

Ultradiscrete Analogue of the Identity of Pfaffians

By

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Abstract

We present an algebraic identity of ultradiscretized hafnians, which is an ultradiscrete form of the identity of pfaffians. The identity stems from a decomposition of a product of the hafnians.

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§ 1. Introduction

Recent progress in the direct method in soliton theory reveals that a soliton equation which exhibits multi-soliton solution is reduced to an identity of pfaffians. The identity of determinants such as the Plücker relation and Jacobi's identity of determinants are special cases of the identity of the pfaffians [1]. Takahashi and the author of the present article have shown that soliton solutions to the box and ball system follow a form of ultradiscretized *permanent* [2]. A permanent is a signature free determinant. Nagai has shown that soliton solutions to the ultradiscrete Toda equation are expressed also by the ultradiscretized permanents [3]. These facts suggest that there must be an identity of ultradiscretized permanents instead of determinants. More generally we expect an identity of ultradiscretized hafnians instead of pfaffians. A hafnian is a signature free pfaffian introduced by Caieniello [4].

We look for the ultradiscretization of the following simple identity of determinants

$$(1.1) \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0,$$

which is one of the Plücker relations.

Let each term in Eq.(1.1) be p_1, p_2 and p_3 , namely

$$(1.2) \quad p_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} = a_1 a_3 b_2 b_4 - a_1 a_4 b_2 b_3 - a_2 a_3 b_1 b_4 + a_2 a_4 b_1 b_3,$$

$$(1.3) \quad p_2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} = a_1 a_2 b_3 b_4 - a_1 a_4 b_2 b_3 - a_2 a_3 b_1 b_4 + a_3 a_4 b_1 b_2,$$

$$(1.4) \quad p_3 = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_1 a_2 b_3 b_4 - a_1 a_3 b_2 b_4 - a_2 a_4 b_1 b_3 + a_3 a_4 b_1 b_2.$$

Then the Plücker relation is written as

$$(1.5) \quad p_1 - p_2 + p_3 = 0,$$

which cannot be ultradiscretized because of negative terms in p_1, p_2 and p_3 .

Now we replace the determinants by the corresponding permanents

$$(1.6) \quad q_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}_+ \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}_+ = a_1 a_3 b_2 b_4 + a_1 a_4 b_2 b_3 + a_2 a_3 b_1 b_4 + a_2 a_4 b_1 b_3,$$

$$(1.7) \quad q_2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}_+ \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}_+ = a_1 a_2 b_3 b_4 + a_1 a_4 b_2 b_3 + a_2 a_3 b_1 b_4 + a_3 a_4 b_1 b_2,$$

$$(1.8) \quad q_3 = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}_+ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}_+ = a_1 a_2 b_3 b_4 + a_1 a_3 b_2 b_4 + a_2 a_4 b_1 b_3 + a_3 a_4 b_1 b_2,$$

where q_1, q_2 and q_3 have no negative terms and can be ultradiscretized. However the corresponding Plücker relation does not hold,

$$(1.9) \quad q_1 - q_2 + q_3 = 2(a_1 a_3 b_2 b_4 + a_2 a_4 b_1 b_3) \neq 0,$$

which indicates that ultradiscrete analogue of the Plücker relation is not easy to find.

We notice comparing the expressions for the products of the determinants, p_1, p_2 and p_3 that they are decomposed into a sum of common terms p_{12}, p_{13} and p_{23} where p_{ij} is the common term of p_i and p_j for $i, j = 1, 2, 3$,

$$(1.10) \quad p_1 = -p_{12} + p_{13}, \quad p_2 = -p_{12} + p_{23}, \quad p_3 = -p_{13} + p_{23},$$

where

$$(1.11) \quad p_{12} = a_1 a_4 b_2 b_3 + a_2 a_3 b_1 b_4,$$

$$(1.12) \quad p_{13} = a_1 a_3 b_2 b_4 + a_2 a_4 b_1 b_3,$$

$$(1.13) \quad p_{23} = a_1 a_2 b_3 b_4 + a_3 a_4 b_1 b_2.$$

The permanents have the same terms as the determinants except the signature. Accordingly we find that the products of the permanents, q_1, q_2 and q_3 are decomposed into a sum of common terms q_{12}, q_{13} and q_{23} , where q_{ij} is the common term of q_i and q_j for $i, j = 1, 2, 3$,

$$(1.14) \quad q_1 = q_{12} + q_{13}, \quad q_2 = q_{12} + q_{23}, \quad q_3 = q_{13} + q_{23},$$

where

$$(1.15) \quad q_{12} = a_1 a_4 b_2 b_3 + a_2 a_3 b_1 b_4,$$

$$(1.16) \quad q_{13} = a_1 a_3 b_2 b_4 + a_2 a_4 b_1 b_3,$$

$$(1.17) \quad q_{23} = a_1 a_2 b_3 b_4 + a_3 a_4 b_1 b_2.$$

The Plücker relation Eq.(1.1) is confirmed by the decomposition of products of determinants,

$$(1.18) \quad p_1 - p_2 + p_3 = -p_{12} + p_{13} - (-p_{12} + p_{23}) - p_{13} + p_{23} = 0.$$

An ultradiscrete analogue of the Plücker relation is obtained as follows.

Replacing the determinants by the corresponding permanents we have

$$(1.19) \quad q_1 + q_3 = q_2.$$

Let

$$(1.20) \quad q_i = \exp(Q_i/\epsilon) \text{ for } i = 1, 2, 3,$$

$$(1.21) \quad q_{ij} = \exp(Q_{ij}/\epsilon) \text{ for } i, j = 1, 2, 3,$$

where ϵ is the ultradiscrete parameter [5]. In the small limit of ϵ we have an ultradiscrete analogue of the Plücker relation, Eq.(1.19),

$$(1.22) \quad Q_2 = \max(Q_1, Q_3),$$

which does not hold in general.

We investigate under what conditions on Q_1, Q_2 and Q_3 Eq.(1.22) does hold. The ultradiscrete form of Eq.(1.14) are

$$(1.23) \quad \begin{aligned} Q_1 &= \max(Q_{12}, Q_{13}), \\ Q_2 &= \max(Q_{12}, Q_{23}), \\ Q_3 &= \max(Q_{13}, Q_{23}). \end{aligned}$$

Substituting these expressions into Eq.(1.22) we obtain

$$(1.24) \quad \max(Q_{12}, Q_{23}) = \max(Q_{12}, Q_{13}, Q_{23}).$$

Obviously Eq.(1.24) does hold if $Q_{13} \leq \max(Q_{12}, Q_{23})$. But it does not hold if $Q_{13} > \max(Q_{12}, Q_{23})$.

However if $Q_{13} > \max(Q_{12}, Q_{23})$ we find, using Eq.(1.23)

$$(1.25) \quad Q_1 = Q_3.$$

Hence we obtain the following algebraic identity of the ultradiscretized permanents,

$$(1.26) \quad [Q_2 - \max(Q_1, Q_3)](Q_1 - Q_3) = 0,$$

which we call “ultradiscrete analogue of the Plücker relation”.

The contents of this article are as follows. In §2, we describe the fundamental properties of determinants, pfaffians, permanents and hafnians. In §3 we prove by induction a decomposition of a product of hafnians into a sum of common terms, which will be used in §4 to obtain an algebraic identity of ultradiscretized hafnians that is a ultradiscrete analogue of the identity of pfaffians.

§ 2. Preliminaries

In this section we describe elementary properties of determinants, permanents, pfaffians and hafnians.

§ 2.1. Determinants, pfaffians, permanents and hafnians

Consider an $n \times n$ matrix $\mathbf{D} = (x_{ij})$ whose determinant is D . We write the n th-order determinant D as

$$(2.1) \quad D = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}_{-}.$$

An n th-order pfaffian denoted by $\text{pf}(1, 2, \dots, 2n)$ is anti-symmetric with respect to the indices, $1, 2, \dots, 2n$, namely

$$(2.2) \quad \text{pf}(1, 2, \dots, i, \dots, j, \dots, 2n) = -\text{pf}(1, 2, \dots, j, \dots, i, \dots, 2n)$$

and is defined by the expansion formula

$$(2.3) \quad \text{pf}(1, 2, \dots, 2n) = \sum_{j=2}^{2n} \text{pf}(1, j)(-1)^j \text{pf}(2, 3, \dots, \hat{j}, \dots, 2n),$$

where $\hat{j}$ means that index j is omitted.

We have, for example,

$$(2.4) \quad \text{pf}(1, 2, 3, 4) = \text{pf}(1, 2)\text{pf}(3, 4) - \text{pf}(1, 3)\text{pf}(2, 4) + \text{pf}(1, 4)\text{pf}(2, 3).$$

An n th-order determinant D is expressed by an n th-order pfaffian as follows

$$(2.5) \quad D = \text{pf}(x_1, x_2, \dots, x_n, n, n-1, \dots, 2, 1)$$

$$(2.6) \quad = (-1)^{n(n-1)/2} \text{pf}(x_1, x_2, \dots, x_n, 1, 2, \dots, n),$$

where the pfaffian entries are defined by

$$(2.7) \quad \text{pf}(x_i, x_j) = 0, \quad \text{pf}(x_i, j) = x_{ij}, \quad \text{for } i, j = 1, 2, \dots, n.$$

A permanent is a signature free determinant, that is the signatures of permutations are not taken into account in its definition.

The permanent of an $n \times n$ matrix $\mathbf{D} = (x_{ij})$ is defined as

$$(2.8) \quad \text{perm}(D) = \sum_{\sigma \in S} \prod_{i=1}^n x_{i, \sigma(i)},$$

where the sum extends over all permutations of number $1, 2, \dots, n$. We write the n th-order permanent P as

$$(2.9) \quad P = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}_{+}.$$

An n th-order hafnian denoted by $\text{hf}(1, 2, \dots, 2n)$ is a signature free pfaffian, which is symmetric with respect to the indices, $1, 2, \dots, 2n$ and is defined by the expansion formula

$$(2.10) \quad \text{hf}(1, 2, \dots, 2n) = \sum_{j=2}^{2n} \text{hf}(1, j) \text{hf}(2, 3, \dots, \hat{j}, \dots, 2n).$$

We have, for example,

$$(2.11) \quad \text{hf}(1, 2, 3, 4) = \text{hf}(1, 2) \text{hf}(3, 4) + \text{hf}(1, 3) \text{hf}(2, 4) + \text{hf}(1, 4) \text{hf}(2, 3).$$

The n th-order permanent P is expressed by the n th-order hafnian.

$$(2.12) \quad P = \text{hf}(x_1, x_2, \dots, x_n, 1, 2, \dots, n),$$

if one define the hafnian entries by

$$(2.13) \quad \text{hf}(x_i, x_j) = 0, \quad \text{hf}(x_i, j) = x_{ij}, \quad \text{for } i, j = 1, 2, \dots, n.$$

§ 2.2. Laplace expansion of permanents

Consider the permanent P_4 of an 4×4 matrix $\mathbf{D} = (a_{ij})$,

$$(2.14) \quad P_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}_+.$$

We have the Laplace expansion formula of P_4 ,

$$(2.15) \quad \begin{aligned} P_4 = & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_+ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}_+ + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}_+ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}_+ \\ & + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}_+ \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}_+ + \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}_+ \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}_+ \\ & + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}_+ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}_+ + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}_+ \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}_+, \end{aligned}$$

which is expressed by the 4th-order hafnians with the entries, $\text{hf}(y_i, y_j) = 0$ and $\text{hf}(y_i, j) = a_{i,j}$ for $i, j = 1, 2, 3, 4$,

$$(2.16) \quad \begin{aligned} P_4 = & \text{hf}(y_1, y_2, y_3, y_4, 1, 2, 3, 4) \\ & = \text{hf}(y_1, y_2, 1, 2) \text{hf}(y_3, y_4, 3, 4) + \text{hf}(y_1, y_2, 3, 4) \text{hf}(y_3, y_4, 1, 2) \\ & + \text{hf}(y_1, y_2, 1, 3) \text{hf}(y_3, y_4, 2, 4) + \text{hf}(y_1, y_2, 2, 4) \text{hf}(y_3, y_4, 1, 3) \\ & + \text{hf}(y_1, y_2, 1, 4) \text{hf}(y_3, y_4, 2, 3) + \text{hf}(y_1, y_2, 2, 3) \text{hf}(y_3, y_4, 1, 4). \end{aligned}$$

Interchanging the index y_2 with the index y_3 in (2.16) we obtain

$$\begin{aligned}
 P_4 &= \text{hf}(y_1, y_3, y_2, y_4, 1, 2, 3, 4) \\
 &= \text{hf}(y_1, y_3, 1, 2)\text{hf}(y_2, y_4, 3, 4) + \text{hf}(y_1, y_3, 3, 4)\text{hf}(y_2, y_4, 1, 2) \\
 &\quad + \text{hf}(y_1, y_3, 1, 3)\text{hf}(y_2, y_4, 2, 4) + \text{hf}(y_1, y_3, 2, 4)\text{hf}(y_2, y_4, 1, 3) \\
 (2.17) \quad &\quad + \text{hf}(y_1, y_3, 1, 4)\text{hf}(y_2, y_4, 2, 3) + \text{hf}(y_1, y_3, 2, 3)\text{hf}(y_2, y_4, 1, 4).
 \end{aligned}$$

Similarly interchanging the index y_2 with the index y_4 in (2.16) we obtain

$$\begin{aligned}
 P_4 &= \text{hf}(y_1, y_4, 1, 2)\text{hf}(y_2, y_3, 3, 4) + \text{hf}(y_1, y_4, 3, 4)\text{hf}(y_2, y_3, 1, 2) \\
 &\quad + \text{hf}(y_1, y_4, 1, 3)\text{hf}(y_2, y_3, 2, 4) + \text{hf}(y_1, y_4, 2, 4)\text{hf}(y_2, y_3, 1, 3) \\
 (2.18) \quad &\quad + \text{hf}(y_1, y_4, 1, 4)\text{hf}(y_2, y_3, 2, 3) + \text{hf}(y_1, y_4, 2, 3)\text{hf}(y_2, y_3, 1, 4).
 \end{aligned}$$

Permanent P_4 is invariant under changing the rows and columns,

$$(2.19) \quad P_4 = \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}_+.$$

Then the Laplace expansion formula of P_4 becomes

$$\begin{aligned}
 P_4 &= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}_+ \begin{vmatrix} a_{33} & a_{43} \\ a_{34} & a_{44} \end{vmatrix}_+ + \begin{vmatrix} a_{31} & a_{41} \\ a_{32} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{23} \\ a_{14} & a_{24} \end{vmatrix}_+ \\
 &\quad + \begin{vmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{vmatrix}_+ \begin{vmatrix} a_{23} & a_{43} \\ a_{24} & a_{44} \end{vmatrix}_+ + \begin{vmatrix} a_{21} & a_{41} \\ a_{22} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{33} \\ a_{14} & a_{34} \end{vmatrix}_+ \\
 (2.20) \quad &\quad + \begin{vmatrix} a_{11} & a_{41} \\ a_{12} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{23} & a_{33} \\ a_{24} & a_{34} \end{vmatrix}_+ + \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{43} \\ a_{14} & a_{44} \end{vmatrix}_+,
 \end{aligned}$$

which is expressed by the 4th-order hafnians with the entries, $\text{hf}(y_i, y_j) = 0$ and $\text{hf}(y_i, j) = a_{ij}$ for $i, j = 1, 2, 3, 4$,

$$\begin{aligned}
 P_4 &= \text{hf}(y_1, y_2, y_3, y_4, 1, 2, 3, 4) \\
 &= \text{hf}(y_1, y_2, 1, 2)\text{hf}(y_3, y_4, 3, 4) + \text{hf}(y_1, y_2, 3, 4)\text{hf}(y_3, y_4, 1, 2) \\
 &\quad + \text{hf}(y_1, y_3, 1, 2)\text{hf}(y_2, y_4, 3, 4) + \text{hf}(y_1, y_3, 3, 4)\text{hf}(y_2, y_4, 1, 2) \\
 (2.21) \quad &\quad + \text{hf}(y_1, y_4, 1, 2)\text{hf}(y_2, y_3, 3, 4) + \text{hf}(y_1, y_4, 3, 4)\text{hf}(y_2, y_3, 1, 2).
 \end{aligned}$$

Interchanging the index 2 with the index 4 in (2.21) we obtain

$$\begin{aligned}
 P_4 &= \text{hf}(y_1, y_2, 1, 4)\text{hf}(y_3, y_4, 2, 3) + \text{hf}(y_1, y_2, 2, 3)\text{hf}(y_3, y_4, 1, 4) \\
 &\quad + \text{hf}(y_1, y_3, 1, 4)\text{hf}(y_2, y_4, 2, 3) + \text{hf}(y_1, y_3, 2, 3)\text{hf}(y_2, y_4, 1, 4) \\
 (2.22) \quad &\quad + \text{hf}(y_1, y_4, 1, 4)\text{hf}(y_2, y_3, 2, 3) + \text{hf}(y_1, y_4, 2, 3)\text{hf}(y_2, y_3, 1, 4).
 \end{aligned}$$

Subtracting the terms, $\text{hf}(y_1, y_2, 1, 4)\text{hf}(y_3, y_4, 2, 3) + \text{hf}(y_1, y_2, 2, 3)\text{hf}(y_3, y_4, 1, 4)$ from Eqs.(2.16) and (2.22) we obtain an identity of the hafnians,

$$\begin{aligned}
 & \text{hf}(y_1, y_2, 1, 2)\text{hf}(y_3, y_4, 3, 4) + \text{hf}(y_1, y_2, 3, 4)\text{hf}(y_3, y_4, 1, 2) \\
 & + \text{hf}(y_1, y_2, 1, 3)\text{hf}(y_3, y_4, 2, 4) + \text{hf}(y_1, y_2, 2, 4)\text{hf}(y_3, y_4, 1, 3) \\
 & = \text{hf}(y_1, y_3, 1, 4)\text{hf}(y_2, y_4, 2, 3) + \text{hf}(y_1, y_3, 2, 3)\text{hf}(y_2, y_4, 1, 4) \\
 (2.23) \quad & + \text{hf}(y_1, y_4, 1, 4)\text{hf}(y_2, y_3, 2, 3) + \text{hf}(y_1, y_4, 2, 3)\text{hf}(y_2, y_3, 1, 4).
 \end{aligned}$$

Interchanging the index 3 with the index 4 in (2.23) we obtain an identity,

$$\begin{aligned}
 & \text{hf}(y_1, y_2, 1, 2)\text{hf}(y_3, y_4, 3, 4) + \text{hf}(y_1, y_2, 3, 4)\text{hf}(y_3, y_4, 1, 2) \\
 & + \text{hf}(y_1, y_2, 1, 4)\text{hf}(y_3, y_4, 2, 3) + \text{hf}(y_1, y_2, 2, 3)\text{hf}(y_3, y_4, 1, 4) \\
 & = \text{hf}(y_1, y_3, 1, 3)\text{hf}(y_2, y_4, 2, 4) + \text{hf}(y_1, y_3, 2, 4)\text{hf}(y_2, y_4, 1, 3) \\
 (2.24) \quad & + \text{hf}(y_1, y_4, 1, 3)\text{hf}(y_2, y_3, 2, 4) + \text{hf}(y_1, y_4, 2, 4)\text{hf}(y_2, y_3, 1, 3).
 \end{aligned}$$

Similarly interchanging the index 2 with the index 3 in (2.24) we obtain an identity,

$$\begin{aligned}
 & \text{hf}(y_1, y_2, 1, 3)\text{hf}(y_3, y_4, 2, 4) + \text{hf}(y_1, y_2, 2, 4)\text{hf}(y_3, y_4, 1, 3) \\
 & + \text{hf}(y_1, y_2, 1, 4)\text{hf}(y_3, y_4, 2, 3) + \text{hf}(y_1, y_2, 2, 3)\text{hf}(y_3, y_4, 1, 4) \\
 & = \text{hf}(y_1, y_3, 1, 2)\text{hf}(y_2, y_4, 3, 4) + \text{hf}(y_1, y_3, 3, 4)\text{hf}(y_2, y_4, 1, 2) \\
 (2.25) \quad & + \text{hf}(y_1, y_4, 1, 2)\text{hf}(y_2, y_3, 3, 4) + \text{hf}(y_1, y_4, 3, 4)\text{hf}(y_2, y_3, 1, 2).
 \end{aligned}$$

These identities will be used in the section 3 in order to decompose $\hat{h}_1, \hat{h}_2$ and $\hat{h}_3$.

§ 2.3. Plücker relations

Let a_i for $i = 1, 2, \dots, N-1$ and b_i for $i = 1, 2, \dots, n$ be N -dimensional vectors and $|\dots|$ express determinants. The Plücker relations are expressed in general by

$$\begin{aligned}
 & \sum_{i=1}^n (-1)^i |a_1, a_2, \dots, a_{N-1}, b_i| \\
 (2.26) \quad & \times |a_1, a_2, \dots, a_{N-n+1}, b_n, b_{n-1}, \dots, \hat{b}_i, \dots, b_1| = 0,
 \end{aligned}$$

where $\hat{b}$ means that symbol b is omitted.

We consider the simplest case, namely $n = 3$ and write the Plücker relation as,

$$(2.27) \quad D_{ab}D_{cd} - D_{ac}D_{bd} + D_{ad}D_{bc} = 0,$$

where $D_{ab}, D_{cd}, D_{ac}, D_{bd}, D_{ad}$ and D_{bc} are n th-order determinants defined by

$$(2.28) \quad D_{ab} = |a, b, x_3, x_4, \dots, x_n|,$$

$$(2.29) \quad D_{ac} = |a, c, x_3, x_4, \dots, x_n|,$$

$$(2.30) \quad D_{ad} = |a, d, x_3, x_4, \dots, x_n|,$$

$$(2.31) \quad D_{bc} = |b, c, x_3, x_4, \dots, x_n|,$$

$$(2.32) \quad D_{bd} = |b, d, x_3, x_4, \dots, x_n|,$$

$$(2.33) \quad D_{cd} = |c, d, x_3, x_4, \dots, x_n|,$$

where a, b, c, d and $x_i (i = 3, 4, \dots, n)$ are n -dimensional vectors.

§ 2.4. Identity of pfaffians

We have the following identity of pfaffians,

$$(2.34) \quad \begin{aligned} \text{pf}(1, 2, \dots, 2n) \text{pf}(5, 6, \dots, 2n) &= \text{pf}(1, 2, 5, 6, \dots, 2n) \text{pf}(3, 4, 5, 6, \dots, 2n) \\ &\quad - \text{pf}(1, 3, 5, 6, \dots, 2n) \text{pf}(2, 4, 5, 6, \dots, 2n) \\ &\quad + \text{pf}(1, 4, 5, 6, \dots, 2n) \text{pf}(2, 3, 5, 6, \dots, 2n), \end{aligned}$$

whose special cases are the Plücker relation and Jacobi's identity.

§ 2.5. Expansion formulae of pfaffians and hafnians

We have the expansion formulae of pfaffians,

$$(2.35) \quad \begin{aligned} &\text{pf}(x_1, x_2, 5, 6, \dots, 2n) \\ &= \sum_{5 \leq i < j \leq 2n} (-1)^{i+j-1} \text{pf}(x_1, x_2, i, j) \text{pf}(5, 6, \dots, \hat{i}, \dots, \hat{j}, \dots, 2n) \end{aligned}$$

and

$$(2.36) \quad \begin{aligned} &\text{pf}(x_1, x_2, x_3, x_4, 5, 6, \dots, 2n) \\ &= \sum_{5 \leq i < j \leq 2n} (-1)^{i+j-1} \text{pf}(x_1, x_2, i, j) \text{pf}(x_3, x_4, 5, 6, \dots, \hat{i}, \dots, \hat{j}, \dots, 2n), \end{aligned}$$

where $\text{pf}(x_j, x_k) = 0$ for $j, k = 1, 2, 3, 4$.

Similar expansion formulae hold for hafnians,

$$(2.37) \quad \begin{aligned} &\text{hf}(x_1, x_2, 5, 6, \dots, 2n) \\ &= \sum_{5 \leq i < j \leq 2n} \text{hf}(x_1, x_2, i, j) \text{hf}(5, 6, \dots, \hat{i}, \dots, \hat{j}, \dots, 2n) \end{aligned}$$

and

$$\begin{aligned}
& \text{hf}(x_1, x_2, x_3, x_4, 5, 6, \dots, 2n) \\
&= \sum_{5 \leq i < j \leq 2n} \text{hf}(x_1, x_2, i, j) \text{hf}(x_3, x_4, 5, 6, \dots, \hat{i}, \dots, \hat{j}, \dots, 2n), \\
&= \sum_{5 \leq i < j \leq 2n} \sum_{5 \leq k < l \leq 2n} \text{hf}(x_1, x_2, i, j) \text{hf}(x_3, x_4, k, l) \\
&\quad \times \text{hf}(5, 6, \dots, \hat{i}, \dots, \hat{j}, \dots, \hat{k}, \dots, \hat{l}, \dots, 2n), \\
&= \sum_{5 \leq i < j < k < l \leq 2n} [\text{hf}(x_1, x_2, i, j) \text{hf}(x_3, x_4, k, l) + \text{hf}(x_1, x_2, k, l) \text{hf}(x_3, x_4, i, j) \\
&\quad + \text{hf}(x_1, x_2, i, k) \text{hf}(x_3, x_4, j, l) + \text{hf}(x_1, x_2, j, l) \text{hf}(x_3, x_4, i, k) \\
&\quad + \text{hf}(x_1, x_2, i, l) \text{hf}(x_3, x_4, j, k) + \text{hf}(x_1, x_2, j, k) \text{hf}(x_3, x_4, i, l)] \\
(2.38) \quad &\quad \times \text{hf}(5, 6, \dots, \hat{i}, \dots, \hat{j}, \dots, \hat{k}, \dots, \hat{l}, \dots, 2n), \\
&= \sum_{5 \leq i < j < k < l \leq 2n} \text{hf}(x_1, x_2, x_3, x_4, i, j, k, l) \\
(2.39) \quad &\quad \times \text{hf}(5, 6, \dots, \hat{i}, \dots, \hat{j}, \dots, \hat{k}, \dots, \hat{l}, \dots, 2n),
\end{aligned}$$

where $\text{hf}(x_i, x_j) = 0$ for $i, j = 1, 2, 3, 4$.

Hereafter we write, for simplicity, n th-order, $(n-1)$ th-order and $(n-2)$ th-order hafnians, respectively as

$$(2.40) \quad \text{hf}(1, 2, \dots, 2n) = \text{hf}(\dots)_n,$$

$$(2.41) \quad \text{hf}(1, 2, \dots, \hat{j}, \dots, \hat{k}, \dots, 2n) = \text{hf}(\hat{j}, \hat{k})_n,$$

$$(2.42) \quad \text{hf}(1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, \hat{k}, \dots, \hat{l}, \dots, 2n) = \text{hf}(\hat{i}, \hat{j}, \hat{k}, \hat{l})_n.$$

Let

$$(2.43) \quad \text{hf}(\dots)_{n-2} = \text{hf}(5, 6, \dots, 2n).$$

Then the expansion formulae of the hafnians are expressed by

$$(2.44) \quad \text{hf}(x_1, x_2, 5, 6, \dots, 2n) = \sum_{5 \leq i < j \leq 2n} \text{hf}(x_1, x_2, i, j) \text{hf}(\hat{i}, \hat{j})_{n-2}$$

and

$$\begin{aligned}
& \text{hf}(x_1, x_2, x_3, x_4, 5, 6, \dots, 2n) \\
(2.45) \quad &= \sum_{5 \leq i < j < k < l \leq 2n} \text{hf}(x_1, x_2, x_3, x_4, i, j, k, l) \text{hf}(\hat{i}, \hat{j}, \hat{k}, \hat{l})_{n-2}.
\end{aligned}$$

A product of the hafnians is then expressed by

$$\begin{aligned}
& \text{hf}(x_1, x_2, 5, 6, \dots, 2n) \text{hf}(x_3, x_4, 5, 6, \dots, 2n) \\
(2.46) \quad &= \sum_{5 \leq i < j \leq 2n} \sum_{5 \leq k < l \leq 2n} \text{hf}(x_1, x_2, i, j) \text{hf}(x_3, x_4, k, l) \text{hf}(\hat{i}, \hat{j})_{n-2} \text{hf}(\hat{k}, \hat{l})_{n-2}.
\end{aligned}$$

Hereafter we omit the character hf denoting hafnians for short. Then the above expansion formula is written simply as

$$(2.47) \quad \begin{aligned} & (x_1, x_2, 5, 6, \dots, 2n)(x_3, x_4, 5, 6, \dots, 2n) \\ &= \sum_{5 \leq i < j \leq 2n} \sum_{5 \leq k < l \leq 2n} (x_1, x_2, i, j)(x_3, x_4, k, l)(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2}. \end{aligned}$$

§ 3. Decomposition of products of the hafnians

Let us consider the following products of the hafnians,

$$(3.1) \quad f_0 = (1, 2, 3, 4, 5, \dots, 2n)(5, \dots, 2n),$$

$$(3.2) \quad f_1 = (1, 2, 5, \dots, 2n)(3, 4, 5, \dots, 2n),$$

$$(3.3) \quad f_2 = (1, 3, 5, \dots, 2n)(2, 4, 5, \dots, 2n),$$

$$(3.4) \quad f_3 = (1, 4, 5, \dots, 2n)(2, 3, 5, \dots, 2n).$$

We shall prove by induction that f_0, f_1, f_2 and f_3 are decomposed into the following forms,

$$(3.5) \quad f_0 = f_{01} + f_{02} + f_{03},$$

$$(3.6) \quad f_1 = f_{01} + f_{12} + f_{13},$$

$$(3.7) \quad f_2 = f_{02} + f_{12} + f_{23},$$

$$(3.8) \quad f_3 = f_{03} + f_{13} + f_{23}.$$

In the above expressions, if one expands f_i and f_j according to the definition(2.10), these two expressions possess common terms, which we denote $f_{i,j}$.

The proof consists of three steps. First we introduce new indices, x_1, x_2, x_3, x_4 in place of the indices 1, 2, 3, 4 and express f_0, f_1, f_2 and f_3 by the following forms,

$$(3.9) \quad f_0 = \bar{f}_1 + \bar{f}_2 + \bar{f}_3 + h_0,$$

$$(3.10) \quad f_1 = \bar{f}_1 + g_1,$$

$$(3.11) \quad f_2 = \bar{f}_2 + g_2,$$

$$(3.12) \quad f_3 = \bar{f}_3 + g_3,$$

where $\bar{f}_j$ is defined by Eqs.(3.67), (3.69) and (3.71) for $j = 1, 2, 3$ respectively, and

$$(3.13) \quad h_0 = (x_1, x_2, x_3, x_4, 5, \dots, 2n)(5, \dots, 2n),$$

$$(3.14) \quad g_1 = (x_1, x_2, 5, \dots, 2n)(x_3, x_4, 5, \dots, 2n),$$

$$(3.15) \quad g_2 = (x_1, x_3, 5, \dots, 2n)(x_2, x_4, 5, \dots, 2n),$$

$$(3.16) \quad g_3 = (x_1, x_4, 5, \dots, 2n)(x_2, x_3, 5, \dots, 2n),$$

and $(x_i, x_j) = 0$ for $i, j = 1, 2, 3, 4$.

Second we introduce new indices y_1, y_2, y_3, y_4 in place of the indices x_1, x_2, x_3, x_4 and express g_1, g_2 and g_3 by the following forms,

$$(3.17) \quad g_1 = \bar{g}_{12} + \bar{g}_{13} + h_1,$$

$$(3.18) \quad g_2 = \bar{g}_{12} + \bar{g}_{23} + h_2,$$

$$(3.19) \quad g_3 = \bar{g}_{23} + \bar{g}_{23} + h_3,$$

where

$$(3.20) \quad h_1 = (y_1, y_2, 5, \dots, 2n)(y_3, y_4, 5, \dots, 2n),$$

$$(3.21) \quad h_2 = (y_1, y_3, 5, \dots, 2n)(y_2, y_4, 5, \dots, 2n),$$

$$(3.22) \quad h_3 = (y_1, y_4, 5, \dots, 2n)(y_2, y_3, 5, \dots, 2n),$$

where $(y_i, y_j) = 0$ for $i, j = 1, 2, 3, 4$ and

$$(3.23) \quad (y_i, j) = a_{i,j} \quad \text{and} \quad a_{i,k}a_{j,k} = 0,$$

for $i, j = 1, 2, 3, 4$ and for $k = 4, 5, \dots, 2n$.

Finally we prove, by induction, decomposition of h_0, h_1, h_2 and h_3 , which gives immediately the decomposition of f_0, f_1, f_2 and f_3 .

§ 3.1. Decomposition of products of $(n-2)$ th-order hafnians

In order to use induction we consider the decomposition of f_0, f_1, f_2 and f_3 for $(n-2)$ th-order hafnians such as

$$(3.24) \quad \begin{aligned} f_0^{n-2} &= (5, 6, 7, 8, 9, 10, \dots, 2n)(9, 10, \dots, 2n), \\ &= f_{01}^{n-2} + f_{02}^{n-2} + f_{03}^{n-2}. \end{aligned}$$

$$(3.25) \quad \begin{aligned} f_1^{n-2} &= (5, 6, 9, 10, \dots, 2n)(7, 8, 9, 10, \dots, 2n), \\ &= f_{01}^{n-2} + f_{12}^{n-2} + f_{13}^{n-2}. \end{aligned}$$

$$(3.26) \quad \begin{aligned} f_2^{n-2} &= (5, 7, 9, 10, \dots, 2n)(6, 8, 9, 10, \dots, 2n), \\ &= f_{02}^{n-2} + f_{12}^{n-2} + f_{23}^{n-2}. \end{aligned}$$

$$(3.27) \quad \begin{aligned} f_3^{n-2} &= (5, 8, 9, 10, \dots, 2n)(6, 7, 9, 10, \dots, 2n), \\ &= f_{03}^{n-2} + f_{13}^{n-2} + f_{23}^{n-2}, \end{aligned}$$

which may be written, using

$$(3.28) \quad (\dots)_{n-2} = (5, 6, 7, 8, 9, 10, \dots, 2n),$$

$$(3.29) \quad (\hat{5}, \hat{6}, \hat{7}, \hat{8})_{n-2} = (9, 10, \dots, 2n),$$

$$(3.30) \quad f_{ij}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) = f_{ij}^{n-2}, \quad \text{for } i, j = 0, 1, 2, 3,$$

as

$$(3.31) \quad \begin{aligned} f_0^{n-2} &= (\hat{5}, \hat{6}, \hat{7}, \hat{8})_{n-2}(\cdots)_{n-2}, \\ &= f_{01}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) + f_{02}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) + f_{03}(\hat{5}, \hat{6}, \hat{7}, \hat{8}), \end{aligned}$$

$$(3.32) \quad \begin{aligned} f_1^{n-2} &= (\hat{5}, \hat{6})_{n-2}(\hat{7}, \hat{8})_{n-2}, \\ &= f_{01}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) + f_{12}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) + f_{13}(\hat{5}, \hat{6}, \hat{7}, \hat{8}), \end{aligned}$$

$$(3.33) \quad \begin{aligned} f_2^{n-2} &= (\hat{5}, \hat{7})_{n-2}(\hat{6}, \hat{8})_{n-2}, \\ &= f_{02}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) + f_{12}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) + f_{23}(\hat{5}, \hat{6}, \hat{7}, \hat{8}), \end{aligned}$$

$$(3.34) \quad \begin{aligned} f_3^{n-2} &= (\hat{5}, \hat{8})_{n-2}(\hat{6}, \hat{7})_{n-2}, \\ &= f_{03}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) + f_{13}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) + f_{23}(\hat{5}, \hat{6}, \hat{7}, \hat{8}). \end{aligned}$$

In general the decomposition of products of $(n-2)$ th-order hafnians are expressed by

$$(3.35) \quad \begin{aligned} f_0^{n-2} &= (\hat{i}, \hat{j}, \hat{k}, \hat{l})_{n-2}(\cdots)_{n-2}, \\ &= f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}), \end{aligned}$$

$$(3.36) \quad \begin{aligned} f_1^{n-2} &= (\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2}, \\ &= f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}), \end{aligned}$$

$$(3.37) \quad \begin{aligned} f_2^{n-2} &= (\hat{i}, \hat{k})_{n-2}(\hat{j}, \hat{l})_{n-2}, \\ &= f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}), \end{aligned}$$

$$(3.38) \quad \begin{aligned} f_3^{n-2} &= (\hat{i}, \hat{l})_{n-2}(\hat{j}, \hat{k})_{n-2}, \\ &= f_{03}(\hat{i}, \hat{l}, \hat{j}, \hat{k}) + f_{13}(\hat{i}, \hat{l}, \hat{j}, \hat{k}) + f_{23}(\hat{i}, \hat{l}, \hat{j}, \hat{k}), \end{aligned}$$

where indices i, j, k, l are members of a list $\{9, 10, \dots, 2n\}$.

We assume the decomposition of products of $(n-2)$ th-order hafnians of the forms, (3.35), (3.36), (3.37) and (3.38) for the use of induction.

§ 3.2. Expressions for f_0, f_1, f_2 and f_3

Consider an entry (j, k) of the hafnian $(1, 2, \dots, 2n)$. Let an index j be a sum of indices $\bar{j}$ and x_j for $j = 1, 2, 3, 4$, namely

$$(3.39) \quad j = \bar{j} + x_j, \quad \text{for } j = 1, 2, 3, 4$$

and introduce new entries of the hafnians,

$$(3.40) \quad (\bar{j}, \bar{k}) = a_{jk},$$

$$(3.41) \quad (x_j, x_k) = 0,$$

$$(3.42) \quad (x_j, \bar{k}) = 0, \quad \text{for } j, k = 1, 2, 3, 4,$$

$$(3.43) \quad (\bar{j}, k) = 0,$$

$$(3.44) \quad (x_j, k) = a_{jk}, \quad \text{for } j = 1, 2, 3, 4 \text{ and } k = 5, 6, \dots, 2n,$$

$$(3.45) \quad (j, k) = a_{jk}, \quad \text{for } j, k = 5, 6, \dots, 2n,$$

so that

$$(3.46) \quad (j, k) = (\bar{j} + x_j, \bar{k} + x_k) = (\bar{j}, \bar{k}) + (\bar{j}, x_k) + (x_j, \bar{k}) + (x_j, x_k)$$

$$(3.47) \quad = a_{j,k}, \quad \text{for } j, k = 1, 2, 3, 4,$$

and

$$(3.48) \quad (j, k) = (\bar{j} + x_j, k) = (\bar{j}, k) + (x_j, k)$$

$$(3.49) \quad = a_{jk}, \quad \text{for } j = 1, 2, 3, 4 \text{ and } k = 5, 6, \dots, 2n.$$

Accordingly we have, for example,

$$(3.50) \quad (1, 2, 5, \dots, 2n) = (\bar{1} + x_1, \bar{2} + x_2, 5, 6, \dots, 2n)$$

$$(3.51) \quad = (\bar{1}, \bar{2}, 5, 6, \dots, 2n) + (\bar{1}, x_2, 5, 6, \dots, 2n)$$

$$(3.52) \quad + (x_1, \bar{2}, 5, 6, \dots, 2n) + (x_1, x_2, 5, 6, \dots, 2n)$$

$$(3.53) \quad = a_{12}(5, 6, \dots, 2n) + (x_1, x_2, 5, 6, \dots, 2n),$$

Then hafnians of order $(n-1)$ and of order n are written as sums of terms of special form, respectively, as follows

$$(3.54) \quad (1, 2, 5, \dots, 2n) = a_{12}(5, \dots, 2n) + (x_1, x_2, 5, \dots, 2n),$$

$$(3.55) \quad (3, 4, 5, \dots, 2n) = a_{34}(5, \dots, 2n) + (x_3, x_4, 5, \dots, 2n),$$

$$(3.56) \quad (1, 3, 5, \dots, 2n) = a_{13}(5, \dots, 2n) + (x_1, x_3, 5, \dots, 2n),$$

$$(3.57) \quad (2, 4, 5, \dots, 2n) = a_{24}(5, \dots, 2n) + (x_2, x_4, 5, \dots, 2n),$$

$$(3.58) \quad (1, 4, 5, \dots, 2n) = a_{14}(5, \dots, 2n) + (x_1, x_4, 5, \dots, 2n),$$

$$(3.59) \quad (2, 3, 5, \dots, 2n) = a_{23}(5, \dots, 2n) + (x_2, x_3, 5, \dots, 2n),$$

$$(3.60) \quad \begin{aligned} (1, 2, 3, 4, 5, \dots, 2n) &= (a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})(5, 6, \dots, 2n) \\ &\quad + a_{12}(x_3, x_4, 5, \dots, 2n) + a_{34}(x_1, x_2, 5, \dots, 2n) \\ &\quad + a_{13}(x_2, x_4, 5, \dots, 2n) + a_{24}(x_1, x_3, 5, \dots, 2n) \\ &\quad + a_{14}(x_2, x_3, 5, \dots, 2n) + a_{23}(x_1, x_4, 5, \dots, 2n) \\ &\quad + (x_1, x_2, x_3, x_4, 5, \dots, 2n). \end{aligned}$$

Then f_0, f_1, f_2, f_3 are written, using the above relations, as

$$(3.61) \quad f_0 = \bar{f}_0 + h_0,$$

$$(3.62) \quad f_1 = \bar{f}_1 + g_1,$$

$$(3.63) \quad f_2 = \bar{f}_2 + g_2,$$

$$(3.64) \quad f_3 = \bar{f}_3 + g_3,$$

where we have used h_0 instead of g_0 for later convenience, and

$$(3.65) \quad \begin{aligned} \bar{f}_0 = & [(a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})(5, 6, \dots, 2n) \\ & + a_{12}(x_3, x_4, 5, \dots, 2n) + a_{34}(x_1, x_2, 5, \dots, 2n) \\ & + a_{13}(x_2, x_4, 5, \dots, 2n) + a_{24}(x_1, x_3, 5, \dots, 2n) \\ & + a_{14}(x_2, x_3, 5, \dots, 2n) + a_{23}(x_1, x_4, 5, \dots, 2n)] \\ & \times (5, 6, \dots, 2n), \end{aligned}$$

$$(3.66) \quad \begin{aligned} h_0 = & (x_1, x_2, x_3, x_4, 5, \dots, 2n)(5, \dots, 2n), \\ \bar{f}_1 = & [a_{12}a_{34}(5, 6, \dots, 2n) \\ & + a_{12}(x_3, x_4, 5, \dots, 2n) + a_{34}(x_1, x_2, 5, \dots, 2n)] \\ & \times (5, 6, \dots, 2n), \end{aligned}$$

$$(3.67) \quad \begin{aligned} g_1 = & (x_1, x_2, 5, \dots, 2n)(x_3, x_4, 5, \dots, 2n), \\ \bar{f}_2 = & [a_{13}a_{24}(5, 6, \dots, 2n) \\ & + a_{13}(x_2, x_4, 5, \dots, 2n) + a_{24}(x_1, x_3, 5, \dots, 2n)] \\ & \times (5, 6, \dots, 2n), \end{aligned}$$

$$(3.68) \quad \begin{aligned} g_2 = & (x_1, x_3, 5, \dots, 2n)(x_2, x_4, 5, \dots, 2n), \\ \bar{f}_3 = & [a_{14}a_{23}(5, 6, \dots, 2n) \\ & + a_{14}(x_2, x_3, 5, \dots, 2n) + a_{23}(x_1, x_4, 5, \dots, 2n)] \\ & \times (5, 6, \dots, 2n), \end{aligned}$$

$$(3.69) \quad g_3 = (x_1, x_4, 5, \dots, 2n)(x_2, x_3, 5, \dots, 2n).$$

Then we find a decomposition of $\bar{f}_0$,

$$(3.70) \quad \bar{f}_0 = \bar{f}_1 + \bar{f}_2 + \bar{f}_3.$$

Accordingly f_0, f_1, f_2, f_3 are expressed by

$$(3.71) \quad f_0 = \bar{f}_1 + \bar{f}_2 + \bar{f}_3 + h_0,$$

$$(3.72) \quad f_1 = \bar{f}_1 + g_1,$$

$$(3.73) \quad f_2 = \bar{f}_2 + g_2,$$

$$(3.74) \quad f_3 = \bar{f}_3 + g_3.$$

§ 3.3. Expressions for g_1, g_2 and g_3

Let $\delta_{i,j}$ be Kronecker's delta function,

$$(3.78) \quad \delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

We have an identity, for $i < j$ and $k < l$,

$$(3.79) \quad \begin{aligned} 1 &= \delta_{i,k}\delta_{j,l} + (\delta_{i,k} + \delta_{i,l} + \delta_{j,k} + \delta_{j,l})(1 - \delta_{i,k}\delta_{j,l}) \\ &+ (1 - \delta_{i,k})(1 - \delta_{i,l})(1 - \delta_{j,k})(1 - \delta_{j,l}). \end{aligned}$$

Accordingly a sum of a product of arbitrary functions, $v(i, j)$ and $w(k, l)$ are written as

$$(3.80) \quad \begin{aligned} \sum_{i < j} \sum_{k < l} v(i, j)w(k, l) &= \sum_{i < j} \sum_{k < l} \delta_{i,k}\delta_{j,l}v(i, j)w(k, l) \\ &+ \sum_{i < j} \sum_{k < l} (\delta_{i,k} + \delta_{i,l} + \delta_{j,k} + \delta_{j,l})(1 - \delta_{i,k}\delta_{j,l})v(i, j)w(k, l) \\ &+ \sum_{i < j} \sum_{k < l} (1 - \delta_{i,k})(1 - \delta_{i,l})(1 - \delta_{j,k})(1 - \delta_{j,l})v(i, j)w(k, l), \end{aligned}$$

which is rewritten as

$$(3.81) \quad \begin{aligned} \sum_{i < j} \sum_{k < l} v(i, j)w(k, l) &= \sum_{i < j} v(i, j)w(i, j) + \sum_{i < j < k} (v(i, j)w(i, k) + v(i, k)w(i, j)) \\ &+ \sum_{i < k < j} (v(i, j)w(k, j) + v(k, j)w(i, j)) + \sum_{i < j < k} (v(i, j)w(j, k) + v(j, k)w(i, j)) \\ &+ \sum_{i < j < k < l} (v(i, j)w(k, l) + v(i, k)w(j, l) + v(i, l)w(j, k) \\ &+ v(k, l)w(i, j) + v(j, l)w(i, k) + v(j, k)w(i, l)). \end{aligned}$$

We expand g_1 ,

$$(3.82) \quad g_1 = (x_1, x_2, 5, 6, \dots, 2n)(x_3, x_4, 5, 6, \dots, 2n)$$

$$(3.83) \quad = \sum_{5 \leq i < j \leq 2n} \sum_{5 \leq k < l \leq 2n} (x_1, x_2, i, j)(x_3, x_4, k, l)(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2}.$$

Let

$$(3.84) \quad u(i, j) = (x_1, x_2, i, j)(\hat{i}, \hat{j}), \quad v(i, j) = (x_3, x_4, i, j)(\hat{i}, \hat{j}).$$

We obtain, using (3.81),

$$\begin{aligned}
g_1 = & \sum_{5 \leq i < j \leq 2n} (x_1, x_2, i, j)(x_3, x_4, i, j)[(\hat{i}, \hat{j})_{n-2}]^2 \\
& + \sum_{5 \leq i < j < k \leq 2n} [(x_1, x_2, i, j)(x_3, x_4, i, k) + (x_1, x_2, i, k)(x_3, x_4, i, j)] \\
& \quad \times (\hat{i}, \hat{j})_{n-2}(\hat{i}, \hat{k})_{n-2} \\
& + \sum_{5 \leq i < k < j \leq 2n} [(x_1, x_2, i, j)(x_3, x_4, k, j) + (x_1, x_2, k, j)(x_3, x_4, i, j)] \\
& \quad \times (\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{j})_{n-2} \\
& + \sum_{5 \leq i < j < k \leq 2n} [(x_1, x_2, i, j)(x_3, x_4, j, k) + (x_1, x_2, j, k)(x_3, x_4, i, j)] \\
& \quad \times (\hat{i}, \hat{j})_{n-2}(\hat{j}, \hat{k})_{n-2} \\
& + \sum_{5 \leq i < j < k < l \leq 2n} [(x_1, x_2, i, j)(x_3, x_4, k, l) + (x_1, x_2, k, l)(x_3, x_4, i, j)] \\
& \quad \times (\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2} \\
& \quad + [(x_1, x_2, i, k)(x_3, x_4, j, l) + (x_1, x_2, j, l)(x_3, x_4, i, k)] \\
& \quad \times (\hat{i}, \hat{k})_{n-2}(\hat{j}, \hat{l})_{n-2} \\
& \quad + [(x_1, x_2, i, l)(x_3, x_4, j, k) + (x_1, x_2, j, k)(x_3, x_4, i, l)] \\
& \quad \times (\hat{i}, \hat{l})_{n-2}(\hat{j}, \hat{k})_{n-2}.
\end{aligned} \tag{3.85}$$

On the other hand we have the following expressions,

$$\begin{aligned}
& (x_1, x_2, i, j)(x_3, x_4, i, j) \\
& = a_{1i}a_{3i}a_{2j}a_{4j} + a_{1j}a_{3j}a_{2i}a_{4i} + a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}, \\
& \text{for } 5 \leq i < j \leq 2n,
\end{aligned} \tag{3.86}$$

$$\begin{aligned}
& (x_1, x_2, i, j)(x_3, x_4, i, k) + (x_1, x_2, i, k)(x_3, x_4, i, j) \\
& = a_{1i}a_{3i}(x_2, x_4, j, k) + a_{2i}a_{4i}(x_1, x_3, j, k) \\
& + a_{1i}a_{4i}(x_2, x_3, j, k) + a_{2i}a_{3i}(x_1, x_4, j, k),
\end{aligned} \tag{3.87}$$

$$\begin{aligned}
& (x_1, x_2, i, j)(x_3, x_4, k, j) + (x_1, x_2, k, j)(x_3, x_4, i, j) \\
& = a_{1j}a_{3j}(x_2, x_4, i, k) + a_{2j}a_{4j}(x_1, x_3, i, k) \\
& + a_{1j}a_{4j}(x_2, x_3, i, k) + a_{2j}a_{3j}(x_1, x_4, i, k),
\end{aligned} \tag{3.88}$$

$$\begin{aligned}
& (x_1, x_2, i, j)(x_3, x_4, j, k) + (x_1, x_2, j, k)(x_3, x_4, i, j) \\
& = a_{1j}a_{3j}(x_2, x_4, i, k) + a_{2j}a_{4j}(x_1, x_3, i, k) \\
& + a_{1j}a_{4j}(x_2, x_3, i, k) + a_{2j}a_{3j}(x_1, x_4, i, k), \\
& \text{for } 5 \leq i < j < k \leq 2n,
\end{aligned} \tag{3.89}$$

which show that all terms in Eq.(3.85) except the last one vanish provided that

$$(3.90) \quad a_{ik}a_{jk} = 0 \quad \text{for } i, j = 1, 2, 3, 4 \text{ and for } k = 5, 6, \dots, 2n.$$

We introduce new indices y_i for $i = 1, 2, 3, 4$ in place of x_i .

Let

$$(3.91) \quad (y_i, y_j) = 0,$$

and

$$(3.92) \quad (y_i, j) = a_{i,j} \quad \text{and} \quad a_{i,k}a_{j,k} = 0,$$

for $i, j = 1, 2, 3, 4$ and for $k = 4, 5, \dots, 2n$.

Accordingly we write g_1 as

$$(3.93) \quad g_1 = \bar{g}_1 + h_1,$$

where

$$(3.94) \quad \begin{aligned} \bar{g}_1 = & \sum_{5 \leq i < j \leq 2n} [a_{1i}a_{3i}a_{2j}a_{4j} + a_{1j}a_{3j}a_{2i}a_{4i} + a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i}, \hat{j})_{n-2}]^2 \\ & + \sum_{5 \leq i < j < k \leq 2n} [a_{1i}a_{3i}(x_2, x_4, j, k) + a_{2i}a_{4i}(x_1, x_3, j, k) \\ & \quad + a_{1i}a_{4i}(x_2, x_3, j, k) + a_{2i}a_{3i}(x_1, x_4, j, k)][(\hat{i}, \hat{j})_{n-2}(\hat{i}, \hat{k})_{n-2}] \\ & + \sum_{5 \leq i < k < j \leq 2n} [a_{1j}a_{3j}(x_2, x_4, i, k) + a_{2j}a_{4j}(x_1, x_3, i, k) \\ & \quad + a_{1j}a_{4j}(x_2, x_3, i, k) + a_{2j}a_{3j}(x_1, x_4, i, k)][(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{j})_{n-2}] \\ & + \sum_{5 \leq i < j < k \leq 2n} [a_{1j}a_{3j}(x_2, x_4, i, k) + a_{2j}a_{4j}(x_1, x_3, i, k) \\ & \quad + a_{1j}a_{4j}(x_2, x_3, i, k) + a_{2j}a_{3j}(x_1, x_4, i, k)][(\hat{i}, \hat{j})_{n-2}(\hat{j}, \hat{k})_{n-2}] \end{aligned}$$

and

$$(3.95) \quad h_1 = (y_1, y_2, 5, 6, \dots, 2n)(y_3, y_4, 5, 6, \dots, 2n),$$

using the condition(3.92).

Interchanging the index 2 with the index 3 in Eq.(3.94) we obtain

$$(3.96) \quad g_2 = (x_1, x_3, 5, 6, \dots, 2n)(x_2, x_4, 5, 6, \dots, 2n) = \bar{g}_2 + h_2,$$

where

$$\begin{aligned}
\bar{g}_2 &= \bar{g}_1|_{2=3}, \\
&= \sum_{5 \leq i < j \leq 2n} [a_{1i}a_{2i}a_{3j}a_{4j} + a_{1j}a_{2j}a_{3i}a_{4i} + a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i}, \hat{j})_{n-2}]^2 \\
&+ \sum_{5 \leq i < j < k \leq 2n} [a_{1i}a_{2i}(x_3, x_4, j, k) + a_{3i}a_{4i}(x_1, x_3, j, k) \\
&\quad + a_{1i}a_{4i}(x_2, x_3, j, k) + a_{2i}a_{3i}(x_1, x_4, j, k)][(\hat{i}, \hat{j})_{n-2}(\hat{i}, \hat{k})_{n-2} \\
&+ \sum_{5 \leq i < k < j \leq 2n} [a_{1j}a_{2j}(x_3, x_4, i, k) + a_{3j}a_{4j}(x_1, x_2, i, k) \\
&\quad + a_{1j}a_{4j}(x_2, x_3, i, k) + a_{2j}a_{3j}(x_1, x_4, i, k)][(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{j})_{n-2} \\
&+ \sum_{5 \leq i < j < k \leq 2n} [a_{1j}a_{2j}(x_3, x_4, i, k) + a_{3j}a_{4j}(x_1, x_2, i, k) \\
&\quad + a_{1j}a_{4j}(x_2, x_3, i, k) + a_{2j}a_{3j}(x_1, x_4, i, k)][(\hat{i}, \hat{j})_{n-2}(\hat{j}, \hat{k})_{n-2}
\end{aligned} \tag{3.97}$$

and

$$(3.98) \quad h_2 = (y_1, y_3, 5, 6, \dots, 2n)(y_2, y_4, 5, 6, \dots, 2n).$$

Similarly interchanging the index 2 with the index 4 in Eq.(3.94) we obtain

$$(3.99) \quad g_3 = (x_1, x_4, 5, 6, \dots, 2n)(x_2, x_3, 5, 6, \dots, 2n) = \bar{g}_3 + h_3,$$

where

$$\begin{aligned}
\bar{g}_3 &= \bar{g}_1|_{2=4}, \\
&= \sum_{5 \leq i < j \leq 2n} [a_{1i}a_{3i}a_{2j}a_{4j} + a_{1j}a_{3j}a_{2i}a_{4i} + a_{1i}a_{2i}a_{3j}a_{4j} + a_{1j}a_{2j}a_{3i}a_{4i}][(\hat{i}, \hat{j})_{n-2}]^2 \\
&+ \sum_{5 \leq i < j < k \leq 2n} [a_{1i}a_{3i}(x_2, x_4, j, k) + a_{2i}a_{4i}(x_1, x_3, j, k) \\
&\quad + a_{1i}a_{2i}(x_3, x_4, j, k) + a_{3i}a_{4i}(x_1, x_2, j, k)][(\hat{i}, \hat{j})_{n-2}(\hat{i}, \hat{k})_{n-2} \\
&+ \sum_{5 \leq i < k < j \leq 2n} [a_{1j}a_{3j}(x_2, x_4, i, k) + a_{2j}a_{4j}(x_1, x_3, i, k) \\
&\quad + a_{1j}a_{2j}(x_3, x_4, i, k) + a_{3j}a_{4j}(x_1, x_2, i, k)][(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{j})_{n-2} \\
&+ \sum_{5 \leq i < j < k \leq 2n} [a_{1j}a_{3j}(x_2, x_4, i, k) + a_{2j}a_{4j}(x_1, x_3, i, k) \\
&\quad + a_{1j}a_{2j}(x_3, x_4, i, k) + a_{3j}a_{4j}(x_1, x_2, i, k)][(\hat{i}, \hat{j})_{n-2}(\hat{j}, \hat{k})_{n-2}
\end{aligned} \tag{3.100}$$

and

$$(3.101) \quad h_3 = (y_1, y_4, 5, 6, \dots, 2n)(y_2, y_3, 5, 6, \dots, 2n).$$

From the expressions of $\bar{g}_1, \bar{g}_2$ and $\bar{g}_3$ we find the following decomposition,

$$(3.102) \quad \bar{g}_1 = \bar{g}_{12} + \bar{g}_{13},$$

$$(3.103) \quad \bar{g}_2 = \bar{g}_{12} + \bar{g}_{23},$$

$$(3.104) \quad \bar{g}_3 = \bar{g}_{13} + \bar{g}_{23},$$

where

$$\begin{aligned} \bar{g}_{12} = & \sum_{5 \leq i < j \leq 2n} [a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i}, \hat{j})_{n-2}]^2 \\ & + \sum_{5 \leq i < j < k \leq 2n} [a_{1i}a_{4i}(x_2, x_3, j, k) + a_{2i}a_{3i}(x_1, x_4, j, k)](\hat{i}, \hat{j})_{n-2}(\hat{i}, \hat{k})_{n-2} \\ & + \sum_{5 \leq i < k < j \leq 2n} [a_{1j}a_{4j}(x_2, x_3, i, k) + a_{2j}a_{3j}(x_1, x_4, i, k)](\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{j})_{n-2} \\ (3.105) \quad & + \sum_{5 \leq i < j < k \leq 2n} [a_{1j}a_{4j}(x_2, x_3, i, k) + a_{2j}a_{3j}(x_1, x_4, i, k)](\hat{i}, \hat{j})_{n-2}(\hat{j}, \hat{k})_{n-2}, \end{aligned}$$

$$(3.106) \quad g_{13} = \bar{g}_{12}|_{3 \Rightarrow 4},$$

$$(3.107) \quad g_{23} = \bar{g}_{12}|_{2 \Rightarrow 4}.$$

Accordingly g_1, g_2, g_3 are decomposed into

$$(3.108) \quad g_1 = \bar{g}_{12} + \bar{g}_{13} + h_1,$$

$$(3.109) \quad g_2 = \bar{g}_{12} + \bar{g}_{23} + h_2,$$

$$(3.110) \quad g_3 = \bar{g}_{23} + \bar{g}_{23} + h_3.$$

§ 3.4. Expressions for h_0, h_1, h_2 and h_3

We have h_0 defined by Eq.(3.66),

$$(3.111) \quad h_0 = (y_1, y_2, y_3, y_4, 5, 6, \dots, 2n)(5, 6, \dots, 2n),$$

which is expanded, using Eq.(2.39), in

$$(3.112) \quad = \sum_{5 \leq i < j < k < l \leq 2n} (y_1, y_2, y_3, y_4, i, j, k, l)(\hat{i}, \hat{j}, \hat{k}, \hat{l})_{n-2}(\dots)_{n-2}.$$

Here we assume that the product $(\hat{i}, \hat{j}, \hat{k}, \hat{l})_{n-2}(\dots)_{n-2}$ is decomposed into

$$(3.113) \quad (\hat{i}, \hat{j}, \hat{k}, \hat{l})_{n-2}(\dots)_{n-2} = f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).$$

The hafnian $(y_1, y_2, y_3, y_4, i, j, k, l)$ in Eq.(3.112) is expanded in three different ways,

$$\begin{aligned}
 & (y_1, y_2, y_3, y_4, i, j, k, l) \\
 &= (y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j) \\
 &+ (y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k) \\
 &+ (y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, i)
 \end{aligned}
 \tag{3.114}$$

$$\begin{aligned}
 &= (y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j) \\
 &+ (y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k) \\
 &+ (y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, i)
 \end{aligned}
 \tag{3.115}$$

$$\begin{aligned}
 &= (y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j) \\
 &+ (y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k) \\
 &+ (y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, i).
 \end{aligned}
 \tag{3.116}$$

Substituting Eqs.(3.114) and (3.113) into Eq.(3.112) we obtain an expression for h_0 ,

$$\begin{aligned}
 h_0 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j) \\
 & + (y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k) \\
 & + (y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, i)] \\
 & \times [f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l})].
 \end{aligned}
 \tag{3.117}$$

On the other hand, replacing x_i in Eq.(3.85) by y_i for $i = 1, 2, 3, 4$ and using Eqs.(3.92) we obtain

$$\begin{aligned}
 h_1 &= (y_1, y_2, 5, 6, \dots, 2n)(y_3, y_4, 5, 6, \dots, 2n), \\
 &= \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)](\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2} \\
 &+ [(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)](\hat{i}, \hat{k})_{n-2}(\hat{j}, \hat{l})_{n-2} \\
 &+ [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)](\hat{i}, \hat{l})_{n-2}(\hat{j}, \hat{k})_{n-2}.
 \end{aligned}
 \tag{3.118}$$

We assume that the products of the $(n-2)$ th-order hafnians are decomposed into

$$(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2} = f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}),
 \tag{3.119}$$

$$(\hat{i}, \hat{k})_{n-2}(\hat{j}, \hat{l})_{n-2} = f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}),
 \tag{3.120}$$

$$(\hat{i}, \hat{l})_{n-2}(\hat{j}, \hat{k})_{n-2} = f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).
 \tag{3.121}$$

Substituting Eqs.(3.119),(3.120) and (3.121) into Eq.(3.118) we obtain an expression for

h_1 ,

$$\begin{aligned}
 h_1 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)] \\
 & \times [f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l})] \\
 & + [(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)] \\
 & \times [f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})] \\
 & + [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)] \\
 & \times [f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})].
 \end{aligned}
 \tag{3.122}$$

Comparing the expression (3.117) for h_0 with the expression (3.122) for h_1 , we obtain common terms h_{01} of h_0 and h_1 ,

$$\begin{aligned}
 h_{01} = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)] \\
 & \times f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
 & + [(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)] \\
 & \times f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
 & + [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)] \\
 & \times f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).
 \end{aligned}
 \tag{3.123}$$

Let the decomposition of h_1 be

$$(3.124) \quad h_1 = h_{01} + \tilde{h}_1.$$

Then we find that

$$\begin{aligned}
 \tilde{h}_1 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)] \\
 & \times [f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l})] \\
 & + [(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)] \\
 & \times [f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})] \\
 & + [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)] \\
 & \times [f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})].
 \end{aligned}
 \tag{3.125}$$

Interchanging y_2 with y_3 in Eqs.(3.117) and (3.122), we obtain common terms h_{02} of h_0

and h_2 ,

$$\begin{aligned}
 h_{02} = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j)] \\
 & \times f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
 & + [(y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k)] \\
 & \times f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
 & + [(y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, l)] \\
 & \times f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).
 \end{aligned}
 \tag{3.126}$$

Interchanging y_2 with y_3 in Eqs.(3.124) and (3.125), we obtain a decomposition of h_2 ,

$$(3.127) \quad h_2 = h_{02} + \tilde{h}_2,$$

where

$$\begin{aligned}
 \tilde{h}_2 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j)] \\
 & \times [f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l})] \\
 & + [(y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k)] \\
 & \times [f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})] \\
 & + [(y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, l)] \\
 & \times [f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})].
 \end{aligned}
 \tag{3.128}$$

Similarly we obtain common terms h_{03} of h_0 and h_3 interchanging y_2 with y_4 in Eqs.(3.117) and (3.122)

$$\begin{aligned}
 h_{03} = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j)] \\
 & \times f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
 & + [(y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)] \\
 & \times f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
 & + [(y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, l)] \\
 & \times f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).
 \end{aligned}
 \tag{3.129}$$

Interchanging y_2 with y_4 in Eqs.(3.124) and (3.125), we obtain a decomposition of h_3 ,

$$(3.130) \quad h_3 = h_{03} + \tilde{h}_3,$$

where

$$\begin{aligned}
 \tilde{h}_3 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j)] \\
 & \times [f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l})] \\
 & + [(y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)] \\
 & \times [f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})] \\
 & + [(y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, l)] \\
 & \times [f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})].
 \end{aligned}
 \tag{3.131}$$

We find, using the expansion formulae (3.114), (3.115), (3.116) and the assumption (3.113), that a simple sum of h_{01} , h_{02} and h_{03} is reduced to h_0 ,

$$\begin{aligned}
 & h_{01} + h_{02} + h_{03} \\
 = & \sum_{5 \leq i < j < k < l \leq 2n} (y_1, y_2, y_3, y_4, i, j, k, l) \\
 & \times [f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l})], \\
 = & \sum_{5 \leq i < j < k < l \leq 2n} (y_1, y_2, y_3, y_4, i, j, k, l)(\hat{i}, \hat{j}, \hat{k}, \hat{l})_{n-2}(\cdots)_{n-2} \\
 = & h_0.
 \end{aligned}
 \tag{3.132}$$

Accordingly h_0 , h_1 , h_2 and h_3 are decomposed into

$$h_0 = h_{01} + h_{02} + h_{03}, \tag{3.133}$$

$$h_1 = h_{01} + \tilde{h}_1, \tag{3.134}$$

$$h_2 = h_{02} + \tilde{h}_2, \tag{3.135}$$

$$h_3 = h_{03} + \tilde{h}_3, \tag{3.136}$$

§ 3.5. Expressions for $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$

We rearrange $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$ as

$$\begin{aligned}
 \tilde{h}_1 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)] \\
 & + [(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
 & + [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)] \\
 & + [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
 & + [(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)] \\
 & + [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}),
 \end{aligned}
 \tag{3.137}$$

$$\begin{aligned}
\tilde{h}_2 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j) \\
& + (y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j) \\
& + (y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, l)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k) \\
& + (y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, l)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l})
\end{aligned}
\tag{3.138}$$

and

$$\begin{aligned}
\tilde{h}_3 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j) \\
& + (y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j) \\
& + (y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, l)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k) \\
& + (y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, l)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).
\end{aligned}
\tag{3.139}$$

We have identities of hafnians, Eqs.(2.23),(2.24) and (2.25), which are written as

$$\begin{aligned}
& (y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j) \\
& + (y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k) \\
& = (y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, l) \\
& + (y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, l),
\end{aligned}
\tag{3.140}$$

$$\begin{aligned}
& (y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j) \\
& + (y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l) \\
& = (y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k) \\
& + (y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)
\end{aligned}
\tag{3.141}$$

and

$$\begin{aligned}
& (y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k) \\
& + (y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l) \\
& = (y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j) \\
& + (y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j).
\end{aligned}
\tag{3.142}$$

Substituting these identities (3.140),(3.141) and (3.142) into Eq.(3.137) we obtain an expression for $\tilde{h}_1$,

$$\begin{aligned}
\tilde{h}_1 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, l) \\
& + [(y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, l)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k) \\
& + (y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j) \\
& + (y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).
\end{aligned}
\tag{3.143}$$

Interchanging y_2 with y_3 in Eq.(3.143) we obtain an expression for $\tilde{h}_2$,

$$\begin{aligned}
\tilde{h}_2 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l) \\
& + [(y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, l)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k) \\
& + (y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j) \\
& + (y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).
\end{aligned}
\tag{3.144}$$

Similarly interchanging y_2 with y_4 in Eq.(3.143) we obtain an expression for $\tilde{h}_3$,

$$\begin{aligned}
\tilde{h}_3 = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, l) \\
& + [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k) \\
& + (y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\
& + [(y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j) \\
& + (y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}).
\end{aligned}
\tag{3.145}$$

From the expressions (3.143),(3.144) and (3.145) for $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$, respectively we obtain the following decompositions,

$$(3.146) \quad \tilde{h}_1 = h_{12} + h_{13},$$

$$(3.147) \quad \tilde{h}_2 = h_{12} + h_{23},$$

$$(3.148) \quad \tilde{h}_3 = h_{13} + h_{23},$$

where

$$(3.149) \quad \begin{aligned} h_{12} = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, l)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\ & + [(y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\ & + [(y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}), \end{aligned}$$

$$(3.150) \quad \begin{aligned} h_{13} = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, l)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\ & + [(y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\ & + [(y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \end{aligned}$$

and

$$(3.151) \quad \begin{aligned} h_{23} = & \sum_{5 \leq i < j < k < l \leq 2n} [(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)] f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\ & + [(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)] f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) \\ & + [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)] f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}). \end{aligned}$$

§ 3.6. Decompositions of f_0, f_1, f_2 and f_3

We have found in the previous subsections the decompositions,

$$(3.152) \quad h_1 = h_{01} + \tilde{h}_1, \quad h_2 = h_{02} + \tilde{h}_2, \quad h_3 = h_{03} + \tilde{h}_3,$$

$$(3.153) \quad \tilde{h}_1 = h_{12} + h_{13}, \quad \tilde{h}_2 = h_{12} + h_{23}, \quad \tilde{h}_3 = h_{13} + h_{23},$$

which together with Eqs.(3.133), give the decompositions of h_0, h_1, h_2 and h_3 ,

$$(3.154) \quad h_0 = h_{01} + h_{02} + h_{03}, \quad h_1 = h_{01} + h_{12} + h_{13},$$

$$(3.155) \quad h_2 = h_{02} + h_{12} + h_{23}, \quad h_3 = h_{03} + h_{13} + h_{23}.$$

On the other hand, we have found the decompositions,

$$(3.156) \quad f_0 = \bar{f}_1 + \bar{f}_2 + \bar{f}_3 + h_0,$$

$$(3.157) \quad f_1 = \bar{f}_1 + g_1, \quad g_1 = g_{12} + g_{13} + h_1,$$

$$(3.158) \quad f_2 = \bar{f}_2 + g_2, \quad g_2 = g_{12} + g_{23} + h_2,$$

$$(3.159) \quad f_3 = \bar{f}_3 + g_3, \quad g_3 = g_{13} + g_{23} + h_3.$$

Combining these decompositions we finally obtain the decompositions,

$$(3.160) \quad f_0 = f_{01} + f_{02} + f_{03}, \quad f_1 = f_{01} + f_{12} + f_{13},$$

$$(3.161) \quad f_2 = f_{02} + f_{12} + f_{23}, \quad f_3 = f_{03} + f_{13} + f_{23},$$

where

$$(3.162) \quad f_{01} = \bar{f}_1 + h_{01}, \quad f_{02} = \bar{f}_2 + h_{02}, \quad f_{03} = \bar{f}_3 + h_{03},$$

$$(3.163) \quad f_{12} = g_{12} + h_{12}, \quad f_{13} = g_{13} + h_{13}, \quad f_{23} = g_{23} + h_{23}.$$

Thus we have proved that the decompositions of the products of n th-order hafnians,

$$(3.164) \quad f_0 = (1, 2, 3, 4, 5, 6, \dots, 2n)(5, 6, \dots, 2n)$$

$$(3.165) \quad f_1 = (1, 2, 5, 6, \dots, 2n)(3, 4, 5, 6, \dots, 2n)$$

$$(3.166) \quad f_2 = (1, 3, 5, 6, \dots, 2n)(2, 4, 5, 6, \dots, 2n)$$

$$(3.167) \quad f_3 = (1, 3, 5, 6, \dots, 2n)(2, 4, 5, 6, \dots, 2n),$$

provided that the decompositions of the products of $(n-2)$ th-order hafnians hold. It is easily proved that the decompositions of the products of fourth-order hafnians hold. Hence we have proved by induction that the decompositions of the products of any order hafnians.

§ 4. Ultradiscrete analogue of the identity of pfaffians

It is known that a variety of soliton equations exhibiting multi-soliton solutions expressed by pfaffians give rise to the following identity of pfaffians,

$$\begin{aligned} & \text{pf}(1, 2, 3, 4, 5, 6, \dots, 2n) \text{pf}(5, 6, \dots, 2n) \\ &= \text{pf}(1, 2, 5, 6, \dots, 2n) \text{pf}(3, 4, 5, 6, \dots, 2n) \\ & - \text{pf}(1, 3, 5, 6, \dots, 2n) \text{pf}(2, 4, 5, 6, \dots, 2n) \\ (4.1) \quad & + \text{pf}(1, 4, 5, 6, \dots, 2n) \text{pf}(2, 3, 5, 6, \dots, 2n). \end{aligned}$$

It is known that the pfaffians can not be ultradiscretized due to negativity terms. A remedy for the problem was found by Takahashi and the author of the present article.

They have expressed the multi-soliton solutions to an ultradiscretized soliton equation called “Box and Ball System” by ultradiscretized *permanents* instead of determinants. Accordingly we consider an ultradiscrete form of the identity of hafnians.

We have

$$(4.2) \quad f_0 = (1, 2, 3, 4, 5, 6, \dots, 2n)(5, 6, \dots, 2n),$$

$$(4.3) \quad f_1 = (1, 2, 5, 6, \dots, 2n)(3, 4, 5, 6, \dots, 2n),$$

$$(4.4) \quad f_2 = (1, 3, 5, 6, \dots, 2n)(2, 4, 5, 6, \dots, 2n),$$

$$(4.5) \quad f_3 = (1, 4, 5, 6, \dots, 2n)(2, 3, 5, 6, \dots, 2n),$$

which were decomposed in the previous section into the following form,

$$(4.6) \quad f_0 = f_{01} + f_{02} + f_{03},$$

$$(4.7) \quad f_1 = f_{01} + f_{12} + f_{13},$$

$$(4.8) \quad f_2 = f_{02} + f_{12} + f_{23},$$

$$(4.9) \quad f_3 = f_{03} + f_{13} + f_{23}.$$

We consider a relation of hafnians,

$$(4.10) \quad f_0 + f_2 = f_1 + f_3,$$

which does holds for pfaffians but not for hafnians.

Let

$$(4.11) \quad f_0 = \exp F_0/\epsilon, \quad f_1 = \exp F_1/\epsilon, \quad f_2 = \exp F_2/\epsilon, \quad f_3 = \exp F_3/\epsilon,$$

$$(4.12) \quad f_{01} = \exp F_{01}/\epsilon, \quad f_{02} = \exp F_{02}/\epsilon, \quad f_{03} = \exp F_{03}/\epsilon,$$

$$(4.13) \quad f_{12} = \exp F_{12}/\epsilon, \quad f_{13} = \exp F_{13}/\epsilon, \quad f_{23} = \exp F_{23}/\epsilon.$$

Taking a limit $\epsilon \rightarrow +0$ in Eq.(4.10), we have an ultradiscrete form of the identity of the hafnians,

$$(4.14) \quad \max(F_0, F_2) = \max(F_1, F_3).$$

By virtue of the decomposition of the hafnians we find that

$$(4.15) \quad F_0 = \max(F_{01}, F_{02}, F_{03}),$$

$$(4.16) \quad F_1 = \max(F_{01}, F_{12}, F_{13}),$$

$$(4.17) \quad F_2 = \max(F_{02}, F_{12}, F_{23}),$$

$$(4.18) \quad F_3 = \max(F_{03}, F_{13}, F_{23}),$$

and Eq.(4.14) is expressed by

$$(4.19) \quad \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23}) = \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23}).$$

We shall investigate whether Eq.(4.19) does hold or not. Equation (4.19) holds in the following six cases,

- (i) $F_{01} \geq \max(F_{02}, F_{03}, F_{12}, F_{13}, F_{23}),$
- (ii) $F_{02} = \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23}),$
- (iii) $F_{03} \geq \max(F_{01}, F_{02}, F_{12}, F_{13}, F_{23}),$
- (iv) $F_{12} \geq \max(F_{01}, F_{02}, F_{03}, F_{13}, F_{23}),$
- (v) $F_{13} = \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23}),$
- (vi) $F_{23} \geq \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{13}),$

except for the following two cases,

- (vii) $F_{02} > \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23}),$
- (viii) $F_{13} > \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23}).$

In the cases (vii) and (viii) we have the relation, $F_0 = F_2$ and $F_1 = F_3$ respectively. Accordingly we find the algebraic identity of the ultradiscretized hafnians,

$$(4.20) \quad (\max(F_0, F_2) - \max(F_1, F_3))(F_0 - F_2)(F_1 - F_3) = 0,$$

where F_0, F_1, F_2 and F_3 are the ultradiscrete form of f_0, f_1, f_2 and f_3 , respectively.

We have thus proved the ultradiscrete analogue of the identity of the pfaffians.

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