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Extremal Problems and Ramsey Properties of Ball, Box or Orthant containing many points in R^d — And Combinatorics of Permutations

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1 Ball and Box

For any points $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d) \in \mathbf{R}^d$, let $Box_d(x, y)$ be the smallest d-dimensional standard box in $\mathbf{R}^d$ which contains the two point x, y, i.e.

$$Box_d(x, y) := \{z = (z_i)_i \in \mathbf{R}^d | x_i \leq z_i \leq y_i \text{ or } x_i \geq z_i \geq y_i \text{ for any } 1 \leq i \leq d\} - \{x, y\}.$$

And let $Ball_d(x, y)$ be the smallest d- dimensinal ball in $\mathbf{R}^d$ which contains the two points $x, y \in \mathbf{R}^d$, i.e.

$$Ball_d(x, y) := \{\frac{1}{2}(x + y) + r \mid \|r\| \leq \frac{1}{2}\|x - y\|\} - \{x, y\},$$

where $\|\cdot\|$ means the euclidean norm.

For any positive integers d, n, if $F = Box$ or $Ball$, then we define $\Pi^F(n, d)$ the largest number which satisfies the condition (*)"For any set P of n points in $\mathbf{R}^d$, there exist two points $x, y \in P$ such that $F_d(x, y)$ contains $\Pi^F(n, d)$ points of P."

When "For any set P " is replaced by "For any convex set P " in (*), we denote $\Pi^F(n, d)$ by $\overline{\Pi}^F(n, d)$.

Clearly, $\Pi^{Box}(n, 1) = \overline{\Pi}^{Box}(n, 1) = \Pi^{Ball}(n, 1) = \overline{\Pi}^{Ball}(n, 1) = n$.

Proposition 1 $\Pi^{Box}(n, 2) = \left\lceil \frac{n-4}{5} \right\rceil, \overline{\Pi}^{Box}(n, 2) = \left\lceil \frac{n}{4} \right\rceil - 1.$

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Theorem 2 $\Pi^{Ball}(n,2) = \overline{\Pi}^{Ball}(n,2) = \lceil \frac{n}{3} \rceil - 1.$

J.Urrutia conjectured $\overline{\Pi}^{Ball}(n) \geq n/2$. Theorem 2 disprove it.

Theorem 3 *For any integer* $n, d(\geq 1)$,

$$\left(\frac{2}{8^{2^{d-1}}}\right) n \leq \Pi^{Box}(n,d) \leq \left(\frac{9.49}{2^{2^{d-1}} 1.47^d}\right) n + 2.$$

Theorem 4 *For any integer* $n, d(\geq 1)$,

$$\left(\frac{2}{8^d}\right) n - 2 \leq \Pi^{Ball}(n,d) \leq \left(\frac{2}{1.15^d}\right) n.$$

It is interesting to compare Theorem 3 with Erdös-Szekeres Theorem($\mathbf{R}^{d+1}$-version).

2 Orthant and Permutation

N.G.de Bruijn extended the Erdös-Szekeres Theorem " *Any sequence of integers of length n contains a monotone subsequence of length* $\lceil\sqrt{n}\rceil$ *(best possible)* " to a result about sequences of d-dimensional vectors, which includes the following proposition:

Let $r(d)$ *be the largest number such that there is a set* P *of* $r(d)$ *points of* $\mathbf{R}^d$ *whose boxes are empty, i.e.* $Box_d(x,y) \cap P = \emptyset$ *for any* $x, y \in P$. *Then* $r(d) = 2^{2^{d-1}}$.

N. Alon, Z. Füredi and M. Katchalski studied a set of n points of $\mathbf{R}^d$ having many empty boxes.

When P is a finite set of points of $\mathbf{R}^d$, for $x = (x_i)_i \in P$ and for $\varepsilon \in \{-1,1\}^d$, consider the εth-orthant having x as the origin,

$$Orth_d(x,\varepsilon) := \{z \in \mathbf{R}^d | \text{ For } \forall i, \text{ if } \varepsilon = 1, z_i \geq x_i, \text{ and if } \varepsilon = -1, z_i \leq x_i\} - \{x\}.$$

Theorem 5 *Let* $l(d)$ *be the largest number such that there is a set* P *of* $l(d)$ *points of* $\mathbf{R}^d$ *whose orthants contains at most one point, i.e.* $|Orth_d(x,\varepsilon) \cap P| \leq 1$ *for* $\forall x \in P$ *and* $\forall \varepsilon \in \{-1,1\}^d$. *Then*

$$1.47^d \leq l(d) \leq c\binom{d}{\lceil d/4 \rceil} < 1.76^d$$

for an absolute constant c and any sufficiently large d. (The lower bound can be shown constructively.)

Let $t, n(t \leq n)$ be positive integers and A a set of n elements. A finite sequence $\sigma = \sigma(1)\sigma(2)\cdots\sigma(t)$ is a *t-permutation* of A if and only if $\sigma(i) \in A$ for any $1 \leq i \leq t$ and $\sigma(i) \neq \sigma(j)$ for $1 \leq \forall i < \forall j \leq t$. The *inverse* of σ is the sequence $\sigma^{-1} = \sigma(t)\sigma(t-1)\cdots\sigma(1)$. Note that the inverse of a t-permutation is a t-permutation. A n-permutation σ of A *contains* a t-parmutation of A if τ is a subsequence of σ. Let $n_t(d)$ [$n_t^*(d)$] be the largest number n having d n-permutations $\{\sigma_1, \sigma_2 \cdots, \sigma_d\}$ of A such that for any t-permutation τ of A, there exists $\sigma_i (1 \leq \exists i \leq d)$ containing τ [τ or τ^{-1}]. A simple argument show that

$$l(d) = n_3^*(d).$$

For example, the five orders 1643275, 2654371, 3615472, 4621573, 5632174 of {1, 2, $\cdots$, 7} yields $n_3^*(5) \geq 7$. We will obtain bounds of $n_t(d)$ and $n_t^*(d)$.

Theorem 6 (i) *For $t \geq 4$ and $d \geq t!$,*

$$\left(1 - \frac{1}{t}\right)\left(\frac{1}{t}\right)^{\frac{1}{t-1}}\left(\frac{t!}{t!-1}\right)^{\frac{d}{t-1}} \leq n_t(d) \leq t - 3 + \left(\frac{d}{\lceil\frac{d}{(t-2)!}\rceil}\right)^{\frac{1}{t-2}}.$$

(ii) *For $t \geq 6$ and $d \geq t!$,*

$$\left(1 - \frac{1}{t}\right)\left(\frac{2}{t}\right)^{\frac{1}{t-1}}\left(\frac{t!}{t!-2}\right)^{\frac{d}{t-1}} \leq n_t^*(d) \leq t - 4 + 2^{\frac{1}{t-3}}\left(\frac{d}{\lceil\frac{d}{(t-3)!}\rceil}\right)^{\frac{1}{t-3}}.$$

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