

The word problem for the braid inverse monoid

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1 Monoid presentations

Let A be an alphabet and A^* the free monoid generated by A . The empty word is denoted by 1. A *(monoid) presentation* is an ordered pair (A, R) where $R \subseteq A^* \times A^*$. A monoid M is *defined* by (A, R) if $M \cong A^* / \equiv_R$ where $\equiv_R$ is the congruence on A^* generated by R . In this situation, we say that M is *generated* by A , A is a *generating set* of M , and R is a set of *defining relations* of M . If a monoid M is generated by an alphabet A , then there is a natural surjection $f : A^* \rightarrow M$. For any $w \in A^*$, the image of w under f is denoted by $[w]$.

If a monoid M has a presentation (A, R) such that both A and R are finite, then we say that M is *finitely presented*.

Let M be a monoid with a finite presentation (A, R) . The *word problem* for M is to decide, given $u, v \in A^*$, whether $u =_R v$.

2 Automatic monoids

In this section, we give definitions and results for automatic monoids and groups. For more information, we refer to [2, 3].

Let M be a monoid with a finite generating set A and let $\pi : A^* \rightarrow M$ be the natural surjection. If there is a regular subset L of A^* such that the restriction $\pi|_L : L \rightarrow M$ is surjective, then the ordered pair (A, L) is called a *rational structure* for M .

Let M be a monoid with a rational structure (A, L) and $\$$ a new symbol such that $\$ \notin A$. Set $A(2, \$) = (A \cup \{\$\}) \times (A \cup \{\$\}) - (\$, \$)$. Define a mapping $\nu : A^* \times A^* \rightarrow A(2, \$)$ by $\nu(1, 1) = 1$ and for $u = a_1 a_2 \cdots a_m$ and $v = b_1 b_2 \cdots b_n$,

$$\nu(u, v) = \begin{cases} (a_1, b_1)(a_2, b_2) \cdots (a_m, b_m)(\$, b_{m+1}) \cdots (\$, b_n) & \text{if } m < n \\ (a_1, b_1)(a_2, b_2) \cdots (a_m, b_m) & \text{if } m = n \\ (a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)(a_{n+1}, \$) \cdots (a_m, \$) & \text{if } m > n \end{cases}$$

Set $L_ = \{\nu(u, v) \mid u, v \in L \text{ such that } [u] = [v] \text{ in } M\}$ and, for $a \in A$, $L_a = \{\nu(u, v) \mid u, v \in L \text{ such that } [ua] = [v] \text{ in } M\}$.

A monoid M is called *automatic* if there is a rational structure (A, L) such that $L_ =$ and L_a for all $a \in A$ are regular subsets of $A(2, \$)$. In this situation, the rational structure (A, L) is called an *automatic structure* for M .

Result 2.1 (see [2, 3]) *Let M be an automatic monoid. Then the word problem for M is solvable in quadratic time. Moreover if M is a group, then M is finitely presented.*

Let M be a monoid with a rational structure (A, L) . For any $w \in A^*$ and non-negative integer t , define a word $w(t) \in A^*$ by

$$w(t) = \begin{cases} \text{the prefix of } w \text{ of length } t & \text{if } t \leq |w|, \\ w & \text{otherwise.} \end{cases}$$

For any $u, v \in A^*$, let $d(u, v) = \min\{|w| \mid w \in A^* \text{ such that } [uw] = [v] \text{ in } M\}$. We say that (A, L) satisfies the *fellow traveler property* if there is a constant k such that $d(u(t), v(t)) < k$ for all $t \geq 0$ whenever $u, v \in L$ and $[ua] = [v]$ in M for some $a \in A \cup \{1\}$.

Result 2.2 (see [3]) *For a group G with a rational structure (A, L) , (A, L) is an automatic structure for G if and only if (A, L) satisfies the fellow traveler property.*

Let M be a monoid with a rational structure (A, L) . (A, L) satisfies the *strong fellow traveler property* if there is a constant k such that, for any $u = a_1 a_2 \cdots a_m, v = b_1 b_2 \cdots b_n \in L$ satisfying $[ua] = [v]$ for some $a \in A \cup \{1\}$, there are $w_1, w_2, \dots, w_\ell \in A^*$ such that $|w_i| < k$ for all i , and $[a_1 w_1] = [b_1], [a_2 w_2] = [w_1 b_2], \dots, [a_\ell w_\ell] = [w_{\ell-1} b_\ell]$ where $\ell = \max\{m, n\}$.

Theorem 2.3 *For a monoid M with a rational structure (A, L) , if (A, L) satisfies the strong fellow traveler property, then M is automatic and finitely presented.*

3 Finite complete presentations

In this section, we one result about monoids with finite complete presentations. For more information on such monoids, we refer to [1].

Let (A, R) be a presentation of a monoid M . We write $u \rightarrow v$ if $(u, v) \in R$. The relation $\rightarrow_R$ on A^* is defined as follows: for $x, y \in A^*$, $x \rightarrow_R y$ if $x = x_1 u x_2$ and $y = x_1 v x_2$ for some $x_1, x_2 \in A^*$ and $u \rightarrow v \in R$. The reflexive transitive closure of $\rightarrow_R$ is denote by $\rightarrow_R^*$. R is *noetherian* if there is no infinite sequence $x_1 \rightarrow_R x_2 \rightarrow_R \cdots \rightarrow_R x_n \rightarrow_R \cdots$. R is *confluent* if, for any $x, y, z \in A^*$ such that $z \rightarrow_R^* x$ and $z \rightarrow_R^* y$, there is $w \in A^*$ such that $x \rightarrow_R^* w$ and $y \rightarrow_R^* w$. Moreover R is *complete* if R is both noetherian and confluent. We set $\text{Left}(R) = \{u \in A^* \mid u \rightarrow v \in R \text{ for some } v \in A^*\}$ and $\text{Irr}(R) = A^* - A^* \cdot \text{Left}(R) \cdot A^*$.

Result 3.1 (see [1]) *Let M be a monoid with a finite complete presentation (A, R) . Then, the word problem for M is solvable and $(A, \text{Irr}(R))$ is a rational structure for M .*

4 Braid groups and its word problem

In this section, we consider braid groups. For more information on braid groups and its word problem, we refer to [3, 4, 5].

A *braid* on n strings is defined as a system of n strings in $\mathbb{R}^2 \times [0, 1] \subset \mathbb{R}^3$. It consists of disjoint intertwining n strings which join n fixed points in the upper plane $\mathbb{R}^2 \times \{0\}$ and n fixed points in the lower plane $\mathbb{R}^2 \times \{1\}$, and intersecting each intermediate plane $\mathbb{R}^2 \times \{t\}$ in exactly n points. A string attached to the upper plane at the i -th position is called the *i -th string*.

By B_n , we denote the set of isotopy classes of braids on n strings. We identify a braid with its isotopy class, and we call an element in B_n simply a braid. B_n has a group structure as follows. The product of two braids β_1 and β_2 , denoted by juxtaposition $\beta_1 \beta_2$, is defined as follows. First attach β_2 under β_1 identifying the upper plane of β_2 and the lower plane of β_1 , and then remove the center plane. The *trivial braid* is the braid in which all strings go straight from the upper plane to the lower

plane. And the *inverse* of a braid is defined as the mirror image of it with respect to the vertical direction.

Result 4.1 (see [3, 5]) B_n has a finite complete presentation and is automatic. Hence, the word problem for B_n is solvable.

5 Braid inverse monoids

A *partial braid* on n strings is defined as a subsystem of a braid on n strings, that is, it consists of disjoint intertwining m strings ($0 \leq m \leq n$) which join m points of the n fixed points in the upper plane $\mathbb{R}^2 \times \{0\}$ and m points of the n fixed points in the lower plane $\mathbb{R}^2 \times \{1\}$, and intersecting each intermediate plane $\mathbb{R}^2 \times \{t\}$ in exactly m points. Accordingly, a partial braid on n strings can be obtained from a braid on n strings by removing some (possibly all or no) strings. For example, in Fig.1, the right-hand side is a partial braid that is obtained from the braid at the left-hand side by removing the fourth string. By BI_n , we denote the set of isotopy classes of partial braids.

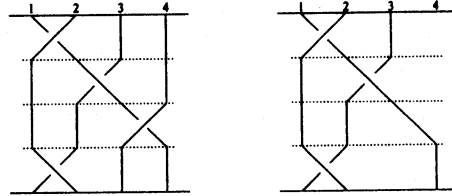


Fig.1 (a braid and a partial braid on 4 strings)

We define the product of two partial braids β_1 and β_2 , denoted by juxtaposition $\beta_1\beta_2$, as follows. First attach β_2 under β_1 identifying the upper plane of β_2 and the lower plane of β_1 . Then remove every string in β_1 (resp. β_2) that has no corresponding string in β_2 (resp. β_1). Lastly remove the center plane. For example, in Fig.2, we remove the second string in β_1 , because it has no corresponding string in β_2 . We also remove the fourth string in β_2 for the same reason.

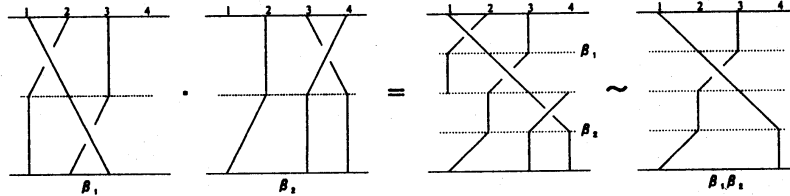


Fig.2 (the product of two partial braids β_1 and β_2 on 4 strings)

Then BI_n is a monoid with this operation and contains B_n as a subgroup.

Result 5.1 (see [6]) BI_n is finitely presented.

6 The word problem for BI_3

In this section, we give a finite complete presentation and an automatic structure for BI_3 using a finite complete presentation and an automatic structure for B_3 .

Let $x, y, [xy], [yx], \delta$ and δ^{-1} be braids as in Fig.3. Let

$$\begin{aligned} A' &= \{x, y, [xy], [yx], \delta, \delta^{-1}\} \text{ and} \\ R' &= \{xy \rightarrow [xy], x[yx] \rightarrow \delta, yx \rightarrow [yx], y[xy] \rightarrow \delta, [xy]x \rightarrow \delta, [xy][xy] \rightarrow x\delta, [yx]y \rightarrow \delta, \\ &\quad [yx][yx] \rightarrow y\delta, \delta x \rightarrow y\delta, \delta y \rightarrow x\delta, \delta[xy] \rightarrow [yx]\delta, \delta[yx] \rightarrow [xy]\delta, \delta^{-1}x \rightarrow y\delta^{-1}, \\ &\quad \delta^{-1}y \rightarrow x\delta^{-1}, \delta^{-1}[xy] \rightarrow [yx]\delta^{-1}, \delta^{-1}[yx] \rightarrow [xy]\delta^{-1}, \delta\delta^{-1} \rightarrow 1, \delta^{-1}\delta \rightarrow 1\}. \end{aligned}$$

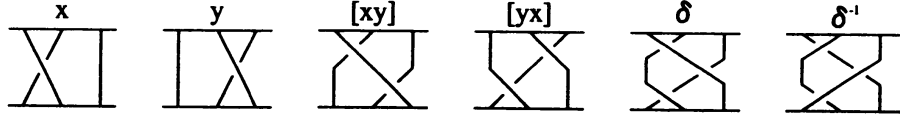


Fig.3

Result 6.1 (see [3, 5]) (A', R') is a finite complete presentation of B_3 and $(A', \text{Irr}(R'))$ is an automatic structure for B_3 .

Let $a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}$ and 0 be partial braids as in Fig.4. Let

$$A = A' \cup \{a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}, z, z^{-1}, 0\}$$

and

$R = R' \cup \{xz \rightarrow z^2, xz^{-1} \rightarrow zz^{-1}, xa \rightarrow a^{-1}a, xb \rightarrow b, xc \rightarrow d, xd \rightarrow c, xa^{-1} \rightarrow aa^{-1}, xb^{-1} \rightarrow ab^{-1}, xc^{-1} \rightarrow zc^{-1}, xd^{-1} \rightarrow zd^{-1}, yz \rightarrow cz, yz^{-1} \rightarrow cz^{-1}, ya \rightarrow b, yb \rightarrow a, yc \rightarrow zz^{-1}, yd \rightarrow dz, ya^{-1} \rightarrow a^{-1}, yb^{-1} \rightarrow b^{-1}, yc^{-1} \rightarrow cc^{-1}, yd^{-1} \rightarrow cd^{-1}, [xy]z \rightarrow dz, [xy]z^{-1} \rightarrow dz^{-1}, [xy]a \rightarrow b, [xy]b \rightarrow a^{-1}a, [xy]c \rightarrow z, [xy]d \rightarrow cz, [xy]a^{-1} \rightarrow aa^{-1}, [xy]b^{-1} \rightarrow ab^{-1}, [xy]c^{-1} \rightarrow dc^{-1}, [xy]d^{-1} \rightarrow dd^{-1}, [yx]z \rightarrow cz^2, [yx]z^{-1} \rightarrow c, [yx]a \rightarrow a^{-1}a, [yx]b \rightarrow a, [yx]c \rightarrow dz, [yx]d \rightarrow zz^{-1}, [yx]a^{-1} \rightarrow ba^{-1}, [yx]b^{-1} \rightarrow bb^{-1}, [yx]c^{-1} \rightarrow czc^{-1}, [yx]d^{-1} \rightarrow czd^{-1}, \delta z \rightarrow dz^2, \delta z^{-1} \rightarrow d, \delta a \rightarrow a, \delta b \rightarrow a^{-1}a, \delta c \rightarrow cz, \delta d \rightarrow z, \delta a^{-1} \rightarrow ba^{-1}, \delta b^{-1} \rightarrow bb^{-1}, \delta c^{-1} \rightarrow dzc^{-1}, \delta d^{-1} \rightarrow dzd^{-1}, \delta^{-1}z \rightarrow d, \delta^{-1}z^{-1} \rightarrow dz^{-2}, \delta^{-1}a \rightarrow a, \delta^{-1}b \rightarrow a^{-1}a, \delta^{-1}c \rightarrow cz^{-1}, \delta^{-1}d \rightarrow z^{-1}, \delta^{-1}a^{-1} \rightarrow ba^{-1}, \delta^{-1}b^{-1} \rightarrow bb^{-1}, \delta^{-1}c^{-1} \rightarrow dz^{-1}c^{-1}, \delta^{-1}d^{-1} \rightarrow dz^{-1}d^{-1}, zx \rightarrow z^2, zy \rightarrow zc^{-1}, z[xy] \rightarrow z^2c^{-1}, z[yx] \rightarrow zd^{-1}, z\delta \rightarrow z^2d^{-1}, z\delta^{-1} \rightarrow d^{-1}, za \rightarrow a^{-1}a, zb \rightarrow 0, zc \rightarrow a, zd \rightarrow a^{-1}a, za^{-1} \rightarrow aa^{-1}, zb^{-1} \rightarrow ab^{-1}, z^{-1}x \rightarrow zz^{-1}, z^{-1}y \rightarrow z^{-1}c^{-1}, z^{-1}[xy] \rightarrow c^{-1}, z^{-1}[yx] \rightarrow z^{-1}d^{-1}, z^{-1}\delta \rightarrow d^{-1}, z^{-1}\delta^{-1} \rightarrow z^{-2}d^{-1}, z^{-1}z \rightarrow zz^{-1}, z^{-1}a \rightarrow a^{-1}a, z^{-1}b \rightarrow 0, z^{-1}c \rightarrow a, z^{-1}d \rightarrow a^{-1}a, z^{-1}a^{-1} \rightarrow aa^{-1}, z^{-1}b^{-1} \rightarrow ab^{-1}, ax \rightarrow aa^{-1}, ay \rightarrow a, a[xy] \rightarrow ab^{-1}, a[yx] \rightarrow aa^{-1}, a\delta \rightarrow ab^{-1}, a\delta^{-1} \rightarrow ab^{-1}, az \rightarrow aa^{-1}, az^{-1} \rightarrow aa^{-1}, a^2 \rightarrow 0, ab \rightarrow 0, ac \rightarrow a, ad \rightarrow 0, ac^{-1} \rightarrow a, ad^{-1} \rightarrow aa^{-1}, bx \rightarrow ba^{-1}, by \rightarrow b, b[xy] \rightarrow bb^{-1}, b[yx] \rightarrow ba^{-1}, b\delta \rightarrow bb^{-1}, b\delta^{-1} \rightarrow bb^{-1}, bz \rightarrow ba^{-1}, bz^{-1} \rightarrow ba^{-1}, ba \rightarrow 0, b^2 \rightarrow 0, bc \rightarrow b, bd \rightarrow 0, bc^{-1} \rightarrow b, bd^{-1} \rightarrow ba^{-1}, cx \rightarrow cz, cy \rightarrow cc^{-1}, c[xy] \rightarrow czc^{-1}, c[yx] \rightarrow cd^{-1}, c\delta \rightarrow czd^{-1}, c\delta^{-1} \rightarrow cz^{-1}d^{-1}, ca \rightarrow b, cb \rightarrow 0, c^2 \rightarrow a^{-1}a, cd \rightarrow b, ca^{-1} \rightarrow a^{-1}, cb^{-1} \rightarrow b^{-1}, dx \rightarrow dz, dy \rightarrow dc^{-1}, d[xy] \rightarrow dzc^{-1}, d[yx] \rightarrow dd^{-1}, d\delta \rightarrow dzd^{-1}, d\delta^{-1} \rightarrow dz^{-1}d^{-1}, da \rightarrow b, db \rightarrow 0, dc \rightarrow a, d^2 \rightarrow b, da^{-1} \rightarrow aa^{-1}, db^{-1} \rightarrow ab^{-1}, a^{-1}x \rightarrow a^{-1}a, a^{-1}y \rightarrow b^{-1}, a^{-1}[xy] \rightarrow a^{-1}a, a^{-1}[yx] \rightarrow b^{-1}, a^{-1}\delta \rightarrow a^{-1}, a^{-1}\delta^{-1} \rightarrow a^{-1}, a^{-1}z \rightarrow a^{-1}a, a^{-1}z^{-1} \rightarrow a^{-1}a, a^{-1}b \rightarrow 0, a^{-1}c \rightarrow 0, a^{-1}d \rightarrow a^{-1}a, a^{-2} \rightarrow 0, a^{-1}b^{-1} \rightarrow 0, a^{-1}c^{-1} \rightarrow b^{-1}, a^{-1}d^{-1} \rightarrow b^{-1}, b^{-1}x \rightarrow b^{-1}, b^{-1}y \rightarrow a^{-1}, b^{-1}[xy] \rightarrow a^{-1}, b^{-1}[yx] \rightarrow a^{-1}a, b^{-1}\delta \rightarrow a^{-1}a, b^{-1}\delta^{-1} \rightarrow a^{-1}a, b^{-1}z \rightarrow 0, b^{-1}z^{-1} \rightarrow 0, b^{-1}a \rightarrow 0, b^{-1}b \rightarrow a^{-1}a, b^{-1}c \rightarrow a^{-1}, b^{-1}d \rightarrow a^{-1}, b^{-1}a^{-1} \rightarrow 0, b^{-2} \rightarrow 0, b^{-1}c^{-1} \rightarrow 0, b^{-1}d^{-1} \rightarrow 0, c^{-1}x \rightarrow d^{-1}, c^{-1}y \rightarrow zz^{-1}, c^{-1}[xy] \rightarrow zd^{-1}, c^{-1}[yx] \rightarrow z, c^{-1}\delta \rightarrow zc^{-1}, c^{-1}\delta^{-1} \rightarrow z^{-1}c^{-1}, c^{-1}z \rightarrow a^{-1}, c^{-1}z^{-1} \rightarrow a^{-1}, c^{-1}a \rightarrow 0, c^{-1}b \rightarrow a, c^{-1}c \rightarrow zz^{-1}, c^{-1}d \rightarrow aa^{-1}, c^{-1}a^{-1} \rightarrow a^{-1}, c^{-1}b^{-1} \rightarrow b^{-1}, c^{-2} \rightarrow a^{-1}a, c^{-1}d^{-1} \rightarrow a^{-1}, d^{-1}x \rightarrow c^{-1}, d^{-1}y \rightarrow zd^{-1}, d^{-1}[xy] \rightarrow zz^{-1}, d^{-1}[yx] \rightarrow zc^{-1}, d^{-1}\delta \rightarrow z, d^{-1}\delta^{-1} \rightarrow z^{-1}, d^{-1}z \rightarrow a^{-1}a, d^{-1}z^{-1} \rightarrow a^{-1}a, d^{-1}a \rightarrow a^{-1}a, d^{-1}b \rightarrow a, d^{-1}c \rightarrow aa^{-1}, d^{-1}d \rightarrow zz^{-1}, d^{-1}a^{-1} \rightarrow 0, d^{-1}b^{-1} \rightarrow 0, d^{-1}c^{-1} \rightarrow b^{-1}, d^{-2} \rightarrow b^{-1}, aa^{-1}a \rightarrow a, a^{-1}aa^{-1} \rightarrow a^{-1}, ba^{-1}a \rightarrow b, a^{-1}ab^{-1} \rightarrow b^{-1}, czc^{-1} \rightarrow c, zz^{-1}c^{-1} \rightarrow c^{-1}, dzz^{-1} \rightarrow d, zz^{-1}d^{-1} \rightarrow d^{-1}, z^2z^{-1} \rightarrow z, zz^{-2} \rightarrow z^{-1}\} \cup \{a0 \rightarrow 0, 0a \rightarrow 0 \mid a \in A\}.$

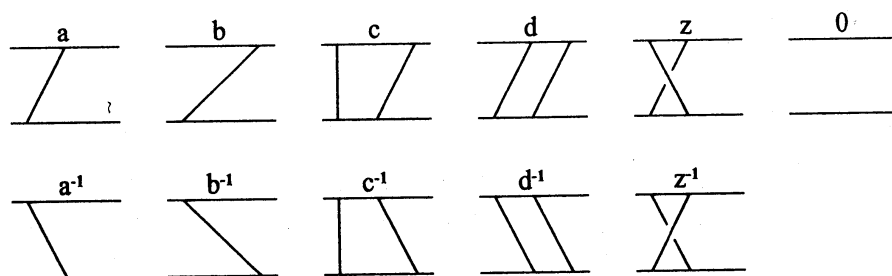


Fig.4

Theorem 6.2 (A, R) is a finite complete presentation of BI_3 .

By the previous theorem and Result 3.1, $(A, \text{Irr}(R))$ is a rational structure for BI_3 . Moreover we have

Theorem 6.3 $(A, \text{Irr}(R))$ satisfies the strong fellow traveler property. Thus by Theorem 2.3, it is an automatic structure for BI_3 .

Hence, we have

Corollary 6.4 The word problem for BI_3 is solvable.

References

- [1] R. V. Book and F. Otto, *String-Rewriting Systems*, Springer-Verlag, New York, 1993.
- [2] C. M. Campbell, E. F. Robertson, N. Ruškuc and R. M. Thomas, *Automatic semigroups*, Theoretical Computer Science, **250** 2001, 365-391.
- [3] J. W. Cannon, D. B. A. Epstein, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett Publishers 1992.
- [4] V. L. Hansen, *Braids and Coverings: selected topics*, London Math. Soc. Student Texts 18, Cambridge Univ. Press, 1989.
- [5] S. M. Hermiller and J. Meier, *Artin groups, rewriting systems and three-manifolds*, Journal of Pure and Applied Algebra, **136** 1999, 141-156.
- [6] I. Inata and Y. Kobayashi, *Monoids with subgroups of finite index and the braid inverse monoid*, SCMJ Special Issue on Computer Science, 2001 JAMS Annual Meeting, to appear. .