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LINEAR REPRESENTATIONS OF A KNOT GROUP OVER A FINITE RING AND ALEXANDER POLYNOMIAL AS AN OBSTRUCTION

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1. Introduction

Let K be a knot in S^3 and $G(K)$ its knot group $\pi_1(S^3-K)$. Here we write $\alpha : G(K) \to \mathbb{Z} = \langle t\rangle$ for the abelianization of $G(K)$. In this paper, we assume

- any presentation of $G(K)$ is a Wirtinger presentation,
- its Alexander polynomial $\Delta_K(t)$ is a polynomial, that is, its lowest degree term is a constant term.

There are many studies on linear representations of $G(K)$ over a finite field or a finite ring. In general it is not easy to see when the set of non commutative $SL(2,\mathbb{Z}/d)$-representations is not empty.

Then we consider the following problem to be easier.

Problem 1.1. *Does there exist a non commutative representation of $G(K)$ in $SL(2,\mathbb{Z}/d)$ for infinitely many integers $d \in \mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n > 0\}$?*

We can prove the following by using zeros of Alexander polynomial.

Theorem 1.2. *If $\Delta_K(t) \neq 1$, then there exists a non commutative representation $G(K) \to GL(2,\mathbb{Z}/d)$ for infinitely many $d \in \mathbb{Z}_+$.*

Further if the Alexander polynomial has a special form, we can prove the following.

Theorem 1.3. *If $\Delta_K(t)$ can be decomposed to a product $f(t)g(t)$ of polynomials with $f(t) = at^2 - bt + a$, $b \geq a > 0$ and $2a - b = \pm 1$, then there exists a non commutative representation $G(K) \to SL(2,\mathbb{Z}/p)$ for infinitely many prime numbers $p \in \mathbb{Z}_+$.*

Remark 1.4. If there exists an epimorphism $G(K) \to G(K')$, then $\Delta_K(t)$ has the form as above.

2. Theorem of de Rham

We recall a formulation of the Alexander polynomial by de Rham from the point of deformations of linear representations.

We fix a Wirtinger presentation of K as

$$G(K) = \langle\ x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1}\ \rangle.$$

Under this presentation, we can assume $\alpha(x_1) = \cdots = \alpha(x_n) = t$.

Any homomorphism $\varphi_0 : G(K) \to \mathbb{C}^* = \mathbb{C} - \{0\}$ can be decomposed to $\varphi_0 = \overline{\varphi}_0 \circ \alpha$ where $\overline{\varphi}_0 : \langle t\rangle \cong \mathbb{Z} \to \mathbb{C}^*$ because $\mathbb{C}^*$ is an abelian group.

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Now we take a map $\varphi : \{x_1, \ldots, x_n\} \to GL(2;\mathbb{C})$ as

$$\varphi(x_1) = \begin{pmatrix} a & b_1 \\ 0 & 1 \end{pmatrix}, \cdots, \varphi(x_n) = \begin{pmatrix} a & b_n \\ 0 & 1 \end{pmatrix}$$

where $a = \varphi_0(x_1) = \cdots = \varphi_0(x_n) \in \mathbb{C}^*$ and $b_1, \ldots, b_n \in \mathbb{C}$.

We consider the problem when φ can be extended to the whole $G(K)$ as a homomorphism. We put

$$\mathbf{b} = {}^t(b_1, \ldots, b_n) \in \mathbb{C}^n.$$

Remark 2.1. If $b_1 = \cdots = b_n = b \in \mathbb{C}$, that is, $\mathbf{b} = {}^t(b, b, \ldots, b)$, then it can be done as an abelian representation, because $\varphi(x_1) = \cdots = \varphi(x_n)$.

From now we assume that $\mathbf{b} \neq {}^t(b, b, \ldots, b)$. Here we define a map $\psi : \{x_1, \ldots, x_n\} \to \mathbb{C}$ by $\psi(x_i) = b_i$.

Definition 2.2. A map $\chi : G(K) \to \mathbb{C}$ is called to be a crossed homomorphism with respect to φ_0 if it satisfies $\chi(xy) = \chi(x) + \varphi_0(x)\chi(y)$ for any $x, y \in G(K)$.

Lemma 2.3. *The above map φ can be extended to $G(K)$ as a homomorpshim if and only if ψ can be extended to $G(K)$ as a crossed homomorphism.*

Hence we consider when ψ can be done to $G(K)$ as a crossed homomorphism.

Let F_n denote the free group of rank n generated by $x_1, \ldots, x_n$. We fix a natural surjection $F_n \to G(K)$.

Definition 2.4. The $\mathbb{Z}F_n$-module DF_n is defined as follows.

- generators:dg $(g \in F_n)$,
- relators:$d(gg') = dg + gdg'$ $(g, g' \in F_n)$.

Remark 2.5. DF_n is the free $\mathbb{Z}F_n$-module generated by $dx_1, \ldots, dx_n$.

The $\mathbb{Z}G(K)$-module $DG(K)$ can be defined similarly by adding relations $dr_1 = \cdots = dr_{n-1} = 0$.

Here we consider a map

$$d : F_n \ni g \mapsto dg \in DF_n.$$

This is a crossed homomorphism with respect to the natural action of F_n on DF_n because $d(gg') = dg + gdg'$ in DF_n. The following equality is well known in the theory of Fox's free derivatives;

$$dg = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i.$$

By the natural map $F_n \to G(K)$, we consider φ and ψ as maps on F_n. We use the same symbol for them.

Now we define $\mathbb{Z}F_n$-homomorphism $\bar{\psi} : DF_n \to \mathbb{C}$ by

$$\begin{aligned} \bar{\psi}\left(\sum_{i=1}^{n} \lambda_i g_i dx_i\right) &= \sum_{i=1}^{n} \lambda_i \varphi_0(g_i)\psi(x_i) \\ &= \sum_{i=1}^{n} \lambda_i \varphi_0(g_i) b_i \end{aligned}$$

for any $\sum_{i=1}^{n} \lambda_i g_i dx_i \in DF_n$ where $\lambda_i \in \mathbb{Z}, g_i \in F_n$.

Lemma 2.6. *The composite map*

$$\psi = \bar{\psi} \circ d : F_n \to \mathbb{C}$$

is a crossed homomorphism with respect to φ_0.

Proof. For any $g, g' \in F_n$, we have

$$\begin{aligned}
\psi(gg') &= \bar{\psi}(d(gg')) \\
&= \bar{\psi}(dg + gdg') \\
&= \bar{\psi}(dg) + \bar{\psi}(gdg') \\
&= \psi(g) + \varphi_0(g)\psi(g').
\end{aligned}$$

□

Lemma 2.7. *The map $\bar{\psi}$ gives a $\mathbb{Z}G(K)$-homomorphism on $DG(K)$ if and only if $\bar{\psi}(dr_i) = 0$ for any relator r_i of $G(K)$. This is also equivalent to*

$$\bar{\psi}\left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j} dx_j\right) = \sum_{j=1}^{n} \varphi_0\left(\frac{\partial r_i}{\partial x_j}\right)\psi(x_j) = 0.$$

Remark 2.8. Here we use the same symbol φ_0 to the extended map $\mathbb{Z}G(K) \to \mathbb{Z}\mathbb{C}^* = \mathbb{C}$ on the integral group ring $\mathbb{Z}G(K)$.

Proof. Recall that any relator of $DG(K)$ can be given from relators of $G(K)$, and

$$dr_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j} dx_j$$

in DF_n. By using them, it is easily seen. □

Now we define an $(n-1) \times n$-matrix $A_{\varphi_0} \in M(n-1, n; \mathbb{C})$ as follows:

$$A_{\varphi_0} = \left(\widetilde{\varphi}_0\left(\frac{\partial r_i}{\partial x_j}\right)\right).$$

It is clear that A_{φ_0} can be obtained from the Alexander matrix of $G(K)$ by substituting $t = a$. From the above argument, the condition for ψ to be a crossed homomorphism as follows.

Lemma 2.9. *ψ is a crossed homomorphism if and only if $A_{\varphi_0}\mathbf{b} = \mathbf{0}$.*

de Rham proved the following [3]. From this theorem, we see that there exists a representation when $\Delta_K(a) = 0$ and can say the Alexander polynomial is an obstruction for the existence of representations. See also [1, 6].

Theorem 2.10 (de Rham). *The map*

$$\varphi : \{x_1, \ldots, x_n\} \ni x_i \mapsto \begin{pmatrix} a & \psi(x_i) \\ 0 & 1 \end{pmatrix} \in GL(2; \mathbb{C})$$

can be extend to $G(K)$ as a homomorphism if and only if $A_{\varphi_0}\mathbf{b} = 0$. In particular then it holds $a = \varphi_0(x_i) = \overline{\varphi}_0(t)$ is a zero of $\Delta_K(t) = 0$.

Proof. First we note that a map φ can be extend to the $G(K)$ as a homomorphism if and only if the image of any relator is the identity matrix E.

For example, we take a relator $r_i = x_i x_j x_i^{-1} x_k^{-1}$. Then the condition is

$$\begin{aligned}\varphi(r_i) &= \varphi(x_i)\varphi(x_j)\varphi(x_i)^{-1}\varphi(x_k)^{-1} \\ &= E.\end{aligned}$$

This is equivalent to $\varphi(x_i)\varphi(x_j) = \varphi(x_k)\varphi(x_i)$. Then we compute the both sides.

$$\varphi(x_i)\varphi(x_j) = \begin{pmatrix} a & b_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b_j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^2 & ab_j + b_i \\ 0 & 1 \end{pmatrix}$$

$$\varphi(x_k)\varphi(x_i) = \begin{pmatrix} a & b_k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^2 & ab_i + b_k \\ 0 & 1 \end{pmatrix}.$$

By comparing entries of the both, we have

$$(1-a)b_i + ab_j - b_k = 0.$$

By Fox's free differential calculus

$$\alpha_*\left(\frac{\partial}{\partial x_i}(x_i x_j - x_k x_i)\right) = 1 - t,$$

$$\alpha_*\left(\frac{\partial}{\partial x_j}(x_i x_j - x_k x_i)\right) = t,$$

$$\alpha_*\left(\frac{\partial}{\partial x_k}(x_i x_j - x_k x_i)\right) = -1.$$

we see the above condition $(1-a)b_i + ab_j - b_k = 0$ is the same with the i-th entry of $A|_{t=a}\mathbf{b}$ equals zero. Therefore the condition to be extended is given by the following linear system

$$A|_{t=a}\mathbf{b} = \mathbf{0}.$$

Hence it is seen that $t = a$ is a zero of $\Delta_K(t) = 0$ and then

$$\varphi : \{x_1, \ldots, x_n\} \ni\mapsto \begin{pmatrix} a & b_i \\ 0 & 1 \end{pmatrix}$$

can be extended to $G(K)$ as a homomorphism. □

Note that the condition for the extension is given by linear equations. Then if φ can be done to $G(K)$ as a homomorphism

$$\varphi_s(x_i) = \begin{pmatrix} a & sb_i \\ 0 & 1 \end{pmatrix}$$

can also be done to $G(K)$ for any $s \in \mathbb{C}^*$. Then φ_s is a deformation of the direct sum of φ_0 and the 1-dimensional trivial representation in $GL(2;\mathbb{C})$.

Now we consider deformations in $SL(2;\mathbb{C})$.

The map

$$\varphi : \{x_1, \ldots, x_n\} \to SL(2;\mathbb{C})$$

is given by $\varphi(x_i) = \begin{pmatrix} a & b_i \\ 0 & a^{-1} \end{pmatrix}$ and

$$\varphi_0 : \{x_1, \ldots, x_n\} \to \mathbb{C}^*$$

by $\varphi_0(x_1) = \cdots = \varphi_0(x_n) = a$.

Starting from this map φ, the condition for φ to be a homorphism on $G(K)$ can be obtaiend as follows.

$$\varphi(x_i)\varphi(x_j) = \begin{pmatrix} a & b_i \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a & b_j \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a^2 & ab_j + a^{-1}b_i \\ 0 & a^{-1} \end{pmatrix},$$

$$\varphi(x_k)\varphi(x_i) = \begin{pmatrix} a & b_k \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a & b_i \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a^2 & ab_i + a^{-1}b_k \\ 0 & a^{-1} \end{pmatrix}.$$

By comparing of entries,

$$(1 - a^2)b_i + a^2 b_j - b_k = 0$$

is obtained as a condition. By similar arguments, we obtain the following condition

$$A|_{t=a^2}\mathbf{b} = \mathbf{0}.$$

In particular we have

$$\Delta_K(a^2) = 0,$$

that is, $t = a^2$ is a zero of $\Delta_K(t) = 0$. Then

$$\varphi_0 \oplus \varphi_0^{-1} : G(K) \ni x \mapsto \begin{pmatrix} \varphi_0(x) & 0 \\ 0 & \varphi_0(x)^{-1} \end{pmatrix} \in SL(2; \mathbb{C})$$

can be deformed in $SL(2; \mathbb{C})$ to $\varphi : G(K) \to SL(2; \mathbb{C})$.

3. Construction of a homomorphism of $G(K)$ into symmetric groups

We can construct a deformation of an abelian represenation in $GL(2, \mathbb{C})$, or $SL(2, \mathbb{C})$. From the above observation, we can get also a homomorphism of $G(K)$ into symmetric groups. This argument was given in [4].

We recall $\Delta_K(t)$ is well defined up to $\pm t^k$. Namely it depends on a Wirtinger presentation of $G(K)$. When we change a presentation, the new one equals to the $\pm t^k$ times old one. It means special value of $\Delta_K(t)$ is not well-defined as a knot invariant in general. However if we substitute an absolute value one complex number $\xi = e^{\sqrt{-1}\theta}$ to t, then its absolute value $|\Delta_K(\xi)|$ gives a knot invariant.

Remark 3.1. The integer $d_K = |\Delta_K(-1)| \in \mathbb{Z}$ is called the determinant of K.

Because the Alexander matrix A of a Wirtinger presentation of $G(K)$ is a matrix over $\mathbb{Z}[t, t^{-1}]$, then by substituting $t = -1$, we have a matrix over the integers

$$A|_{t=-1} \in M((n-1) \times n; \mathbb{Z}).$$

Then a linear equation system for the extension is defined over $\mathbb{Z}$. When we consider

$$A|_{t=-1}\mathbf{b} = \mathbf{0}$$

over $\mathbb{Z}/d_K$, any $(n-1) \times (n-1)$-minor $A|_{t=-1}$ is zero mod d_K. Hence there exits the solution

$$\mathbf{b} = {}^t(b_1, \ldots, b_n) \in (\mathbb{Z}/d_K)^n.$$

At that time a representation

$$\bar{\varphi} : G(K) \to GL(2; \mathbb{Z}/d_K)$$

over $\mathbb{Z}/d_K$ can be given by

$$\bar{\varphi}(x_i) = \begin{pmatrix} -1 & b_i \\ 0 & 1 \end{pmatrix}.$$

Here an affine transformation

$$\bar{\varphi}(x_i) = \begin{pmatrix} -1 & b_i \\ 0 & 1 \end{pmatrix}$$

can be consider a permutation

$$\mathbb{Z}/d_K \ni m \mapsto -m + b_i \in \mathbb{Z}/d_K$$

on $\mathbb{Z}/d_K$. Therefore we obtain a homomorphism into a symmetric group,

$$G(K) \to \mathfrak{S}_{d_K}.$$

Next we consider to substitute any positive integer to t. If we put $t = m \in \mathbb{Z}$ and consider the linear sytem mod $d_{K,m} = |\Delta_K(m)|$, then

$$A|_{t=m}\mathbf{b} \equiv 0 \bmod d_{K,m}$$

has a solution over $\mathbb{Z}/d_{K,m}$. Of course $d_{K,m}$ depends on the choice of a Wirtinger presentation. However we can obtain a representation defined by using a fixed presentation.

Then we obtain a representation

$$\bar{\varphi} : G(K) \to GL(2; \mathbb{Z}/d_{K,m}).$$

For any generator, its image is given by

$$G(K) \ni x_i \mapsto \begin{pmatrix} m & b_i \\ 0 & 1 \end{pmatrix}$$

and it gives

$$\mathbb{Z}/d_{K,m} \ni k \mapsto mk + b_i \in \mathbb{Z}/d_{K,m}.$$

Therefore we obtain a homomorphism of $G(K)$ into the symmetric group of degree $d_{K,m}$

$$G(K) \to \mathfrak{S}_{d_{K,m}}.$$

Problem 3.2. *What kind of property does the above homomorphism $G(K) \to \mathfrak{S}_{d_{K,m}}$ have ?*

4. $SL(2, \mathbb{Z}/d)$-REPRESENATION OF $G(K)$

In this section, we give a proof of Theorem 1.2.

We assume that the Alexander polynomial of K is given by

$$\Delta_K(t) = a_{2k}t^{2k} + a_{2k-1}t^{2k-1} + \cdots + a_1 t + a_0,$$

where $a_{2k} = a_0 > 0, \sum_{i=0}^{2k} a_i = \pm 1$, and it can be defined by the Wirtinger presentation

$$\langle x_1, \ldots, x_n \mid r_1. \ldots, r_n \rangle.$$

If we substitute $t = p^2$ for $\Delta_K(t)$, then

$$d_{p^2} = \Delta_K(p^2) = a_{2k}p^{4k} + a_{2k-1}p^{4k-2} + \cdots + a_1 p^2 + a_0.$$

If p is a sufficient large prime number,

$$d_{p^2} = \Delta_K(p^2) > p^2 > p.$$

Further we put the condition $(a_0, p) = 1$, then

$$(d_{p^2}, p) = 1.$$

Then for any prime number p as above, p is a unit in $\mathbb{Z}/d_{p^2}$.

Since there exists a solution

$$A|_{t=p^2}\mathbf{b} = 0 \bmod \mathbb{Z}/d_{p^2},$$

then an abelian representation

$$\rho : G(K) \ni x_i \mapsto \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \in SL(2, \mathbb{Z}/d_{p^2})$$

can be deformed to a non commutative representation

$$\tilde{\rho} : G(K) \ni x_i \mapsto \begin{pmatrix} p & b_i \\ 0 & p^{-1} \end{pmatrix} \in SL(2, \mathbb{Z}/d_{p^2})).$$

Therefore we obtain the following.

Theorem 4.1. *There exits a non commutative representation $G(K) \to SL(2, \mathbb{Z}/d_{p^2})$ for infinitely many $d_{p^2} = |\Delta_K(p^2)|$.*

Remark 4.2. It is not easy to see which d_{p^2} is a prime number or not.

5. $GL(2, \mathbb{Z}/p)$-REPRESENTATION OF $G(K)$

If d_p is not a prime number, then we cannot consider the twisted Alexander polynomial [7] for the representation as above. Then we want to consider the following problem.

Problem 5.1. *Does there exit a non commutative representation $G(K) \to SL(2, \mathbb{Z}/p)$ for infinitely many prime number p ?*

In this section we prove the existence of $GL(2, \mathbb{Z}/p)$-representations by using the Alexander polynomial.

For any knot with the Alexander polynomial of degree 2, we can prove the problem for $GL(2, \mathbb{Z}/p)$-representations. We assume that the Alexander polynomial of K is given by

$$\Delta_K(t) = at^2 - bt + a,$$

where $b \geq a > 0, \Delta_K(1) = 2a - b = \pm 1$. Then by the condition $2a - b = \pm 1$, $a = \dfrac{b \pm 1}{2}$.

Theorem 5.2. *There exits a non commutative representation $G(K) \to GL(2, \mathbb{Z}/p)$ for infinitely many prime number p.*

If we can prove the following proposition, for such a prime number p and $t = n$, an abelian representation of $G(K)$ over $\mathbb{Z}/p$

$$\rho : G(K) \ni x_i \mapsto \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}/p)$$

can be deformed to a non commutative representation

$$\tilde{\rho} : G(K) \ni x_i \mapsto \begin{pmatrix} n & b_i \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}/p),$$

and we get the theorem.

Proposition 5.3. *There exists a solution of $\Delta_K(t) \equiv 0$ mod p for infinitely many prime numbers p.*

Let us consider the congruence

$$at^2 - bt + a \equiv 0 \bmod p.$$

When we consider the equation

$$at^2 - bt + a = 0$$

over $\mathbb{C}$, then

$$t = \frac{b \pm \sqrt{b^2 - 4a^2}}{2a}$$

is the solutions. Here if $D = b^2 - 4a$ is a square number mod p, that is, a quadratic residue mod p, then there exists a solution of the above congruence.

Definition 5.4. For an integer k and a prime number p, the Legendre symbol $\left(\frac{k}{p}\right)$ is defined as follows.

$$\left(\frac{k}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv k \bmod p \text{ has a solution} \\ -1 & \text{if } x^2 \equiv k \bmod p \text{ has no solution} \end{cases}$$

By using $2a - b = \pm 1$, we can eliminate a in $D = b^2 - 4a^2$ and obtain $D = \pm 2b - 1$. Then we put $D_+ = 2b - 1$ and $D_- = -2b - 1$ for the both. By using Legendre symbol, we prove the following.

Proposition 5.5. *For infinitely many prime numbers p, each of Legendre symbols of $D_\pm$ is*

$$\left(\frac{D_\pm}{p}\right) = 1.$$

We treat separately D_+ and D_-.

1. The case of $D_+ = 2b - 1$.

Here we assume that

$$p = 4(2b - 1)n + 1$$

is a prime number and not a divisor of a.

Remark 5.6. By the theorem of Dirichlet, there exisit infinitely many prime number as above.

If p is a divisor of $2b - 1$, then $D_+ \equiv 0 \bmod p$. Hence there exists a solution of $\Delta_K(t) \equiv 0 \bmod p$.

Assume that p is not a divisor of $2b - 1$. By the reciprocity law of the Jacobi symbol,

$$\begin{aligned}\left(\frac{2b-1}{p}\right)\left(\frac{p}{2b-1}\right) &= (-1)^{\frac{p-1}{2}\frac{2b-1-1}{2}} \\ &= (-1)^{2(2b-1)n(b-1)} \\ &= 1.\end{aligned}$$

Therefore we have

$$\begin{aligned}\left(\frac{2b-1}{p}\right) &= \left(\frac{p}{2b-1}\right)\\ &= \left(\frac{4(2b-1)n+1}{2b-1}\right)\\ &= \left(\frac{1}{2b-1}\right)\\ &= 1.\end{aligned}$$

2. The case of $D_- = -2b-1$

Now assume that

$$p = 4(2b+1)n+1$$

is a prime number and not a divisor of a.

Now

$$\left(\frac{-2b-1}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2b+1}{p}\right).$$

By the quadratic reciprocity law,

$$\begin{aligned}\left(\frac{-1}{p}\right) &= (-1)^{\frac{p-1}{2}}\\ &= (-1)^{2(2b+1)n}\\ &= 1.\end{aligned}$$

Hence

$$\left(\frac{-2b-1}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2b+1}{p}\right) = \left(\frac{2b+1}{p}\right).$$

By using the reciprocity law of the Jacobi symbol,

$$\begin{aligned}\left(\frac{2b+1}{p}\right)\left(\frac{p}{2b+1}\right) &= (-1)^{\frac{p-1}{2}\frac{2b+1-1}{2}}\\ &= (-1)^{2(2b+1)nb}\\ &= 1.\end{aligned}$$

Therefore we have

$$\begin{aligned}\left(\frac{2b+1}{p}\right) &= \left(\frac{p}{2b+1}\right)\\ &= \left(\frac{4(2b+1)n+1}{2b+1}\right)\\ &= \left(\frac{1}{2b+1}\right)\\ &= 1.\end{aligned}$$

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