

σ -finite acims for weakly expanding random maps with uniformly contractive branches

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1 Introduction

For a given measurable dynamical system or measurable random dynamical system over a probability space, the existence of an absolutely continuous finite or σ -finite infinite invariant measure (acim, for abbreviation) with respect to the reference measure is important and fundamental to analyze statistical properties of the system. See [A97, ANV15, BB16, F69, F80, S95] and references therein. In [T1], the author gave some necessary and sufficient conditions for the existence of a finite or σ -finite acim for a given system over a probability space (in terms of Markov operators). In this note, based on [T2], we introduce a certain model of (annealed type) random dynamical systems and estimate the density functions of their acim.

We recall definitions and notations which are needed in this note. Let $(X, \mathcal{B}, \lambda)$ be a probability space and $T : X \rightarrow X$ be a measurable and non-singular transformation (i.e., $T^{-1}\mathcal{B} \subset \mathcal{B}$ and the pushforward measure $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ). Then a measure μ on $(X, \mathcal{B})$ is called an *acim* for T provided that μ is absolutely continuous with respect to λ and $\mu \circ T^{-1} = \mu$. μ is called a *finite* (resp. σ -finite) *acim* when μ is a finite (resp. σ -finite) measure. Analogously, for an annealed type random map we define an acim as follows. Let I be a non-empty and at most countable set and T_i be a measurable and non-singular transformation from X into itself for each $i \in I$. For a given probability vector $\{p_i\}_{i \in I}$, a *random iteration* (of $\{T_i\}_{i \in I}$ with probabilities $\{p_i\}_{i \in I}$) $\{T_i, p_i : i \in I\}$ is the Markov process given by the following transition probability:

$$\mathbb{P}(x, A) = \sum_{i \in I} p_i 1_A(T_i x)$$

for λ -almost every $x \in X$ and any $A \in \mathcal{B}$. Then a λ -absolutely continuous finite (resp. σ -finite) measure μ is called a *finite* (resp. σ -finite) *acim* if μ satisfies

$$\mu = \sum_{i \in I} p_i \mu \circ T_i^{-1}.$$

We also recall that a measure μ is called *ergodic* if for any measurable set with $\mathbb{P}(x, E) = 1_E(x)$ for μ -almost every $x \in X$ we have either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$. For more general cases of random dynamics including the case when I is uncountable or position dependent cases (i.e., p_i is also a function of $x \in X$), we refer to [GB03, I12, I20] and references therein.

Now we are addressed to define the model of our random dynamical systems on the unit interval. Set arbitrary $\alpha > 0$. We define the partition of $X = [0, 1]$ into $\mathcal{Q}_\alpha = \{X_n = ((n+1)^{-1/\alpha}, n^{-1/\alpha}]\}_{n \geq 1}$. Then we have $\lambda(X_n)/\lambda(X_{n+1})$ monotonically decrease to 1 as $n \rightarrow \infty$ where λ is Lebesgue measure on X . Let I be a non-empty countable set and for each $i \in I$, J_i be a non-empty subset of $\mathbb{N}$. We consider a family of transformations $\{T_i : i \in I\}$ on X (see Figure 1 below for an example) which are piecewise monotone and piecewise linear on the partition $\mathcal{Q}_\alpha$ satisfying

- (a-1) For each $i \in I$, $T_i|_{X_n} : X_n \rightarrow X_{n-1}$, for any $n \geq 2$, monotonically increasing given by

$$T_i|_{X_n}(x) = \frac{(n-1)^{-1/\alpha} - n^{-1/\alpha}}{n^{-1/\alpha} - (n+1)^{-1/\alpha}}x - \frac{(n+1)^{-1/\alpha}(n-1)^{-1/\alpha} - n^{-2/\alpha}}{n^{-1/\alpha} - (n+1)^{-1/\alpha}},$$

namely, all T_i are identical on $X \setminus X_1$;

- (a-2) For each $i \in I$, $T_i|_{X_1} : X_1 \rightarrow \bigcup_{k \in J_i} X_k$, monotonically increasing and surjective which is piecewise linear in the sense

$$T'_i|_{X_1} = \frac{\sum_{k \in J_i} \lambda(X_k)}{\lambda(X_1)}$$

whenever the derivative can be defined.

We call a family of transformations $\{T_i : i \in I\}$ with the above conditions (a-1) and (a-2) *piecewise linear intermittent Markov maps* (with the index $\{J_i\}_{i \in I}$).

Then, for a given probability vector p_i on I , we consider a *random iteration of piecewise linear intermittent Markov maps with uniformly contractive part* such that

- (b-1) $\{T_i : i \in I\}$ is piecewise linear intermittent Markov maps;
- (b-2) On X_1 , the random iteration $\{T_i, p_i : i \in I\}$ is uniformly contractive on average:

$$\sum_{i \in I} \frac{p_i}{|T'_i|_{X_1}|} = \sum_{i \in I} \frac{p_i \lambda(X_1)}{\sum_{k \in J_i} \lambda(X_k)} > 1.$$

Remark 1.1. The requirement that I is countable is just for simplicity of notation. In fact, we can define the model even when I is an uncountable set and p is a probability measure on I . If I is uncountable then we integrate over I instead of summation over I . See [T2] for more detail.

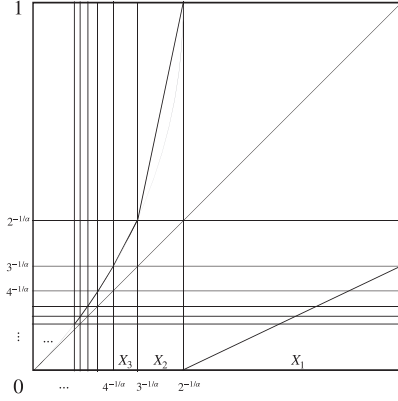


Figure 1: A graph of T_i for some $i \in I$ of piecewise linear intermittent Markov maps. For this T_i , we can see $J_i = \{n \in \mathbb{N} : n \geq 3\}$ since $T_i X_1 = \bigcup_{n \geq 3} X_n$.

By the definition, random iterations of piecewise linear intermittent Markov maps with uniformly contractive part do not satisfy expanding on average even if we remove a small neighborhood of a common indifferent fixed point 0 in the sense [I20]. In the next section we will see that this model always admits a finite/ σ -finite acim and show the criterion whether an acim is finite or infinite. Moreover, under some mild condition, the acim is conservative, ergodic and hence unique up to multiplicative constants.

2 Main Result

In this section, we show the existence of a σ -finite acim for any random iteration of piecewise linear intermittent Markov maps with uniformly contractive part defined in the last section. For weakly expanding case (excluding the case like (b-2)), sufficient conditions for the existence of a σ -finite acim was already shown in [I20] via the inducing scheme. The main result of this note is as follows.

Theorem 2.1. *Any random iteration $\{T_i, p_i : i \in I\}$ of piecewise linear intermittent Markov maps with uniformly contractive part, which satisfies (b-1) and (b-2), admits a σ -finite acim. The invariant density $d\mu/d\lambda$ is given by the following formula:*

$$\frac{d\mu}{d\lambda} = \sum_{i \in I} \frac{p_i}{\sum_{j \in J_i} \lambda(X_j)} \sum_{n \geq 0} \sum_{\substack{j \in J_i \\ j > n}} \frac{\lambda(X_j)}{\lambda(X_{j-n})} 1_{X_{j-n}}. \quad (2.1)$$

Consequently, $\mu(X) < \infty$ if and only if

$$\sum_{i \in I} \frac{p_i}{\sum_{j \in J_i} \lambda(X_j)} \sum_{n \geq 0} \sum_{\substack{j \in J_i \\ j > n}} \lambda(X_j) < \infty.$$

Remark 2.2. (i) The statements of Theorem 2.1 are still true when I is an uncountable set and p is a probability on I , by replacing the summation by integration over I . See Theorem 3.1 and Corollary 3.1 in [T2].

(ii) For the case of nonlinear random maps like LSV map or Manneville–Pomeau map with uniformly contractive part, we also may be able to show the existence of σ -finite acims and to estimate its densities if the branches on X_1 is linear in the sense (a-2). Otherwise, it would be possible to estimate the invariant density under certain distortion property on X_1 . That will be studied in the other paper.

The following corollary shows the uniqueness of the acim given in Theorem 2.1 under mild condition.

Corollary 2.3. Suppose that $\#J_i \geq 2$ for each $i \in I$ with $p_i > 0$. Then the acim μ for a random iteration $\{T_i, p_i : i \in I\}$ given by (2.1) is an ergodic measure. Consequently, the acim μ is unique up to multiplicative constant.

In the rest of this note, we give several examples of random iterations of piecewise linear intermittent Markov maps with uniformly contractive part.

The first example is a random iteration of piecewise linear intermittent Markov maps with “thin branches” which return to the indifferent fixed point. The σ -finite acim of the following example varies at $\alpha = 1$ from finite to infinite, which is same as the deterministic case [Th80].

Example 2.4. Let $\alpha > 0$, $I = \{1, 2\}$ and $J_i = \{(10i)n : n \geq 1\}$ for $i \in I$. Set $p_1 = p$ (and $p_2 = 1 - p$). (See Figure 2 below for the corresponding map.) Then the random iteration of piecewise linear intermittent Markov maps with uniformly contractive part $\{T_i; p_i : i \in I\}$ has the unique ergodic σ -finite acim μ by Theorem 2.1. We also calculate that $\mu(X) < \infty$ if and only if $\alpha < 1$, same as deterministic transformation case.

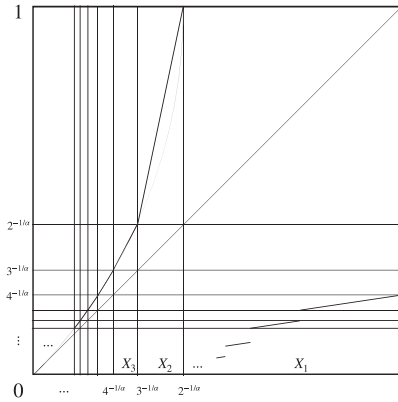


Figure 2: A graph of T_1 in Example 2.4

The next example is a random iteration which always admits a finite acim for any α of the order of tangency at the indifferent fixed point, because all points will never return to a neighborhood of the indifferent fixed point.

Example 2.5. Let $\alpha > 0$, $I \subset \mathbb{N}$ and J_i satisfy that $\bigcup_{i \in I} J_i$ is a finite set. Then, from Theorem 2.1, for any probability vector $\{p_i\}_{i \in I}$, the random iteration of $\{T_i; p_i : i \in I\}$ admits a finite acim. Further, the invariant density for this random iteration is bounded above even around the point 0.

The following example also admits a finite acim, although many points of positive measure will return to an enough small neighborhood of the indifferent fixed point with positive probability.

Example 2.6. Let $\alpha > 0$, $I = \mathbb{N}$ and $J_i = \{2, 3, \dots, i+1\}$ for $i \in I$. See Figure 3 for the maps $\{T_i : i \in I\}$. If we put $p_i = 1/2^i$ for $i \in I$, then the random iteration $\{T_i, p_i : i \in I\}$ admits the ergodic σ -finite acim μ by Theorem 2.1. We can also see $\mu(X) < \infty$ for any $\alpha > 0$. That is, the invariant measure μ is always finite.

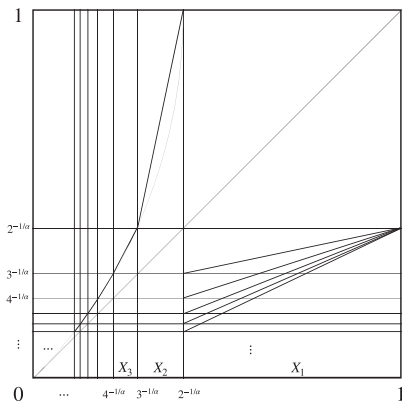


Figure 3: A graph of transformations in Example 2.6 and 2.7 (infinitely many branches on X_1 correspond to $T_i|_{X_1}$, $i \in I$, respectively)

We modify Example 2.6 and we show that the modified random dynamics admit both finite and σ -finite infinite acim depending on the parameter $\alpha > 0$. The critical point of α is different from the deterministic case.

Example 2.7. Let $\alpha > 0$ and $k \geq 2$ a natural number. Set $I = \mathbb{N}$ and $p_i = 1/2^i$. Then we put $J_i = \{2, 3, \dots, 2^{k(i+1)}\}$. As the same way in Example 2.6, we have the ergodic σ -finite acim μ for this random iteration. Furthermore we can calculate that μ is finite if and only if $\alpha < k/(k-1)$.

Remark 2.8. For random iterations of non-uniformly expanding maps as Example 2.7 and 2.9, the critical value of α where the invariant measure varies from finite to infinite

can be different from deterministic case ([Th80]). Similar example which is expanding on average except a small neighborhood of an indifferent fixed point can be seen in Example 6.2 of [I20].

In the last two examples, most of points will rarely return to the indifferent fixed point 0 so that the acim hard to become infinite. Then, conversely, we will see that the following example makes the acim tend to become an infinite measure.

Example 2.9. Let $\alpha > 0$, $I = \mathbb{N}$ and $k(i) \geq 2$ increasing natural numbers of $i \in I$. We set $J_i = \{j \in \mathbb{N} : j \geq k(i)\}$. Then for any probability vector $\{p_i\}_{i \in I}$, the ergodic σ -finite acim μ for this random iteration satisfies that μ is finite if and only if $\alpha < 1$ and $\sum_i p_i k(i) < \infty$. Thus, for example, if $p_i = 6/(i^2 \pi^2)$ and $k(i) = 2 + [i^\gamma]$ for some $\gamma > 0$, then μ is finite if and only if $\alpha < 1$ and $0 < \gamma < 1$.

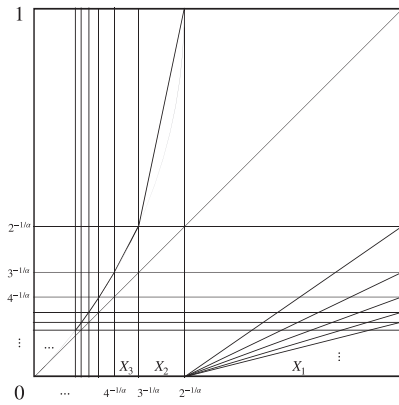


Figure 4: A graph of transformations in Example 2.9 (infinitely many branches on X_1 correspond to $T_i|_{X_1}$, $i \in I$, respectively)

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