

Extensions of partitions of unity and covers

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1. Introduction

By a space we mean a topological space and γ denotes an infinite cardinal. Let X be a space and A a subspace of X . By Shapiro [13], A is said to be P^γ -embedded in X if every γ -separable continuous pseudo-metric on A can be extended to a continuous pseudo-metric on X . A subspace A is said to be P -embedded in X if A is P^γ -embedded in X for every γ . Recently, Dydak [5] defined that A is $P^\gamma(\text{locally-finite})$ -embedded in X if for every locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|_A = f_\alpha$ for every $\alpha \in \Omega$. A subspace A is said to be $P(\text{locally-finite})$ -embedded in X if A is $P^\gamma(\text{locally-finite})$ -embedded in X for every γ .

It was proved in [5] that $P^\gamma(\text{locally-finite})$ -embedding implies P^γ -embedding. This fact is also verified from characterizations of P^γ -embedding and $P^\gamma(\text{locally-finite})$ -embedding as the following. On Theorem 1.1, (1) $\Leftrightarrow$ (2) is well-known (cf. [1]), and (1) $\Leftrightarrow$ (3) is in [5] or [11].

Theorem 1.1 ([1], [5], [11]). *For a space X and a subspace A of X , the following statements are equivalent:*

- (1) A is P^γ -embedded in X ;
- (2) for every locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of X such that $V_\alpha \cap A \subset U_\alpha$ for every $\alpha \in \Omega$;
- (3) for every locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a (not necessarily locally finite) partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|_A = f_\alpha$ for every $\alpha \in \Omega$.

Theorem 1.2 ([14]). *For a space X and a subspace A of X , the following statements are equivalent:*

- (1) A is $P^\gamma(\text{locally-finite})$ -embedded in X ;
- (2) for every locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$.

Notice that the space Z given in [11, Example 3] admits a P - but not $P^\omega(\text{locally-finite})$ -embedded subspace (cf. [14]).

The first purpose of this talk is to characterize P^γ -embedding under the viewpoint of exactly extending cozero-set covers such as in Theorem 1.2. The second one is to investigate for P^ω (point-finite)-embedding (see Section 3 for the definition) under the same viewpoint to Theorem 1.2, and apply it to prove that the rationals $\mathbb{Q}$ of the Michael line $\mathbb{R}_\mathbb{Q}$ is not P^ω (point-finite)-embedded in $\mathbb{R}_\mathbb{Q}$.

A collection $\{f_\alpha : \alpha \in \Omega\}$ of continuous functions $f_\alpha : X \rightarrow [0, 1]$, $\alpha \in \Omega$, is said to be a *partition of unity* on X if $\sum_{\alpha \in \Omega} f_\alpha(x) = 1$ for every $x \in X$. A partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on X is said to be *locally finite* (resp. *point-finite* [5], or *uniformly locally finite*) if $\{f_\alpha^{-1}((0, 1]) : \alpha \in \Omega\}$ is locally finite (resp. point-finite, or uniformly locally finite) in X . Here, a collection $\mathcal{F}$ of subsets of X is said to be *uniformly locally finite* (resp. *uniformly discrete*) in X if there exists a normal open cover $\mathcal{U}$ of X such that every $U \in \mathcal{U}$ meets at most finitely many members (resp. at most one member) of $\mathcal{F}$ ([9], [10], [3]).

2. Exact extensions of cozero-set covers and P -embedding

Our main result in this section is the following; Alò-Shapiro proved in [1] the equivalence (1) $\Leftrightarrow$ (3) assuming that X is normal and A is closed in X .

Theorem 2.1 (Main). *For a space X and a subspace A of X , the following statements are equivalent:*

- (1) A is P^γ -embedded in X ;
- (2) for every uniformly locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a uniformly locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|_A = f_\alpha$ for every $\alpha \in \Omega$;
- (3) for every uniformly locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a uniformly locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$.

We apply Theorem 2.1 to give another characterization of P -embedding by exactly extending zero-set collections. Blair [3] essentially proved that: *A subspace A of a space X is P^γ -embedded in X if and only if for every uniformly discrete zero-set collection $\{Z_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a uniformly discrete zero-set collection $\{F_\alpha : \alpha \in \Omega\}$ of X such that $F_\alpha \cap A = Z_\alpha$ for every $\alpha \in \Omega$.* In our case, we give the following:

Theorem 2.2. *For a space X and a subspace A of X , the following statements are equivalent:*

- (1) A is P^γ -embedded in X ;

(2) every uniformly locally finite zero-set collection $\{Z_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \leq \gamma$, there exists a uniformly locally finite zero-set collection $\{F_\alpha : \alpha \in \Omega\}$ of X such that $F_\alpha \cap A = Z_\alpha$ for every $\alpha \in \Omega$.

As another application of Theorem 2.1, we give some results concerning locations of spaces around functionally Katětov spaces. Let γ, κ be infinite cardinals. In [15], a space X is said to be (γ, κ) -Katětov if X is normal and for every closed subspace A of X and every locally finite κ^+ -open cover $\{U_\alpha : \alpha < \gamma\}$ of A , there exists a locally finite κ^+ -open cover $\{V_\alpha : \alpha < \gamma\}$ of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha < \gamma$. Here, a subspace U of X is said to be κ^+ -open set if U can be expressed as the union of κ many cozero-sets of X . When X is (γ, ω) -Katětov for every γ , X is said to be *functionally Katětov* (cf. [7], [11], [15]). Similarly, when X is (γ, κ) -Katětov for every γ and κ (resp. (ω, κ) -Katětov for every κ , or (ω, ω) -Katětov), X is said to be *Katětov* (resp. *countably Katětov*, or *countably functionally Katětov*). Note that γ -collectionwise normal countably paracompactness implies being (γ, κ) -Katětov, and the latter implies γ -collectionwise normality (cf. [7], [15]). Moreover they were proved in [11] that every hereditarily normal space is countably Katětov, and that Rudin's Dowker space is functionally Katětov but not countably Katětov. In [11], they were essentially proved that every collectionwise normal P -space is functionally Katětov and that every normal P -space is countably functionally Katětov; here a space is said to be a P -space if every cozero-set is closed. A space X is said to be *hereditarily basically disconnected* if for every subspace A of X , the closure of a cozero-set of A in A is open in A .

With the aid of Theorem 2.1, we slightly generalize the result mentioned above in the following:

Lemma 2.3. *Let X be a γ -collectionwise normal space. Assume that for every closed subspace A of X , every locally finite κ^+ -open cover, with card $\leq \gamma$, of A is uniformly locally finite in A . Then, X is (γ, κ) -Katětov.*

Hence we have:

Theorem 2.4. *Every γ -collectionwise normal and hereditarily basically disconnected space is (γ, ω) -Katětov.*

It also follows from Lemma 2.3 that: *If X is a collectionwise normal and hereditarily extremally disconnected space, then X is Katětov*; where X is said to be *hereditarily extremally disconnected* if for every subspace A of X , the closure of an open set of A in A is open in A . The author does not know the assumption of X above implies countable paracompactness of X .

3. $P(\text{point-finite})$ -embeddings and covers

Let X be a space and A a subspace of X . On exactly extending partitions of unity, consider the following conditions:

- (i) for every partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|_A = f_\alpha$ for every $\alpha \in \Omega$;
- (ii) for every point-finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a point-finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|_A = f_\alpha$ for every $\alpha \in \Omega$;
- (iii) for every locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|_A = f_\alpha$ for every $\alpha \in \Omega$;
- (iv) for every uniformly locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \leq \gamma$, there exists a uniformly locally finite partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|_A = f_\alpha$ for every $\alpha \in \Omega$.

Dydak proved in [5] that (i) equals that A is P^γ -embedded in X , and Theorem 2.1 shows that (iv) also equals that A is P^γ -embedded in X . The condition (iii) is precisely the definition of $P^\gamma(\text{locally-finite})$ -embedding; as was already commented in the introduction, (iii) is strictly stronger than the P^γ -embedding. By Dydak [5], the above condition (ii) is said to be that A is $P^\gamma(\text{point-finite})$ -embedded in X and it is proved in [5] that this condition is also strictly stronger than the P^γ -embedding (cf. Theorem 3.4 below).

Recall Theorem 1.2 and (2) $\Leftrightarrow$ (3) of Theorem 2.1. Then, we see that P^γ -embedding and $P^\gamma(\text{locally-finite})$ -embedding can be stated by extensions of cozero-set covers as well as extensions of partitions of unity. On the other hand, for $P^\gamma(\text{point-finite})$ -embedding, we have the following theorem and examples.

Theorem 3.1 (Main). *For a space X and a subspace of A , the following statements are equivalent:*

- (1) A is $P^\omega(\text{point-finite})$ -embedded in X ;
- (2) for every point-finite countable cozero-set cover $\{U_n : n \in \mathbb{N}\}$ of A , there exists a point-finite countable cozero-set cover $\{V_n : n \in \mathbb{N}\}$ of X such that $V_n \cap A = U_n$ for every $n \in \mathbb{N}$.

The following examples show that Theorem 3.1 need not hold on uncountable cardinal cases.

Example 3.2. *Let γ be an uncountable cardinal. There exist a space X and a closed subspace A of X such that every point-finite cozero-set cover*

of A can be extended to a point-finite cozero-set cover of X , but A is not P^γ -embedded in X .

Sketch of the construction. We use notations as in [2] and [8]. In particular, we assume the uncountable set P in [2] as $|P| = \gamma$. Let F , f_p and F_p be the same as in [2]. Let G be the space in [8], namely,

$$G = F_p \cup \{f \in F : f(q) = 0 \text{ except for finitely many } q \in Q\}.$$

Consider the space introduced in the last part of [8, Example 2] and denote it X , namely,

$$X = (F_p \times \{0\}) \cup (G \times \{1/i : i \in \mathbb{N}\})$$

taking as a base at a point $(y, 0)$ the sets $\{(y, 0)\} \cup (U \times \{1/i : i \geq j\})$, where U is a neighborhood of y in G and $j \in \mathbb{N}$, and other points be isolated. Let $A = F_p \times \{0\}$.

Example 3.3. *There exist a space X and a closed subspace A of X such that A is P (point-finite)-embedded in X , but that A has a point-finite cozero-set cover which can not be extended to a cozero-set cover of X .*

Sketch of the construction. Consider the product space $Z = L(\omega_1) \times (\omega + 1) \times (\omega_2 + 1)$, where $L(\omega_1)$ is the set $\omega_1 + 1$ taking a base at the point ω_1 as $\{[\beta, \omega_1] : \beta < \omega_1\}$ and other points be isolated; and $\omega + 1$ and $\omega_2 + 1$ have the usual order topology. Let $X = Z - \{(\omega_1, \omega, \omega_2)\}$ and $A = L(\omega_1) \times (\omega + 1) \times \{\omega_2\} - \{(\omega_1, \omega, \omega_2)\}$ a subspace of X .

We give an application of Theorem 3.1. Let $\mathbb{R}_\mathbb{Q}$ be the Michael line and $\mathbb{Q}$ be the rationals. Dydak commented in [5] that “we do not know if $\mathbb{Q}$ is P (point-finite)-embedded in $\mathbb{R}_\mathbb{Q}$ ” and constructed his own example of a P -embedding which is not P (point-finite)-embedding. Answering his question, we have the following:

Theorem 3.4. $\mathbb{Q}$ is not P^ω (point-finite)-embedded in $\mathbb{R}_\mathbb{Q}$.

Finally we give a result that three extension properties equal under a condition only for the subspace A .

Theorem 3.5. *Let X be a space, A a subspace of X and γ an infinite cardinal. If A is a P -space, then the following statements are equivalent:*

- (1) A is P^γ -embedded in X ;
- (2) A is P^γ (locally-finite)-embedded in X ;
- (3) A is P^γ (point-finite)-embedded in X .

Note that every closed subspace of Rudin’s Dowker space is P (point-finite)-embedded; it can be proved by combining some results in [5], [6] and [12]. This fact can also be seen by the above theorem directly.

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