

Commutators of Singular Integral Operators on Morrey Spaces with General Growth Functions

山形大学理学部 水原 昂廣 (Takahiro Mizuhara)

Abstract of the Talk

The talk will be concerned with the boundedness of the commutators of Calderon-Zygmund singular integral operators on Morrey spaces $L^{p,\Phi}(R^n)$ with growth functions $\Phi(x, r)$ satisfying the condition ; there exists a constant C , independent of $(x, r) \in R_+^{n+1}$, such that for any $(x, r) \in R_+^{n+1}$

$$(1) \quad \int_r^\infty [\Phi(x, t)/t^{a+1}]dt \leq C\Phi(x, r)/r^a, \quad \text{for some } a > 0.$$

In this case, we write $\Phi \in G_a$ simply. We denote by $L^{p,\Phi}(R^n)$, $0 < p < \infty$, the space of locally integrable functions f , defined on R^n , for which there exists a constant C , independent of balls $B = B(x, r)$, such that

$$(2) \quad \int_{B(x,r)} |f(y)|^p dy \leq C^p \Phi(x, r)$$

for all balls $B = B(x, r)$. Let $\|f\|_{p,\Phi}$ be the smallest constant C satisfying (2). Then the space $L^{p,\Phi}(R^n)$ becomes a quasi-Banach space with quasi-norm $\|\cdot\|_{p,\Phi}$. In particular, if $1 \leq p < \infty$, then the space $L^{p,\Phi}(R^n)$ becomes a Banach space with norm $\|\cdot\|_{p,\Phi}$.

Let $BMO(R^n)$ be the space of all functions of bounded mean oscillation and let $\Lambda_\alpha(R^n)$, $0 < \alpha < n$, be the space of all Lipschitz continuous functions of order α . Let M be the Hardy-Littlewood maximal operator. We need two variants of M . For $0 < q < \infty$ let $M_q f(x) = \{(M|f|^q)(x)\}^{1/q}$. The sharp maximal function $f^\sharp(x)$ is defined by

$$f^\sharp(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y) - f_B| dy, \quad \text{where } f_B = |B|^{-1} \int_B f(y) dy.$$

Let T be a Calderon-Zygmund singular integral operator defined by $Tf = k * f$ with the kernel k satisfying the conditions :

$$\|\hat{k}\|_\infty \leq C, \quad |k(x)| \leq C|x|^{-n} \quad \text{for } 0 \neq x \in R^n,$$

$$|k(x) - k(x-y)| \leq C|y|/|x|^{n+1} \quad \text{for } |y| \leq |x|/2.$$

Let I_α , $0 < \alpha < n$, be the Riesz potential of order α defined by

$$(I_\alpha f)(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Related to $I_\alpha f$, the fractional maximal function $M_{\alpha/n}^* f(x)$ is defined by

$$M_{\alpha/n}^* f(x) = f_{\alpha,1}^*(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy.$$

For a locally integrable function b and an operator S , we define the commutator $[b, S]$, between the operator S and the multiplication operator by b , by $[b, S] = bS - Sb$.

We have proved the following ([Miz₂]);

Theorem 1(Theorem 2.1.) *Let $0 < p < \infty$. We assume that $\Phi \in G_n$. Then there exists a constant $C = C(p, \Phi) > 0$, independent of f , such that for all $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(3) \quad \|Mf\|_{p,\Phi} \leq C \|f^\sharp\|_{p,\Phi}$$

where $L_c^\infty(R^n)$ be the set of all essentially bounded functions on R^n with compact support.

We use the method due to Di Fazio and Ragusa. Our method is based on weighted maximal inequality due to Garcia-Cuerva and Rubio de Francia.

From this and the pointwise estimate due to Strömberg ;

$$\{[b, T](f)\}^\sharp(x) \leq C \|b\|_* \{M_q(Tf)(x) + (M_s f)(x)\}, \quad 1 < q, s < \infty,$$

for almost all $x \in R^n$, we obtain the boundedness of the commutators $[b, T]$ on Morrey spaces ([Miz₂]) ;

Theorem 2(Theorem 2.2.) *Let $1 < p < \infty$ and $b \in BMO(R^n)$. We assume that $\Phi \in G_n$. Then the commutator $[b, T]$ is bounded in $L^{p,\Phi}$. More precisely, there exists a constant $C = C(p, \Phi) > 0$, independent of b and f , such that for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(4) \quad \|[b, T](f)\|_{p,\Phi} \leq C \|b\|_* \|f\|_{p,\Phi}.$$

Also we can observe the following ([Miz₂]) ;

Theorem 3(Theorem 2.3.) *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, T]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of b and f , such that for all $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(5) \quad \|[b, T]f\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha(R^n)} \|f\|_{p,\Phi}.$$

This follows from the result (due to Naki [N]) of the boundedness of Riesz potential on Morrey spaces and the pointwise estimate ;

$$|([b, T]f)(x)| \leq \int_{R^n} |b(x) - b(y)| |k(x-y)| |f(y)| dy \leq C \|b\|_{\Lambda_\alpha(R^n)} I_\alpha(|f|)(x).$$

Further we obtain the following result ([Miz₂]) from the boundedness of the fractional maximal operator $M_{\alpha/n}^*$ on Morrey spaces and the pointwise estimate due to Strömberg ;

$$\{[b, I_\alpha](f)\}^\sharp(x) \leq C \|b\|_* \{M_u(I_\alpha f)(x) + (M_{\alpha t/n}^* |f|^t)^{1/t}(x)\}$$

for almost all $x \in R^n$, where $1 < u, t < p < n/\alpha$.

Theorem 4(Theorem 3.1.) *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$ and $\Phi^{q/p} \in G_n$. If $b \in BMO(R^n)$, then the commutator $[b, I_\alpha]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of b and f , such that for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(6) \quad \|[b, I_\alpha]f\|_{q,\Phi^{q/p}} \leq C \|b\|_* \|f\|_{p,\Phi}.$$

Similarly we can show the following ([Miz₂]) ;

Theorem 5(Theorem 3.2.) *Let $1 < p < q < \infty$, $0 < \alpha, \beta$, $0 < \alpha + \beta = n(1/p - 1/q) < n$, $1 < p < n/(\alpha + \beta)$. We assume that $\Phi \in G_{n-p(\alpha+\beta)}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, I_\beta]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of b and f , such that for all $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(7) \quad \|[b, I_\beta]f\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha(R^n)} \|f\|_{p,\Phi}.$$

Our results (Theorems 1, 2 and 4) generalize partly the classical results due to Di Fazio and Ragusa [DiFRag]. Also we obtain the new results (Theorems 3 and 5).

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1. Introduction.

Let $\Phi = \Phi(x, r)$, be a growth function on $R_+^{n+1} = R^n \times R_+$, that is, a positive and non-decreasing function with respect to $r > 0$. We say that the growth function $\Phi(x, r)$ satisfies the Δ_2 -condition (or doubling condition) for $r > 0$ if there exists constant $D = D(\Phi) \geq 1$, independent of (x, r) , such that

$$(1.1) \quad \Phi(x, 2r) \leq D\Phi(x, r), \quad (x, r) \in R_+^{n+1},$$

or equivalently,

$$\Phi(x, 2r)/D \leq \Phi(x, r) \leq \Phi(x, 2r), \quad (x, r) \in R_+^{n+1}.$$

In this case, we write $\Phi \in \Delta_2$ simply. We consider the following functions in Δ_2 ;

$$\Phi(x, r) = \Psi(x)r^\lambda \{\log(1+r)\}^\mu, \quad \Psi(x) \in L^\infty(R^n), \quad 0 \leq \lambda < \infty, \quad -\infty < \mu < \infty.$$

Remark. Nakai [Nak] assumed a slightly weak condition on $\Phi(x, r)$ replacing (1.1) ; there exists a constant $C > 0$ such that, for all $(x, r) \in R_+^{n+1}$,

$$(1.2) \quad r \leq t \leq 2r \implies C^{-1} \leq \Phi(x, t)/\Phi(x, r) \leq C.$$

However, for simplicity, we describe the results on the assumption of (1.1). Of course our results are also valid under the condition (1.2).

Function Spaces. Let R^n be the n -dimensional Euclidean space and let $B = B(x, r)$ be the ball centered at $x \in R^n$ and with radius $r > 0$. Let $Q = Q(x, r)$ be the cube centered at $x \in R^n$ and with sides of length $r > 0$, where the cube will always mean a compact cube with sides parallel to the axes and nonempty interior. $|B|$ and $|Q|$ stand for the Lebesgue measures of ball B and cube Q , respectively. Let $0 < p < \infty$.

Definition 1.1 (Morrey spaces). (Confer Mizuhara [Miz₁]). We denote by $L^{p,\Phi} = L^{p,\Phi}(R^n)$ the space of locally integrable functions f , defined on R^n , for which there exists a constant C , independent of balls $B = B(x, r)$, such that

$$(1.3) \quad \int_{B(x,r)} |f(y)|^p dy \leq C^p \Phi(x, r)$$

for all balls $B = B(x, r)$. Let $\|f\|_{p,\Phi}$ be the smallest constant C satisfying (1.3). Then the space $L^{p,\Phi}$ becomes a quasi-Banach space with quasi-norm $\|\cdot\|_{p,\Phi}$ in the sense of Triebel [Tri]. In particular, if $1 \leq p < \infty$, then the space $L^{p,\Phi}$ becomes a Banach space with norm $\|\cdot\|_{p,\Phi}$. The balls $B = B(x, r)$ in (1.3) can be replaced by cubes $Q = Q(x, r)$.

When $\Phi(x, r) = r^\lambda$, $\lambda \geq 0$, then $L^{p,\lambda}$ is the classical Morrey space denoted by $L^{p,\lambda}$ simply. The classical Morrey spaces $L^{p,\lambda}$, $0 < \lambda < n$, were originally introduced by Morrey [Mor] in 1938 and used by himself and the others in the problems related to the calculus of variations and the theory of elliptic PDE's. We refer to Campanato [Cam], Giaginta [Gia], Kufner-John-Fučik [KufJohFuc] and Peetre [P₂].

The some properties of $L^{p,\lambda}$ are known ; If $1 \leq p < \infty$, then $L^{p,0} = L^p(R^n)$ and $L^{p,n} = L^\infty(R^n)$ isometrically. If $n < \lambda$, then $L^{p,\lambda} = \{0\}$. If $1 \leq p < \infty$ and $0 < \lambda < n$, then $L^{p,\lambda}$ does not include nonzero constants. Hence, in the classical Morrey spaces, $L^{p,\lambda}$ for $0 < \lambda < n$ is interesting. Also Hölder's inequality implies the imbedding theorem ; if $(n - \lambda)/q = (n - \mu)/p$, $p \leq q$, then $L^{q,\lambda} \subset L^{p,\mu}$.

Let $BMO(R^n)$ be the John-Nirenberg space of all functions of bounded mean oscillation (see John-Nirenberg [JoN]), that is, $BMO(R^n)$ is a Banach space, modulo constants, with norm $\|\cdot\|_*$ defined by

$$\|b\|_* = \sup_B |B|^{-1} \int_B |b(y) - b_B| dy, \quad \text{where } b_B = |B|^{-1} \int_B b(y) dy.$$

The space $BMO(R^n)$ is identified with the dual space of the Hardy space $H^1(R^n)$ in the sense of Fefferman-Stein ([FeS₂]).

Let $\Lambda_\alpha(R^n)$, $0 < \alpha < n$, be the space of all Lipschitz continuous functions of order α on R^n . The space $\Lambda_\alpha(R^n)$ is homogeneous in the sense of dilations. The dual space of $H^p(R^n)$ can be identified with the Lipschitz space $\Lambda_\alpha(R^n)$, $\alpha = n(1/p - 1)$.

Classical operators. Let f be a locally integrable function on R^n . The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy$$

where the supremum is taken over all balls B containing x and $|B|$ is the volume of the ball B . We introduce two variants of M . Let $0 < q < \infty$ and

$$M_q f(x) = \{(M|f|^q)(x)\}^{1/q}.$$

Then Hölder's inequality shows that $Mf = M_1 f \leq M_q f$ if $1 \leq q < \infty$ and $M_q f \leq M_1 f = Mf$ if $0 < q \leq 1$. The sharp maximal function $f^\sharp(x)$ is defined by

$$f^\sharp(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y) - f_B| dy, \quad \text{where } f_B = |B|^{-1} \int_B f(y) dy.$$

Let T be a Calderon-Zygmund singular integral operator $Tf = k * f$ defined by the kernel k satisfying the conditions;

$$\|\hat{k}\|_{\infty} \leq C, \quad |k(x)| \leq C|x|^{-n} \quad \text{for } 0 \neq x \in R^n,$$

$$|k(x) - k(x - y)| \leq C|y|/|x|^{n+1} \quad \text{for } |y| \leq |x|/2.$$

For $\epsilon > 0$, put

$$T_{\epsilon}f(x) = \int_{|y|>\epsilon} k(y)f(x-y)dy \quad \text{and} \quad T^*f(x) = \sup_{\epsilon>0} |T_{\epsilon}f(x)|.$$

Let I_{α} , $0 < \alpha < n$, be the fractional integral operator (or Riesz potential operator) of order α defined by

$$(I_{\alpha}f)(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

for a suitable function f . Related to $I_{\alpha}f$, the fractional maximal function $M_{\alpha/n}^*f(x)$, which appeared in [MucWhe] as $f_{\alpha,1}^*(x)$, is defined by

$$M_{\alpha/n}^*f(x) = f_{\alpha,1}^*(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy.$$

We define the commutator $[b, S]$ between an operator S and the multiplication operator by a locally integrable function b , by $[b, S] = bS - Sb$.

In this note we show the boundedness of the commutator $[b, T]$, for $b \in BMO(R^n)$ or $b \in \Lambda_{\alpha}(R^n)$, on Morrey spaces $L^{p,\Phi}(R^n)$ with some growth function Φ . Our results (Theorems 2.1, 2.2, 3.1) generalize partly the recent results due to Di Fazio and Ragusa [DiFRag] on the classical Morrey spaces $L^{p,\lambda}(R^n)$, $0 < \lambda < n$, $1 < p < \infty$. Further we obtain the new results (Theorems 2.3 and 3.2). The letters C 's will denote positive constants, which may have different values in each line.

2. Commutators between Calderon-Zygmund singular integral operators and multiplication operator by a function $b \in BMO(R^n) \cup \Lambda_{\alpha}(R^n)$.

G_{α} -condition. We consider the following condition on growth function $\Phi(x, r)$;

$$\frac{\Phi(x, t)}{t^{\alpha}} \in L^1([r, \infty), dt/t)$$

for all $r > 0$ and any $x \in R^n$, and, in addition, there exists a constant C , independent of $(x, r) \in R_+^{n+1}$, such that

$$(2.1) \quad \int_r^{\infty} [\Phi(x, t)/t^{\alpha+1}] dt \leq C\Phi(x, r)/r^{\alpha}, \quad (x, r) \in R_+^{n+1},$$

for some $a > 0$. In this case, we write $\Phi \in G_a$ simply.

We can observe the following property of G_a , $a > 0$;

Lemma 2.1. (i) If $0 < a < a' < n$, then $G_a \subset G_{a'} \subset G_n \subset \Delta_2$.

(ii) If $\Phi \in \Delta_2$ with doubling constant D , $1 \leq D < 2^n$, then $\Phi \in G_n$.

(iii) If $a > 0$, then $G_a \subset G_{a\gamma}$ for some γ , $0 < \gamma < 1$. More precisely, if $\Phi \in G_a$, $a > 0$, there exist constants $\gamma = \gamma(C, a)$, $0 < \gamma < 1$, and $C' = C'(C, a, \gamma) > 0$ such that for any $(x, r) \in R_+^{n+1}$

$$(2.2) \quad \int_r^\infty [\Phi(x, t)/t^{a\gamma+1}] dt \leq C' \Phi(x, r)/r^{a\gamma}.$$

Proof. (i), (ii) These are easy to see.

(iii) Let

$$\Phi_a(x, r) = \int_r^\infty [\Phi(x, t)/t^{a+1}] dt.$$

Then (2.1) implies

$$\Phi_a(x, r) \leq C \Phi(x, r)/r^a.$$

For $0 < r < R$, we have, integrating by parts and using (2.1),

$$\begin{aligned} \int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt &= \int_r^R [\Phi(x, t)/t^{a+1} t^{a(\gamma-1)}] dt \\ &= [-\Phi_a(x, t) t^{a(1-\gamma)}]_r^R - \int_r^R [-\Phi_a(x, t) a(1-\gamma) t^{a(1-\gamma)-1}] dt \\ &= -\Phi_a(x, R) R^{a(1-\gamma)} + \Phi_a(x, r) r^{a(1-\gamma)} + a(1-\gamma) \int_r^R [\Phi_a(x, t) t^{a(1-\gamma)-1}] dt \\ &\leq C \Phi(x, r)/r^{a\gamma} + a(1-\gamma) C \int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt. \end{aligned}$$

Hence we obtain

$$\int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt \leq \frac{C}{1 - a(1-\gamma)C} \Phi(x, r)/r^{a\gamma},$$

and we have (2.2) with

$$C' = \frac{C}{1 - a(1-\gamma)C} > 0.$$

Thus we have (2.2) for some γ such that $1 - (1/aC) < \gamma < 1$.

Q.E.D.

First using this Lemma, we show the following ;

Theorem 2.1. *Let $0 < p < \infty$. We assume that $\Phi \in G_n$. Then there exists a constant $C > 0$, independent of f , such that*

$$(2.3) \quad \|Mf\|_{p,\Phi} \leq C \|f^\sharp\|_{p,\Phi}$$

for all $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$, where $L_c^\infty(R^n)$ is the set of all essentially bounded functions on R^n with compact support.

Proof. We use the method due to Di Fazio-Ragusa [DifRag]. We recall the weighted version of the maximal inequality due to Fefferman-Stein [FS₂]; there exists a constant C such that

$$(2.4) \quad \int_{R^n} \{Mf(x)\}^p w(x) dx \leq C \int_{R^n} \{f^\sharp(x)\}^p w(x) dx$$

for all $w \in A_\infty$ and all $f \in L_w^p(R^n)$, for $0 < p < \infty$ (see Garcia-Cuerva-Rubio de Francia [GarRub; p.410]) where A_q , $1 \leq q \leq \infty$, is the Muckenhoupt class of weight functions.

Let $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$ and B a ball. We take $w(x)$ as $(M\chi)^\gamma \in A_1$, $0 < \gamma < 1$, where $\chi = \chi_B(x)$ is the characteristic function of the ball $B = B(x_0, r)$.

Then we get by (2.4),

$$\begin{aligned} \int_B \{Mf(x)\}^p dx &= \int_{R^n} \{Mf(x)\}^p \chi_B(x) dx \\ &\leq \int_{R^n} \{Mf(x)\}^p \{M\chi_B(x)\}^\gamma dx \leq C \int_{R^n} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx \\ &= C \int_{B(x_0,r)} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx \\ &\quad + C \sum_{k=1}^{\infty} \int_{B(x_0,2^k r) - B(x_0,2^{k-1}r)} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx \\ &\leq C \left\{ \int_{B(x_0,r)} \{f^\sharp(x)\}^p dx + \sum_{k=1}^{\infty} (2^{-k n \gamma}) \int_{B(x_0,2^k r)} \{f^\sharp(x)\}^p dx \right\} \\ &\leq C \|f^\sharp\|_{p,\Phi}^p \left\{ \Phi(x_0, r) + \sum_{k=1}^{\infty} (2^{-k})^{n\gamma} \Phi(x_0, 2^k r) \right\} \\ &\leq C \|f^\sharp\|_{p,\Phi}^p \sum_{k=0}^{\infty} \frac{\Phi(x_0, 2^k r)}{2^{kn\gamma}} \sim C \|f^\sharp\|_{p,\Phi}^p r^{n\gamma} \int_r^\infty \frac{\Phi(x_0, t)}{t^{n\gamma+1}} dt. \end{aligned}$$

Since, by Lemma 2.1, the last term is bounded by

$$C \|f^\sharp\|_{p,\Phi}^p \Phi(x_0, r),$$

we have

$$\|Mf\|_{p,\Phi} \leq C\|f^\sharp\|_{p,\Phi}.$$

Thus we have (2.3) for some $C > 0$, independent of $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Q.E.D.

Our second aim is to show the following ;

Theorem 2.2. *Let $1 < p < \infty$, $b \in BMO(R^n)$ and T be a Calderon-Zygmund singular integral operator. We assume that $\Phi \in G_n$. Then the commutator $[b, T]$ is bounded in $L^{p,\Phi}$. More precisely, there exists constant C , independent of b and f , such that*

$$(2.5) \quad \|[b, T](f)\|_{p,\Phi} \leq C\|b\|_* \|f\|_{p,\Phi}$$

for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

To prove the theorem we need Theorem 2.1 and the following three lemmas ;

Lemma 2.2. *Let $1 < q, s < \infty$, $b \in BMO(R^n)$ and T be a Calderon-Zygmund singular integral operator. Then there exists constant C independent of b and f such that*

$$\{[b, T](f)\}^\sharp(x) \leq C\|b\|_* \{M_q(Tf)(x) + (M_s f)(x)\}$$

for almost all $x \in R^n$ and all $f \in L_c^\infty(R^n)$

Proof. This is the pointwise estimate due to Strömberg (see [Tor, p.418.] and Janson [Jan; pp.268-269.]).

Q.E.D.

Lemma 2.3. *Let $0 < q < p < \infty$. We assume that $\Phi \in G_n$. Then the maximal operator M_q is a bounded operator in $L^{p,\Phi}(R^n)$ and*

$$\|M_q f\|_{p,\Phi} \leq C\|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}(R^n)$.

Proof. The proof depends on the weighted maximal inequality due to Fefferman-Stein [FefSte₁]. In the restricted case $1 \leq q < p < \infty$, the corresponding result is proved by Nakai [Nak; Theorem 1]. It is not difficult to extend the result to the case $0 < q < p < \infty$. Confer also Chiarenza-Frasca [ChiFra] and Mizuhara [Miz₁].

Q.E.D.

Lemma 2.4. *Let $1 < p < \infty$. We assume that $\Phi \in G_n$. Then the Calderon-Zygmund singular integral operator T is a bounded operator in $L^{p,\Phi}(R^n)$ and*

$$(2.6) \quad \|Tf\|_{p,\Phi} \leq C\|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}(R^n)$.

Proof. This is the result due to Nakai [Nak; Theorem 2] in the setting of more general growth functions. Confer also Peetre [Pee₁], Chiarenza-Frasca [ChiFra] and Mizuhara [Miz₁].

We note that we can give a short proof following the method of the author [Miz₁] which depends on the weighted maximal inequality due to Cordoba-Fefferman [CorFef] (see also [GarRub]);

there exists constant C , depending only on T, p and $0 < \gamma < 1$, such that

$$(2.7) \quad \int_{R^n} |Tf(x)|^p \phi(x) dx \leq C \int_{R^n} |f(x)|^p (M\phi)^\gamma(x) dx$$

for all f and $\phi(x) \geq 0$. A standard proof using (2.7) implies (2.6).

Q.E.D.

Proof of Theorem 2.2. We apply the method of Di Fazio-Ragusa [DifRag] to our case. We suppose that $b \in BMO(R^n)$. Then Theorem 2.1 and Lemma 2.2 imply that, for $1 < q, s < p < \infty$,

$$\begin{aligned} \|[b, T](f)\|_{p,\Phi} &\leq \|M\{[b, T](f)\}\|_{p,\Phi} \\ &\leq C\|([b, T](f))^\sharp\|_{p,\Phi} \leq C\|b\|_* \{ \|M_q(Tf)\|_{p,\Phi} + \|M_s f\|_{p,\Phi} \}. \end{aligned}$$

Since, Lemma 2.3 and Lemma 2.4 imply

$$\|M_q(Tf)\|_{p,\Phi} \leq C\|Tf\|_{p,\Phi} \leq C\|f\|_{p,\Phi} \quad \text{and} \quad \|M_s f\|_{p,\Phi} \leq C\|f\|_{p,\Phi},$$

we obtain

$$\|[b, T](f)\|_{p,\Phi} \leq C\|b\|_* \|f\|_{p,\Phi}$$

for $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$. Thus we have (2.5).

Q.E.D.

When $b \in \Lambda_\alpha(R^n)$, $0 < \alpha < n$, we obtain the following :

Theorem 2.3. Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, T]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and

$$(2.8) \quad \|[b, T]f\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha(R^n)} \|f\|_{p,\Phi}$$

for some constant C independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

To prove the theorem we need the following lemma ;

Lemma 2.5. Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. Then the fractional integral operator I_α is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and

$$(2.9) \quad \|I_\alpha f\|_{q,\Phi^{q/p}} \leq C \|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}$.

Proof. This is the result due to Nakai [Nak; Theorem 3].

Q.E.D.

Proof of Theorem 2.3. Let $b \in \Lambda_\alpha(R^n)$. Then

$$\begin{aligned} |([b, T]f)(x)| &\leq \int_{R^n} |b(x) - b(y)| |k(x-y)| |f(y)| dy \\ &\leq C \|b\|_{\Lambda_\alpha} \int_{R^n} |x-y|^\alpha |x-y|^{-n} |f(y)| dy = C \|b\|_{\Lambda_\alpha} I_\alpha(|f|)(x). \end{aligned}$$

Hence we have, by Lemma 2.5,

$$\|[b, T]f\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha} \|I_\alpha(|f|)\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha} \|f\|_{p,\Phi}.$$

Thus we have (2.8) for some $C > 0$, independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Q.E.D.

3. Commutators between the fractional integral operator and multiplication operator by a function $b \in BMO(R^n) \cup \Lambda_\alpha(R^n)$.

In this section first we show the following when $b \in BMO(R^n)$;

Theorem 3.1. *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$ and $\Phi^{q/p} \in G_n$. If $b \in BMO(R^n)$, then the commutator $[b, I_\alpha]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and*

$$(3.1) \quad \|[b, I_\alpha]f\|_{q,\Phi^{q/p}} \leq C \|b\|_* \|f\|_{p,\Phi}$$

for some constant C independent of $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

To prove the theorem, we need the following pointwise estimate and the bounbedness of the fractional maximal operator ;

Lemma 3.1. *Let $0 < \alpha < n$, $1 < u$, $t < n/\alpha$ and $b \in BMO(R^n)$. Then there exists constant C independent of b and f such that*

$$\{[b, I_\alpha](f)\}^\sharp(x) \leq C \|b\|_* \{M_u(I_\alpha f)(x) + (M_{\alpha t/n}^* |f|^t)^{1/t}(x)\}$$

for almost all $x \in R^n$ and all $f \in L_c^\infty(R^n)$

Proof. This is the pointwise estimate due to Strömberg (see [Tor, p.419.] and Di Fazio-Ragusa [DifRag; p.326, Lemma 2.]).

Q.E.D.

Lemma 3.2. *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. Then the fractional maximal operator $M_{\alpha/n}^*$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and*

$$(3.2) \quad \|M_{\alpha/n}^* f\|_{q,\Phi^{q/p}} \leq C \|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}$.

Proof. Let $B(z, r)$ be any ball centered at z and with radius $r > 0$ such that $x \in B(z, r)$. Since

$$\begin{aligned} I_\alpha(|f|)(x) &:= \int_{R^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \geq \int_{B(z,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \\ &\geq \frac{C'}{r^{n-\alpha}} \int_{B(z,r)} |f(y)| dy \simeq \frac{C''}{|B|^{1-\alpha/n}} \int_{B(z,r)} |f(y)| dy, \end{aligned}$$

we have the pointwise estimate ;

$$(M_{\alpha/n}^* f)(x) \leq I_\alpha(|f|)(x)$$

for almost all $x \in R^n$ and all $f \in L^{p,\Phi}$. Hence Lemma 2.5 implies the result.

Q.E.D.

Proof of Theorem 3.1. Let $b \in BMO(R^n)$. Then Theorem 2.1 and Lemma 3.1 imply that, for $1 < u, t < q < p < \infty$,

$$\begin{aligned} \| [b, I_\alpha](f) \|_{q,\Phi^{q/p}} &\leq \| M\{[b, I_\alpha](f)\} \|_{q,\Phi^{q/p}} \leq \| \{([b, I_\alpha](f))^\# \} \|_{q,\Phi^{q/p}} \\ &\leq C \| b \|_* \{ \| M_u(I_\alpha f) \|_{q,\Phi^{q/p}} + \| (M_{\alpha t/n}^* |f|^t)^{1/t} \|_{q,\Phi^{q/p}} \}. \end{aligned}$$

Also, under the assumption on Φ , Lemma 2.3 and Lemma 2.5 imply

$$\| M_u(I_\alpha f) \|_{q,\Phi^{q/p}} \leq C \| I_\alpha f \|_{q,\Phi^{q/p}} \leq C \| f \|_{p,\Phi}$$

and Lemma 3.2 imply

$$\begin{aligned} \| (M_{\alpha t/n}^* |f|^t)^{1/t} \|_{q,\Phi^{q/p}} &= \| M_{\alpha t/n}^* |f|^t \|_{q/t,\Phi^{q/p}}^{1/t} \\ &\leq C \| (|f|^t) \|_{p/t,\Phi}^{1/t} = C \| f \|_{p,\Phi}. \end{aligned}$$

Hence we obtain

$$\| [b, I_\alpha](f) \|_{q,\Phi^{q/p}} \leq C \| b \|_* \| f \|_{p,\Phi}$$

for $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$. Thus we have (3.1).

Q.E.D.

We close this section showing the following ;

Theorem 3.2. Let $1 < p < q < \infty$, $0 < \alpha, \beta$, $0 < \alpha + \beta = n(1/p - 1/q) < n$, $1 < p < n/(\alpha + \beta)$. We assume that $\Phi \in G_{n-p(\alpha+\beta)}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, I_\beta]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and

$$(3.3) \quad \| [b, I_\beta]f \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha} \| f \|_{p,\Phi}$$

for some constant C independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Proof of Theorem 3.2. Let $b \in \Lambda_\alpha(R^n)$. Then

$$|([b, I_\beta]f)(x)| \leq C \| b \|_{\Lambda_\alpha} I_{\alpha+\beta}(|f|)(x).$$

for almost all $x \in R^n$. Hence we have, by Lemma 2.5,

$$\| [b, I_\beta]f \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha} \| I_{\alpha+\beta}(|f|) \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha} \| f \|_{p,\Phi}.$$

Thus we have (3.3) for some $C > 0$, independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Q.E.D.

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TAKAHIRO MIZUHARA :

Department of Mathematical Sciences,
Faculty of Science, Yamagata University,
Yamagata 990-8560, Japan.