'_2) = (1, 0),$
- $(l_1, l'_1) = (0, 1), (l_2, l'_2) = (1, 1),$
- $(l_1, l'_1) = (1, 0), (l_2, l'_2) = (1, 1).$

Before diving into these cases, let us recall the structure of the fields after the discrete Fourier transform (3.79). We are interested in the $k = 1$ sector, where

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)} = \frac{1}{2} \left(-\psi_{L,[1]}^1 - \psi_{L,[2]}^1 + \psi_{L,[1]}^2 + \psi_{L,[2]}^2 \right), \quad (\text{A.49a})$$

$$\tilde{\psi}_{L,-\frac{1}{2}}^{(1)} = \frac{1}{2} \left(+\psi_{L,[1]}^1 - \psi_{L,[2]}^1 - \psi_{L,[1]}^2 + \psi_{L,[2]}^2 \right). \quad (\text{A.49b})$$

Case $(l_1, l'_1) = (0, 0), (l_2, l'_2) = (1, 0)$. In this case we have $\psi_{L,[2]}^1(z+1) = \psi_{L,[2]}^2(z)$ or equivalently $\psi_{L,[2]}^2(z+1) = \psi_{L,[2]}^1(z)$, and hence $\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}$ and $\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}$ are no longer independent fermions:

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z+1) = -\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z), \quad (\text{A.50a})$$

$$\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z+1) = -\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z), \quad (\text{A.50b})$$

so the spatial period is doubled and the modulus of the torus becomes $\tau/2$.¹ Moreover, bosonization is no longer independent, and the replica-twist operators become

¹The cutoff scale on the torus is also halved.

correlated. This leads to

$$\begin{aligned}
 Z_{(0,0)(1,0)}^{\text{NS},k=1}(L) &= Z \cdot \left\langle \prod_{p=-(2-1)/2}^{(2-1)/2} e^{i\frac{p}{2}\tilde{\varphi}_{L,p}^{(1)}(L)} e^{-i\frac{p}{2}\tilde{\varphi}_{L,p}^{(1)}(0)} \right\rangle \cdot (\text{right mover}) \\
 &= Z \cdot \left\langle e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,-\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(0)} e^{+i\frac{1}{4}\tilde{\varphi}_{L,-\frac{1}{2}}^{(1)}(0)} \right\rangle \cdot (\text{right mover}) \\
 &= Z \cdot \left\langle e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L+1)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(0)} e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(1)} \right\rangle \Big|_{\tau/2} \cdot (\text{right mover}),
 \end{aligned} \tag{A.51}$$

which is reduced to a four-point function. The anti-holomorphic part is obtained by taking the complex conjugate and replacing $L \rightarrow R$. This computation can be performed using eq. (A.39). Since the spatial direction is of twice the original length, we extract the θ_2 contribution with $\eta = \eta' = 0$, and obtain

$$Z_{(0,0)(1,0)}^{\text{NS},k=1}(L) = \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{L}{2}|\frac{\tau}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{1}{2}|\frac{\tau}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{L+1}{2}|\frac{\tau}{2})} \right|^{-\frac{1}{4}} \frac{1}{|\eta(\frac{\tau}{2})|^2} \left| \theta_2\left(\frac{\tau}{2}\right) \right|^2, \tag{A.52}$$

as desired.

Case $(l_1, l'_1) = (0, 1), (l_2, l'_2) = (1, 1)$. In this case we need to add the $l' = 1$ orbifold twist to the four-point function just discussed. This is equivalent to setting $\eta' = \frac{1}{2}$ in the previous computation, which yields

$$Z_{(0,1)(1,1)}^{\text{NS},k=1}(L) = \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{L}{2}|\frac{\tau}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{1}{2}|\frac{\tau}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|\frac{\tau}{2})}{\theta_1(\frac{L+1}{2}|\frac{\tau}{2})} \right|^{-\frac{1}{4}} \frac{1}{|\eta(\frac{\tau}{2})|^2} \left| \theta_1\left(\frac{\tau}{2}\right) \right|^2 = 0, \tag{A.53}$$

so this contribution vanishes.

Case $(l_1, l'_1) = (0, 0), (l_2, l'_2) = (0, 1)$. In this case, $\psi_{L,[2]}^1(z+\tau) = \psi_{L,[2]}^2(z)$ or $\psi_{L,[2]}^2(z+\tau) = \psi_{L,[2]}^1(z)$, so $\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}$ and $\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}$ are again not independent:

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z+\tau) = -\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z), \tag{A.54a}$$

$$\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z+\tau) = -\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z), \tag{A.54b}$$

so now the period in the τ -direction is doubled and the modulus of the torus becomes 2τ . Again, bosonization ceases to be independent and the replica-twist operators are

correlated, leading to

$$\begin{aligned}
 Z_{(0,0)(0,1)}^{\text{NS},k=1}(L) &= Z \cdot \left\langle \prod_{p=-(2-1)/2}^{(2-1)/2} e^{i\frac{p}{2}\tilde{\varphi}_{L,p}^{(1)}(L)} e^{-i\frac{p}{2}\tilde{\varphi}_{L,p}^{(1)}(0)} \right\rangle \cdot (\text{right mover}) \\
 &= Z \cdot \left\langle e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,-\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(0)} e^{+i\frac{1}{4}\tilde{\varphi}_{L,-\frac{1}{2}}^{(1)}(0)} \right\rangle \cdot (\text{right mover}) \\
 &= Z \cdot \left\langle e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L+\tau)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(0)} e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(\tau)} \right\rangle \Big|_{2\tau} \cdot (\text{right mover}),
 \end{aligned} \tag{A.55}$$

which can be computed using eq. (A.39). Since the τ -direction is now twice as long, we extract the θ_4 contribution with $\eta = \eta' = 0$ and obtain

$$Z_{(0,0)(0,1)}^{\text{NS},k=1}(L) = \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(L|2\tau)} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(\tau|2\tau)} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(L+\tau|2\tau)} \right|^{-\frac{1}{4}} \frac{1}{|\eta(2\tau)|^2} |\theta_4(2\tau)|^2, \tag{A.56}$$

as desired.

Case $(l_1, l'_1) = (1, 0), (l_2, l'_2) = (1, 1)$. In this case we must add the $l = 1$ orbifold twist to the four-point function above, which amounts to taking $\eta = \frac{1}{2}$. We then obtain

$$Z_{(1,0)(1,1)}^{\text{NS},k=1}(L) = \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(L|2\tau)} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(\tau|2\tau)} \right|^{\frac{1}{4}} \left| \frac{\partial\theta_1(0|2\tau)}{\theta_1(L+\tau|2\tau)} \right|^{-\frac{1}{4}} \frac{1}{|\eta(2\tau)|^2} |\theta_1(2\tau)|^2 = 0, \tag{A.57}$$

so this contribution vanishes.

Case $(l_1, l'_1) = (0, 0), (l_2, l'_2) = (1, 1)$. In this case we have

$$\begin{aligned}
 \psi_{L,[2]}^1(z+1) &= \psi_{L,[2]}^2(z), & \psi_{L,[2]}^2(z+1) &= \psi_{L,[2]}^1(z), \\
 \psi_{L,[2]}^1(z+\tau) &= \psi_{L,[2]}^2(z), & \psi_{L,[2]}^2(z+\tau) &= \psi_{L,[2]}^1(z),
 \end{aligned}$$

so again $\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}$ and $\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}$ are not independent:

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z+1) = -\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z), \tag{A.58a}$$

$$\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z+1) = -\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z), \tag{A.58b}$$

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z+\tau) = -\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z), \tag{A.58c}$$

$$\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z+\tau) = -\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z), \tag{A.58d}$$

and the modulus of the torus becomes $(\tau + 1)/2$.² Repeating the same steps, we have

$$\begin{aligned}
 Z_{(0,0)(1,1)}^{\text{NS},k=1}(L) &= Z \cdot \left\langle \prod_{p=-(2-1)/2}^{(2-1)/2} e^{i\frac{p}{2}\tilde{\varphi}_{L,p}^{(1)}(L)} e^{-i\frac{p}{2}\tilde{\varphi}_{L,p}^{(1)}(0)} \right\rangle \cdot (\text{right mover}) \\
 &= Z \cdot \left\langle e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,-\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(0)} e^{+i\frac{1}{4}\tilde{\varphi}_{L,-\frac{1}{2}}^{(1)}(0)} \right\rangle \cdot (\text{right mover}) \\
 &= Z \cdot \left\langle e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(L+1)} e^{-i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(0)} e^{+i\frac{1}{4}\tilde{\varphi}_{L,+\frac{1}{2}}^{(1)}(1)} \right\rangle \Big|_{\frac{\tau+1}{2}} \cdot (\text{right mover}),
 \end{aligned} \tag{A.59}$$

so from eq. (A.39) we must now extract the θ_1 contribution with $\eta = \eta' = 0$:

$$Z_{(0,0)(1,1)}^{\text{NS},k=1}(L) = \left| \frac{\partial \theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{L}{2}|\frac{\tau+1}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial \theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{1}{2}|\frac{\tau+1}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial \theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{L+1}{2}|\frac{\tau+1}{2})} \right|^{-\frac{1}{4}} \frac{1}{|\eta(\frac{\tau+1}{2})|^2} (-) \left| \theta_1\left(\frac{\tau+1}{2}\right) \right|^2 = 0, \tag{A.60}$$

and again the contribution vanishes.

Case $(l_1, l'_1) = (0, 1), (l_2, l'_2) = (1, 0)$. In this case we have

$$\begin{aligned}
 \psi_{L,[2]}^1(z+1) &= \psi_{L,[2]}^2(z), & \psi_{L,[2]}^2(z+1) &= \psi_{L,[2]}^1(z), \\
 \psi_{L,[1]}^1(z+\tau) &= \psi_{L,[1]}^2(z), & \psi_{L,[2]}^2(z+\tau) &= \psi_{L,[2]}^1(z),
 \end{aligned}$$

so $\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}$ and $\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}$ transform as

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z+1) = -\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z), \tag{A.61a}$$

$$\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z+1) = -\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z), \tag{A.61b}$$

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z+\tau) = +\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z), \tag{A.61c}$$

$$\tilde{\psi}_{L,-\frac{1}{2}}^{(1)}(z+\tau) = +\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z), \tag{A.61d}$$

with the modulus again given by $(\tau + 1)/2$.³ Although the correlator to be computed is the same as in eq. (A.59), the fermions now satisfy

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z+2) = +\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z), \tag{A.62a}$$

$$\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z+1+\tau) = -\tilde{\psi}_{L,+\frac{1}{2}}^{(1)}(z), \tag{A.62b}$$

²The cutoff scale on the torus is halved.

³The cutoff scale on the torus is halved.

and therefore in eq. (A.39) we must extract the θ_2 contribution with $\eta = \eta' = 0$:

$$Z_{(0,1)(1,0)}^{\text{NS},k=1}(L) = \left| \frac{\partial \theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{L}{2}|\frac{\tau+1}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial \theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{1}{2}|\frac{\tau+1}{2})} \right|^{\frac{1}{4}} \left| \frac{\partial \theta_1(0|\frac{\tau+1}{2})}{\theta_1(\frac{L+1}{2}|\frac{\tau+1}{2})} \right|^{-\frac{1}{4}} \frac{1}{|\eta(\frac{\tau+1}{2})|^2} \left| \theta_2\left(\frac{\tau+1}{2}\right) \right|^2. \quad (\text{A.63})$$

Results.

We now have all the contributions $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS},k=1}(L)$, and we can summarize the result for $\text{Tr}_A[\rho_A^2]$ as in eq. (A.46):

$$\begin{aligned} & \left(Z_{\text{Cyc}^2\mathcal{D}}^{\text{NS}} \right)^2 \cdot \text{Tr}_A[\rho_A^2] \\ &= \left| \frac{\partial \theta_1(0|\tau)}{\theta_1(L|\tau)} \right|^{\frac{1}{4}} \frac{1}{|\eta(\tau)|^4} \left| \theta_3\left(\frac{1}{4}L|\tau\right) \right|^2 \left| \theta_3\left(-\frac{1}{4}L|\tau\right) \right|^2 \cdot \left(\frac{1}{2}\right)^2 \sum_{l_1, l'_1=0}^1 \sum_{l_2, l'_2=0}^1 Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS},k=1}(L), \end{aligned}$$

with $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS},k=1}(L)$ given by eqs. (A.47), (A.48), (A.52), (A.53), (A.56), (A.57), (A.60), and (A.63).

We now verify that this result satisfies the thermal Rényi relation (3.29) for $n = 2$. To this end, we set $L = \sigma, 1 - \sigma$ and examine the behavior of $Z_{(l_1, l'_1)(l_2, l'_2)}^{\text{NS},k=1}(L)$ in the limit $\sigma \rightarrow 0$.⁴ Factoring out the universal singularity $|\sigma|^{-1/4}$, we obtain the following limits as $L \rightarrow 0$ and $L \rightarrow 1$.

Limit $L \rightarrow 0$. For the diagonal contributions we find

$$\begin{aligned} Z_{(0,0)(0,0)}^{\text{NS},k=1}(0 + \sigma) &= \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_3(\tau)}{\eta(\tau)} \right|^4, & Z_{(0,1)(0,1)}^{\text{NS},k=1}(0 + \sigma) &= \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_4(\tau)}{\eta(\tau)} \right|^4, \\ Z_{(1,0)(1,0)}^{\text{NS},k=1}(0 + \sigma) &= \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_2(\tau)}{\eta(\tau)} \right|^4, & Z_{(1,1)(1,1)}^{\text{NS},k=1}(0 + \sigma) &= 0, \end{aligned} \quad (\text{A.64})$$

⁴In some of these computations, note that changes in the modulus of the torus imply corresponding changes in the cutoff scale.

while for the off-diagonal contributions, using (A.47), we obtain

$$\begin{aligned}
 Z_{(0,0)(1,0)}^{\text{NS},k=1}(0+\sigma) &= \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_2(\frac{\tau}{2})}{\eta(\frac{\tau}{2})} \right|^2 = \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \frac{|\theta_2(\tau)\theta_3(\tau)|^2}{|\eta(\tau)|^4}, \\
 Z_{(0,0)(0,1)}^{\text{NS},k=1}(0+\sigma) &= \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_4(2\tau)}{\eta(2\tau)} \right|^4 = \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \frac{|\theta_3(\tau)\theta_4(\tau)|^2}{|\eta(\tau)|^4}, \\
 Z_{(0,1)(1,0)}^{\text{NS},k=1}(0+\sigma) &= \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_2(\frac{\tau+1}{2})}{\eta(\frac{\tau+1}{2})} \right|^2 = \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \frac{|\theta_2(\tau)\theta_4(\tau)|^2}{|\eta(\tau)|^4}, \\
 Z_{(0,1)(1,1)}^{\text{NS},k=1}(0+\sigma) &= Z_{(1,0)(1,1)}^{\text{NS},k=1}(0+\sigma) = Z_{(0,0)(1,1)}^{\text{NS},k=1}(0+\sigma) = 0,
 \end{aligned} \tag{A.65}$$

and therefore

$$\left(Z_{\text{Cyc}^2\mathcal{D}}^{\text{NS}} \right)^2 \cdot \text{Tr}_A[\rho_A^2] = \frac{1}{|\sigma|^{\frac{1}{2}}} \cdot \left(\left| \frac{\theta_3(\tau)}{\eta(\tau)} \right|^2 \right)^2 \cdot \left(\frac{1}{2} \cdot \frac{|\theta_3(\tau)|^2 + |\theta_2(\tau)|^2 + |\theta_4(\tau)|^2}{|\eta(\tau)|^2} \right)^2. \tag{A.66}$$

Limit $L \rightarrow 1$. For the diagonal contributions we obtain

$$\begin{aligned}
 Z_{(0,0)(0,0)}^{\text{NS},k=1}(1-\sigma) &= Z_{(0,1)(0,1)}^{\text{NS},k=1}(1-\sigma) = \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_3(2\tau)}{\eta(2\tau)} \right|^4, \\
 Z_{(1,0)(1,0)}^{\text{NS},k=1}(1-\sigma) &= Z_{(1,1)(1,1)}^{\text{NS},k=1}(1-\sigma) = \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_2(2\tau)}{\eta(2\tau)} \right|^4,
 \end{aligned} \tag{A.67}$$

while for the off-diagonal contributions we obtain

$$\begin{aligned}
 Z_{(0,0)(0,1)}^{\text{NS},k=1}(1-\sigma) &= Z_{(0,1)(0,0)}^{\text{NS},k=1}(1-\sigma) = \frac{1}{|\sigma|^{\frac{1}{4}}} \cdot \left| \frac{\theta_4(2\tau)}{\eta(2\tau)} \right|^4, \\
 (\text{others}) &= 0,
 \end{aligned} \tag{A.68}$$

and hence

$$\left(Z_{\text{Cyc}^2\mathcal{D}}^{\text{NS}} \right)^2 \cdot \text{Tr}_A[\rho_A^2] = \frac{1}{|\sigma|^{\frac{1}{2}}} \cdot \left| \frac{\theta_3(2\tau)}{\eta(2\tau)} \right|^2 \cdot \left(\frac{1}{2} \cdot \frac{|\theta_3(2\tau)|^2 + |\theta_2(2\tau)|^2 + |\theta_4(2\tau)|^2}{|\eta(2\tau)|^2} \right). \tag{A.69}$$

Consistency condition. From these results we can confirm that

$$\lim_{\sigma \rightarrow 0} \left[S_A^{(2)}(1-\sigma) - S_A^{(2)}(\sigma) \right] = \frac{1}{1-2} \log \frac{Z_{\text{Cyc}^2\mathcal{D}}^{\text{NS}}(2\tau, 2\bar{\tau})}{\left(Z_{\text{Cyc}^2\mathcal{D}}^{\text{NS}}(\tau, \bar{\tau}) \right)^2}, \tag{A.70}$$

as expected. Here we have used the expression

$$Z_{\text{Cyc}^2\mathcal{D}}^{\text{NS}}(\tau, \bar{\tau}) = \left| \frac{\theta_3(\tau)}{\eta(\tau)} \right|^2 \cdot \left(\frac{1}{2} \cdot \frac{|\theta_3(\tau)|^2 + |\theta_2(\tau)|^2 + |\theta_4(\tau)|^2}{|\eta(\tau)|^2} \right), \quad (\text{A.71})$$

which follows from eq. (3.76).

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