

ON THE SMOOTHING PROPERTIES OF DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper gives an overview of a new approach to global smoothing problems for dispersive and non-dispersive evolution equations based on the global canonical transforms and the underlying global microlocal analysis. The paper discusses the equivalence of known smoothing estimates for different equations, gives new estimates for equations with homogeneous and non-homogeneous symbols, and gives an overview of global L^2 -boundedness properties of Fourier integral operators.

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1. INTRODUCTION

The analysis of nonlinear evolution partial differential equations usually relies on the global analysis for linearised equations. For this purpose, one tries to construct and analyse global solutions for corresponding linear equations. In general, partial differential equations of different types lead to representation of solutions in different forms. For example, elliptic partial differential equations lead to parametrices in the form of pseudo-differential operators. On the other hand, propagators for hyperbolic equations can be constructed in the form of Fourier integral operators. Other equations, like Schrödinger equation or linearised Korteweg-de Vries equation lead to more general oscillatory integrals. Solutions to Schrödinger equations can be viewed in the

Date: October 5, 2008.

The first author was partly supported by the EPSRC Leadership Fellowship EP/G007233/1.

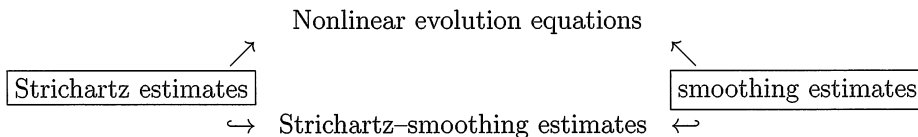
form of Legendrian oscillatory integrals while more general evolution partial differential equations give rise to oscillatory integrals of more general types. Elements of the required analysis usually include methods of representation of solutions, calculus of solution operators and of propagators, global weighted L^2 and other estimates, spectral properties, functional analytic properties, etc.

In this paper we briefly overview several approaches to linearisations of nonlinear evolution equations as well as approaches to smoothing estimates. We give several examples of equations and corresponding problems. The main issue discussed here are the smoothing estimates for linear evolution equations. We are mainly interested in evolution equations of dispersive and non-dispersive types. For this purpose, we will discuss their normal forms and canonical transforms which can be used for the reduction of general equations to these normal forms, and introduce comparison principles which can be used to obtain further information about equations in their normal form.

We will also give an overview of related problems such as necessary estimates for Fourier integral and pseudo-differential operators, estimates in weighted L^2 -spaces for pseudo-differential and Fourier integral operators under minimal conditions, global calculus of these operators and other aspects and applications.

2. EVOLUTION EQUATIONS

Main approaches to nonlinear evolution equations can be summarised as



In both of these approaches one looks for global estimates for linearised equations. Strichartz estimates here are essentially $L^p L^q$ space-time estimates while smoothing estimates are essentially Sobolev L^2 space-time estimates. Since 1980's different spectral methods were developed to establish smoothing estimates for a variety of equations, starting perhaps from Kato [Ka2], and since 1990's methods of harmonic analysis were also used. The purpose of this paper is to describe also methods of the microlocal analysis, namely those of “canonical transforms” and “comparison principles”.

For real-valued function $a(\xi) \in \mathbb{R}$ and $D_x = -i\partial_x$, consider evolution equation

$$\begin{cases} (i\partial_t + a(D_x)) u(t, x) = 0, \\ u(0, x) = \varphi(x). \end{cases}$$

The basic question for smoothing estimates is when its solution $u(t, x) = e^{ita(D_x)}\varphi(x)$ has the space-time estimate

$$(2.1) \quad \|Au(t, x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)},$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and for what operators $A = A(X, D_x)$. We note that this gives also the estimate for the equation

$$\begin{cases} (\partial_t^2 + a(D_x)^2) u(t, x) = 0, \\ u(0, x) = \varphi_0(x), \quad \partial_t u(0, x) = \varphi_1(x), \end{cases}$$

since its solution $u(t, x)$ can be expressed as a linear combination of terms

$$e^{\pm ita(D_x)}\varphi_0, \quad \frac{e^{\pm ita(D_x)}}{a(D_x)}\varphi_1.$$

The space-time estimate (2.1) is one of the “fundamental estimates” for (nonlinear) equations that arise in various problems in different sciences, e.g.

- $a(\xi) = |\xi|^2 \implies$ Schrödinger equation $i\partial_t u - \Delta_x u = 0$;
- $a(\xi) = |\xi| \implies$ Wave equation $\partial_t^2 u - \Delta_x u = 0$;
- $a(\xi) = \sqrt{|\xi|^2 + 1} \implies$ Relativistic Schrödinger equation

$$i\partial_t u + \sqrt{1 - \Delta_x} u = 0;$$

- $a(\xi) = \sqrt{|\xi|^2 + 1} \implies$ Klein–Gordon equation $\partial_t^2 u - \Delta_x u + u = 0$;
- $a(\xi) = \xi^3$ ($n = 1$) $\implies$ Korteweg–de Vries (shallow water waves)

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0;$$

- $a(\xi) = |\xi|\xi$ ($n = 1$) $\implies$ Benjamin–Ono equation (deep water waves)

$$\partial_t u - \partial_x |D_x| u + u\partial_x u = 0;$$

- $a(\xi) = \xi_1^2 - \xi_2^2$ ($n = 2$) $\implies$ Davey–Stewartson system

$$\begin{cases} i\partial_t u - \partial_x^2 u + \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x v, \\ \partial_x^2 v - \partial_y^2 v = \partial_x |u|^2; \end{cases}$$

(shallow water wave in 2D)

Some other higher order/dimensional cases are also of interest, e.g.

- $a(\xi) =$ polynomial of order 3 ($n = 2$), e.g. Shrira equation where $a(\xi)$ has normal form

$$\xi_1^3 + \xi_2^3, \quad \xi_1^3 + 3\xi_2^2, \quad \xi_1^2 + \xi_1\xi_2^2$$

(gravitational waves in 2D);

- $a(\xi) =$ quadratic form ($n \geq 3$) $\implies$ Zakharov–Schulman equation describing the interaction of small amplitude high frequency wave and acoustic wave.

The main idea of our approach is that instead of looking at particular equations, we would like to look at relations between different equations. Thus, our plan can be summarised as

$$\boxed{\text{Estimate for Eq1}} + \boxed{\text{Relation Eq1} \Leftrightarrow \text{Eq2}} \Rightarrow \boxed{\text{Estimate for Eq2}}$$

where Eq2 is a general equation for which we want to have an estimate. For this, to have an estimate for Eq2, we need to find equation Eq1 which is a model equation in a normal form, and a corresponding estimate for it. The relation between equations Eq1 and Eq2 is then realised as a canonical transform which we can view as a Fourier integral operator on $\mathbb{R}^n$.

Thus, among other things, we need

- to find normal forms and operators to reduce equations to normal forms;
- to find operators transforming equations into each other.

Both things are done by using Fourier integral operators in $\mathbb{R}^n$ for which we need to develop global calculus and global weighted estimates in $\mathbb{R}^n$. However, Fourier integral operators of the form

$$Tu(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) \widehat{u}(\xi) d\xi$$

work best if the phase $\phi(x,\xi)$ is essentially of order one in ξ (e.g. homogeneous, or SG-order one, etc.) Thus, we can reduce equation of order m to a normal form of the same order m , and the subsequent question becomes of how to relate equations of different orders. For this, we introduce comparison principles for normal forms, so that we have

$$\begin{array}{c} \boxed{\text{Estimate for Eq1mod}} + \boxed{\text{Relation Eq1mod} \Leftrightarrow \text{Eq1}} \Leftrightarrow \boxed{\text{Estimate for Eq1}} \\ \Downarrow \text{ (comparison principles) } \\ \boxed{\text{Estimate for Eq2mod}} + \boxed{\text{Relation Eq2mod} \Leftrightarrow \text{Eq2}} \Leftrightarrow \boxed{\text{Estimate for Eq2}} \end{array}$$

Comparison principles allow to relate model equations Eq1mod and Eq2mod, thus relating equations Eq1 and Eq2 of different orders. For example, we can relate estimates for wave, Schrödinger, and KdV equations, in particular showing that they are in fact equivalent.

3. SMOOTHING ESTIMATES

As one model, let us first consider the following Schrödinger equation:

$$(3.1) \quad \begin{cases} (i\partial_t + \Delta_x) u(t, x) = 0, \\ u(0, x) = \varphi(x) \in L^2(\mathbb{R}^n). \end{cases}$$

By Plancherel's theorem, we know that the solution $u(t, x) = e^{it\Delta}\varphi(x)$ preserves the L^2 -norm of initial data φ , that is, for any fixed time $t \in \mathbb{R}$, we have

$$\|u(t, \cdot)\|_{L^2(\mathbb{R}_x^n)} = \|\varphi\|_{L^2(\mathbb{R}^n)}.$$

However, integration in both (t, x) gives smoothing by $|D_x|^{1/2}$ in x , i.e. we have

$$\|\chi(t, x)|D_x|^{1/2}u\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)},$$

for all $\chi \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$. Thus, integration in t gives an extra gain of regularity of order $1/2$ in x . One of our subsequent objectives is to relate this smoothing effect for Schrödinger equation to that of other evolution equations. It turns out that in fact it implies smoothing estimates for other a-priori completely unrelated equations. For example, we will show that it implies the gain of one derivative for solutions to the KdV equation.

For the Schrödinger equation (3.1), for dimensions $n \geq 2$, we have smoothing estimates

$$\|Au\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|g\|_{L^2(\mathbb{R}_x^n)},$$

where A is one of the following operators:

- [1] $A = \langle x \rangle^{-s}|D_x|^{1/2} \quad (s > 1/2)$
- [2] $A = |x|^{\alpha-1}|D_x|^\alpha \quad (1 - n/2 < \alpha < 1/2)$
- [3] $A = \langle x \rangle^{-1}\langle D_x \rangle^{1/2} \quad (n > 2)$

The type [1] was given by Ben-Artzi and Klainerman [BK] for $n \geq 3$, and by Chihara [Ch] for $n \geq 2$. The type [2] was given by Kato and Yajima [KY] for $n \geq 3$ and $0 \leq \alpha < 1/2$, and by Sugimoto [Su1, Su2] for $n \geq 2$ and $1 - n/2 < \alpha < 1/2$. It is known that it is not true for $\alpha = 1/2$ (see e.g. Watanabe [W]). The type [3] was given by Kato and Yajima [KY]. Walther [Wa1, Wa2] also gave another approach to this estimate using spherical harmonics.

Each proof was carried out by proving one of the following estimates (or their variants):

$$\|\widehat{A^*f}|_{\rho S^{n-1}}\|_{L^2(\rho S^{n-1})} \leq C\sqrt{\rho}\|f\|_{L^2(\mathbb{R}^n)} \quad (\text{Restriction theorem})$$

$$\sup_{\operatorname{Im} z > 0} |(R(z)A^*f, A^*f)| \leq C\|f\|_{L^2(\mathbb{R}^n)}^2 \quad (\text{Resolvent estimate})$$

where $R(z) = (-\Delta - z)^{-1}$. In fact, we have equality

$$\operatorname{Im} (R(\rho^2 + i0)f, f) = \frac{1}{4(2\pi)^{n-1}\rho} \|\hat{f}|_{\rho S^{n-1}}\|_{L^2(\rho S^{n-1})}^2.$$

By duality, both of these estimates easily imply the smoothing estimate

$$\|Au\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|g\|_{L^2(\mathbb{R}_x^n)}.$$

In comparison to these methods, our approach will be instead based on the geometric analysis, relying on the development of the global analysis of Fourier integral operators in $\mathbb{R}^n$, and weighted estimates for these operators in Sobolev spaces.

We note that the smoothing effect of evolution partial differential equations has been studied for more than 20 years. Kato [Ka2] showed a local gain of one derivative for the KdV equation. A gain of half derivative for Schrödinger equation and related problems were analysed by Ben-Artzi and Devinatz [BD1, BD2], Constantin and Saut [CS], Sjölin [Sj], Vega [V], Kato and Yajima [KY], Ben-Artzi and Klainerman [KY], etc. The smoothing effect for different dispersive equations was studied by Kenig, Ponce and Vega [KPV1]–[KPV6], Walther [Wa1, Wa2], Ben-Artzi and Devinatz [BD1, BD2], Ben-Artzi and Nemirovsky [BN], Linares and Ponce [LP], Watanabe [W], Hoshiro [Ho1, Ho2], Chihara [Ch], Sugimoto [Su1, Su2], Ben-Artzi, Koch and Saut [BKS], etc. Smoothing properties of equations on manifolds were also analysed, see e.g. Doi [Do1], Burq, Gerard and Tzvetkov [BGT], etc.

For simplicity, let us first consider general equations with homogeneous symbols:

$$(3.2) \quad \begin{cases} (i\partial_t + a(D_x)) u(t, x) = 0, \\ u(0, x) = \varphi(x) \in L^2(\mathbb{R}^n), \end{cases}$$

where $a(\xi)$ consists of only principal part $a_m(\xi)$ and is dispersive in the following sense

$$(H) \quad a(\xi) = a_m(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \neq 0),$$

where

- $a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ real-valued;
- $a_m(\lambda\xi) = \lambda^m a_m(\xi)$ ($\lambda > 0, \xi \neq 0$).

The dispersiveness assumption (H) means that the classical orbit, that is, the solution curve to

$$\begin{cases} \dot{x}(t) = (\nabla a)(\xi(t)), & \dot{\xi}(t) = 0 \\ x(0) = 0, & \xi(0) = \xi_0, \end{cases}$$

escapes to infinity as $t \rightarrow \infty$. We will also consider more general dispersive equations. Let $a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$: $a_m(\lambda\xi) = \lambda^m a_m(\xi)$, $\lambda > 0, \xi \neq 0$, be as before. We will consider operators $a(D)$ having a_m as the homogeneous principal part. Thus, we can introduce another assumption:

$$(L) \quad a(\xi) \in C^\infty(\mathbb{R}^n), \quad \nabla a(\xi) \neq 0 \text{ for all } \xi, \text{ and } a - a_m \in S^{m-1}.$$

For example, function $a(\xi) = \xi_1^3 + \dots + \xi_n^3 + \xi_1$ satisfies (L) with $m = 3$.

There are ways to generalise these conditions further, including classes of non-dispersive equations. In fact, the dispersiveness condition sometimes breaks down (e.g. often for systems), but our methods give some (quite sharp) smoothing estimates in these cases as well, see [RS6] for details.

4. IDEA OF THE APPROACH

The main idea of our approach is that if we have a relation $a(\xi) = (\sigma \circ \psi)(\xi)$ for some $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to be specified later, then the Fourier integral operator T with phase function $x \cdot \xi - y \cdot \psi(\xi)$ gives

$$a(D) = T \circ \sigma(D) \circ T^{-1}.$$

Consequently, suppose that we have estimate

$$\|\langle x \rangle^{-\kappa} \rho(D_x) e^{it\sigma(D_x)} g(x)\|_{L_{t,x}^2} \leq C \|g\|_{L_x^2}.$$

Then we can rewrite it as

$$\|\langle x \rangle^{-\kappa} T^{-1}(T \rho(D_x) T^{-1})(T e^{it\sigma(D_x)} T^{-1}) T g(x)\|_{L_{t,x}^2} \leq \|g\|_{L_x^2},$$

which implies

$$\|\langle x \rangle^{-\kappa} T^{-1}(\rho \circ \psi)(D_x) e^{ita(D_x)} \tilde{g}(x)\|_{L_{t,x}^2} \leq C \|T^{-1} \tilde{g}\|_{L_x^2},$$

with $\tilde{g} = Tg$. If we combine this with weighted estimates for T and T^{-1} in Sobolev spaces with weight $\langle x \rangle^{-\kappa}$, we obtain the corresponding smoothing estimate for $e^{ita(D_x)}$, namely that

$$\|\langle x \rangle^{-\kappa} (\rho \circ \psi)(D_x) e^{ita(D_x)} \tilde{g}(x)\|_{L_{t,x}^2} \leq C \|\tilde{g}\|_{L_x^2}.$$

Thus, canonical transforms allow us to reduce general estimates to estimates in the following normal forms (of $a(D_x)$). Such transformations may be microlocal in ξ , but they are global in x . Without going much into detail, we have:

- elliptic : $a(\xi) \neq 0$. In this case we can transform $a(D)$ to $|D_1|^m$;
- principal type : $\nabla a(\xi) \neq 0$. Here we can transform $a(D)$ to $|D_1|^{m-1} D_2$;
- non – principal type : $\nabla a(\xi_0) = 0$. In this case we can transform $a(D)$ to some normal forms determined by the Hessian $D_\xi^2 a(\xi_0)$.

We note that in general these normal forms are different from those of Duistermaat and Hörmander [DH], and that in general, $a(\xi)$ does not have to be homogeneous (as under assumption (L)). The non-homogeneous behaviour is absorbed in the phase function - this is a difference with usual Fourier integral operators.

In order to pursue this, we first have to establish estimates for model cases and to relate them to each other. Consequently, we need to develop a global theory of Fourier integral operators including operators with phases coming from these canonical transformations.

The argument above shows that if we have weighted estimates for Fourier integral operators, we can freely insert and take them out of smoothing estimates. Moreover, one can use Fourier integral operators to reduce general smoothing estimates to smoothing estimates for operators in normal forms in one and two dimensions.

The next question is whether we can further relate estimates for equations in normal. If we could, we would obtain comprehensive relations between smoothing

estimates for very different equations. The answer is that indeed, we can. For this, we will introduce several comparison principles for evolution partial differential equations that can be used for this purpose. Altogether, this allows us to relate smoothing estimates for a-priori unrelated equations and to show that a variety of known smoothing estimates are in fact equivalent to each other.

5. COMPARISON PRINCIPLES

For solutions $u(t, x) = e^{itf(D_x)}\varphi(x)$ and $v(t, x) = e^{itg(D_x)}\varphi(x)$ to two equations

$$\begin{cases} (i\partial_t + f(D_x))u(t, x) = 0, \\ u(0, x) = \varphi(x), \end{cases} \quad \text{and} \quad \begin{cases} (i\partial_t + g(D_x))v(t, x) = 0, \\ v(0, x) = \varphi(x), \end{cases}$$

the comparison principle is the general rule to derive time-space estimate for u

$$(5.1) \quad \left\| \left\| w(x)\sigma(D_x) \overbrace{e^{itf(D_x)}\varphi(x)}^{u(t,x)} \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \right\|_{L^2(\mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)}$$

from time-space estimate for v

$$(5.2) \quad \left\| \left\| w(x)\tau(D_x) \underbrace{e^{itg(D_x)}\varphi(x)}_{v(t,x)} \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \right\|_{L^2(\mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Thus, if we know estimate (5.2) for $v(t, x)$, how can we get estimate (5.1) for $u(t, x)$? Let us start with the one-dimensional situation.

Theorem 5.1. *Let $f, g \in C^1(\mathbb{R})$ be real-valued and strictly monotone. If $\sigma, \tau \in C^0(\mathbb{R})$ satisfy*

$$(5.3) \quad \frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leq A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}},$$

then we have

$$(5.4) \quad \|\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A\|\tau(D_x)e^{itg(D_x)}\varphi(\tilde{x})\|_{L^2(\mathbb{R}_t)}$$

for all $x, \tilde{x} \in \mathbb{R}$. Consequently, for any measurable weight $w(x)$, we have

$$(5.5) \quad \|w(x)\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq A\|w(x)\tau(D_x)e^{itg(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)}.$$

We note that since the value of the constant A in this theorem is the same, one can also keep track of constants. For example, for $n \geq 3$ and $m > 0$, we get that

$$\| |x|^{-1} |D_x|^{m/2-1} e^{it|D_x|^m} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq \sqrt{\frac{2\pi}{m(n-2)}} \|\varphi\|_{L^2(\mathbb{R}_x^n)},$$

where the constant $\sqrt{\frac{2\pi}{m(n-2)}}$ is sharp. Indeed, for $m = 2$ this was calculated by Simon [Si] using Kato's theory, and we can use Theorem 5.1 to deduce it for all $m > 0$ from this.

Moreover, the use of this comparison principle simplifies proofs of many smoothing estimates. For example, the comparison principle immediately yields the equality

$$\sqrt{m} \| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \|_{L^2(\mathbb{R}_t)} = \sqrt{l} \| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(\tilde{x}) \|_{L^2(\mathbb{R}_t)}$$

for all $l, m > 0$, all $x, \tilde{x} \in \mathbb{R}$, and all φ such that $\text{supp } \widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$. Using this with $l = 1$, we get

$$\begin{aligned} \sqrt{m} \| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \|_{L^2(\mathbb{R}_t)} &= \| e^{it|D_x|} \varphi(\tilde{x}) \|_{L^2(\mathbb{R}_t)} \\ &= \| \varphi(\tilde{x} + t) \|_{L^2(\mathbb{R}_t)} = \| \varphi \|_{L^2(\mathbb{R}_x)}. \end{aligned}$$

Hence, multiplying by $\langle x \rangle^{-s}$ with $s > 1/2$ and integrating, we get

$$\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \| \varphi \|_{L^2(\mathbb{R}_x)}$$

which is the smoothing estimate in the 1D model case.

We have the full range of such estimates in 1D, 2D, and radially symmetric model cases, and comprehensive relations among them, for details of which we refer to [RS5]. Here, let us only give the statement in the radially symmetric case:

Theorem 5.2. *Let $f, g \in C^1(\mathbb{R}_+)$ be real-valued and strictly monotone. If $\sigma, \tau \in C^0(\mathbb{R}_+)$ satisfy*

$$\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \leq A \frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}},$$

then we have estimate

$$\| \sigma(|D_x|) e^{itf(|D_x|)} \varphi(x) \|_{L^2(\mathbb{R}_t)} \leq A \| \tau(|D_x|) e^{itg(|D_x|)} \varphi(x) \|_{L^2(\mathbb{R}_t)}$$

for all $x \in \mathbb{R}^n$. Especially, for any measurable weight $w(x)$, we have

$$\| w(x) \sigma(|D_x|) e^{itf(|D_x|)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq A \| w(x) \tau(|D_x|) e^{itg(|D_x|)} \varphi(x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)}.$$

Since our normal forms in Section 4 are in 1D (elliptic case) and in 2D (dispersive case), and some estimates reduce to the radially symmetric case, these comparison principles cover all normal forms and all types that we need, which we will show later. There are also extensions of these principles to equations with time-dependent coefficients, for which we refer to [RS6].

We note also that the comparison principle works in both ways, namely if we have reverse the inequality in (5.3), we must also reverse inequalities in (5.4) and (5.5). This follows simply by relabeling pairs σ, f and τ, g . From this it follows, in particular, that the smoothing estimates for the wave ($m = 1$), Schrödinger ($m = 2$) and KdV ($m = 3$) equations are all equivalent to each other.

As another example, we can relate Klein–Gordon, relativistic Schrödinger, standard Schrödinger, and wave equations among each other by the comparison principle that yields

$$\begin{aligned} \left\| e^{-it\sqrt{1-\Delta_x}} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} &= \sqrt{2} \left\| \langle D_x \rangle^{1/2} e^{it\Delta_x} \varphi(\tilde{x}) \right\|_{L^2(\mathbb{R}_t)} \\ &= \left\| |D_x|^{-1/2} \langle D_x \rangle^{1/2} e^{\pm it\sqrt{-\Delta_x}} \varphi(\tilde{x}) \right\|_{L^2(\mathbb{R}_t)}. \end{aligned}$$

This implies the equivalence of the corresponding smoothing estimates.

In comparison principle we can take any further norm with respect to x .

Corollary 5.3. *Let functions f, g, σ, τ be as in the comparison principle Theorem 5.1. Let $0 < p \leq \infty$. Then, for any measurable function w on $\mathbb{R}^n$, we have*

$$\begin{aligned} \|w(x)\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))} \\ \leq A \|w(x)\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))}. \end{aligned}$$

From this it follows, in particular, that for all $-\infty < p \leq \infty$, quantities

$$\|e^{it\sqrt{-\Delta}}\varphi\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))},$$

$$\||D_x|^{1/2}e^{-it\Delta}\varphi\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))},$$

and

$$\||D_x|e^{it(-\Delta)^{3/2}}\varphi\|_{L^p(\mathbb{R}_x^n, L^2(\mathbb{R}_t))}$$

for propagators of the wave, Schrödinger, and linearised KdV type equations, respectively, are equivalent. We also note that Minkowski's inequality implies that for $p_1 \leq 2 \leq p_2$ we have

$$\|f\|_{L^2(\mathbb{R}_t, L^{p_1}(\mathbb{R}_x^n))} \leq C \|f\|_{L^{p_1}(\mathbb{R}_x^n, L^2(\mathbb{R}_t))}, \quad \|f\|_{L^{p_2}(\mathbb{R}_x^n, L^2(\mathbb{R}_t))} \leq C \|f\|_{L^2(\mathbb{R}_t, L^{p_2}(\mathbb{R}_x^n))}.$$

The $L^2(\mathbb{R}_t)$ -norm in time is often critical and, also, Strichartz estimates with $p = \infty$ may fail, so the smaller $L^\infty(\mathbb{R}_x^n, L^2(\mathbb{R}_t))$ -norms may be a good substitute in some situations.

6. FURTHER ESTIMATES

We can not only relate estimates, but also get new ones for new and old equations. Results of this section are proved in [RS1, RS5].

Let $u(t, x)$ be the solution of equation (3.2), that is of

$$(6.1) \quad \begin{cases} (i\partial_t + a(D_x))u(t, x) = 0, \\ u(0, x) = \varphi(x) \in L^2(\mathbb{R}^n), \end{cases}$$

Theorem 6.1. *Condition (H) with $n \geq 1, m > 0, s > 1/2$, or condition (L) with $n \geq 1, m \geq 1, s > 1/2$, imply the estimate*

$$(6.2) \quad \|\langle x \rangle^{-s} |D_x|^{(m-1)/2} u(t, x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Estimate (6.2) was previously obtained by Ben-Artzi and Klainerman [BK] in the case $a(\xi) = |\xi|^2$ and $n \geq 3$, and by Chihara [Ch] under condition (H) and $m > 1$. We note that in particular Theorem 6.1 also covers the case $m = 1$ of hyperbolic equations.

Theorem 6.2. *Assume condition (H) and $m > 1$. Assume $n > m + 1$ or assume $n > m > 1$ for elliptic a . Then we have estimate*

$$\left\| \langle x \rangle^{-m/2} \langle D_x \rangle^{(m-1)/2} u(t, x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Alternatively, assume condition (L), $m > 0$ and $s > 1/2$. Then we have

$$\left\| \langle x \rangle^{-s} \langle D_x \rangle^{(m-1)/2} u(t, x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

These estimates were previously obtained by Kato and Yajima [KY] in the case $a(\xi) = |\xi|^2$ and $n \geq 3$, and later by Walther [Wa1, Wa2] in the case $a(\xi) = |\xi|^m$ and $n > m > 1$.

We can get similar results for equations with time dependent coefficients, for operators $c(t)a(D_x)$, where $c(t) > 0$ a.e. and continuous, see [RS6].

By using canonical transforms, we can also get the critical case of derivatives and weights in the Agmon–Hörmander’s limiting absorption principle, see [RS3]. Consequently, we get the critical case of the Kato–Yajima’s smoothing estimate, and we get the critical case $s = \frac{1}{2}$ of restriction/trace theorems for

$$\|\widehat{f}|_{\rho\Sigma}\|_{L^2(\rho\Sigma)} \leq C(\rho)\|f\|_{H^s},$$

with corresponding formulae for $C(\rho)$, see [RS3].

Indeed, for any operator $A = A(X, D_x)$ acting on the variable x , the smoothing estimate

$$\|Ae^{ita(D_x)}\varphi\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}$$

is equivalent to the restriction estimate

$$\left\| \widehat{A^* f}|_{\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a; \rho^{n-1}d\omega/|\nabla a|)} \leq C\sqrt{\rho}\|f\|_{L^2(\mathbb{R}_x^n)},$$

where $\rho > 0$, $\rho\Sigma_a = \{\rho\omega : \omega \in \Sigma_a\}$, and $\Sigma_a = \{\xi \in \mathbb{R}^n : a(\xi) = 1\}$.

Let $a(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ be real-valued and satisfy $a(\xi) > 0$ and $a(\lambda\xi) = \lambda^2 a(\xi)$ for $\lambda > 0$ and $\xi \neq 0$. Then using different weights in smoothing estimates we get

$$\begin{aligned} \|f|_{\rho\Sigma_a}\|_{L^2(\rho\Sigma_a, \rho^{n-1}d\omega)} &\leq C\|f\|_{H^s(\mathbb{R}^n)} \quad (s > 1/2), \\ \|f|_{\rho\Sigma_a}\|_{L^2(\rho\Sigma_a, \rho^{n-1}d\omega)} &\leq C\rho^{s-1/2}\|f\|_{\dot{H}^s(\mathbb{R}^n)} \quad (n/2 > s > 1/2). \end{aligned}$$

For $\rho = 1$ this gives a trace theorem, and for all ρ , the global smoothing estimate yields a growth order of norms in trace theorems. If we in addition assume that the

Gaussian curvature of Σ_a is non-vanishing, then we have also the critical cases:

$$\begin{aligned} \left\| \left(\frac{\nabla a(x)}{|\nabla a(x)|} \wedge \frac{D_x}{|D_x|} \right) f|_{\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a, \rho^{n-1}d\omega)} &\leq C\|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)}, \\ \left\| \left(\frac{x}{|x|} \wedge \frac{\nabla a^*(D_x)}{|\nabla a^*(D_x)|} \right) f|_{\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a, \rho^{n-1}d\omega)} &\leq C\|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)}. \end{aligned}$$

Here the dual function $a^*(x)$ is determined by the relation

$$\Sigma_{a^*} = \Sigma_a^*$$

where the dual hypersurface is defined by

$$\Sigma_a^* = \{\nabla a(\xi) : \xi \in \Sigma_a\}.$$

These results can be further applied to the global in time well-posedness for derivative nonlinear Schrödinger equations, see [RS7].

7. INVARIANT ESTIMATES

Let us now consider equations of the form

$$\begin{cases} (i\partial_t + a(t, D_x)) u(t, x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\ u(0, x) = \varphi(x) & \text{in } \mathbb{R}_x^n. \end{cases}$$

We can conjecture the following estimate (e.g. for type [1])

$$(7.1) \quad \left\| \langle x \rangle^{-s} |\nabla a(t, D_x)|^{1/2} e^{i \int_0^t a(t, D_x) dt} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2).$$

First we can note that in the dispersive case it is equivalent to the usual type [1] estimate. On the other hand, it still continues to hold for a variety of non-dispersive equations, where $\nabla a(\xi)$ may become zero on some set and when the usual estimate fails. In fact, it does take into account zeros of the gradient $\nabla a(\xi)$, which is also responsible for the interface between dispersive and non-dispersive zone (e.g. how quickly the gradient vanishes). Moreover, estimate (7.1) is invariant under canonical transforms of the equation, it is scaling invariant or “almost” invariant in the non-homogeneous case. Moreover, estimate (7.1) is sharp with respect to the order and to the non-degeneracy of ∇a . For the detailed explanation of these properties we refer to [RS6].

8. ESTIMATES FOR FOURIER INTEGRAL OPERATORS

In this section we discuss global estimates for Fourier integral operators in L^2 . The class of operators arising in the smoothing problems is not covered by Asada–Fujiwara [AF] and by other results which are to be mentioned later. We will also discuss the global calculus of Fourier integral operators and pseudo-differential operators on $\mathbb{R}^n$ under minimal assumptions on phases and amplitudes – this includes the SG-calculus of Coriasco [11] and other calculi (e.g. the one by Boggiatto, Buzano and Rodino [4],

etc.). Moreover, we will discuss global estimates for Fourier integral operators and pseudo-differential operators in weighted Sobolev spaces in $\mathbb{R}^n$, also under minimal assumptions on phases and amplitudes.

We consider Fourier integral operators of the form

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(y, \xi))} a(x, y, \xi) u(y) d\xi dy,$$

with real-valued phase function $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. The case of pseudo-differential operators is a special case with $\phi(y, \xi) = -y \cdot \xi$.

Similar results will also hold for the adjoint operators. In particular, this includes operators of the form

$$Su(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)} a(x, \xi) \widehat{u}(\xi) d\xi,$$

which appear as propagators for hyperbolic equations.

Local L^2 estimates for non-degenerate Fourier integral operators are well known since Eskin [E] and Hörmander [H]. Local L^p estimates for non-degenerate Fourier integral operators have been studied over the years, see e.g. Seeger, Sogge and Stein [SSS] and references therein.

The global estimates for Fourier integral operators have been less studied although global L^2 -boundedness of pseudo-differential operators have been thoroughly analysed e.g. by Calderon and Vaillancourt [5], Coifman and Meyer [8], Cordes [9], etc.

Global L^2 -boundedness of Fourier integral operators of zero order was analysed by Asada [A1, A2], Asada and Fujiwara [AF], Kumano-go [Ku], under conditions on infinitely many derivatives of ϕ . Also, they require $\partial_\xi \partial_\xi \phi$ to be bounded, which fails in many important situations, in particular for operators arising in smoothing problems.

For example, in a typical application to smoothing problem, the canonical transform has the phase of the form $x \cdot \xi - y \cdot \psi(\xi)$, so $\phi(y, \xi) = y \cdot \psi(\xi)$, where ψ is positively homogeneous of order one. But then

$$\partial_\xi \partial_\xi \phi(y, \xi) = y \cdot \partial_\xi \partial_\xi \psi(\xi)$$

is unbounded on $\mathbb{R}^n \times \mathbb{R}^n$.

Boulkhemair [Bo1, Bo2] analysed global L^2 -boundedness of Fourier integral operators under conditions on symbols in Sobolev-Kato spaces, which relax conditions on the differentiability of amplitudes, but made similar assumptions on $\partial_\xi \partial_\xi \phi$ to be uniformly bounded, as above.

Thus, one of the aims of the following results is to remove the assumption on the global boundedness of $\partial_\xi \partial_\xi \phi$ to be able to apply such results to smoothing problems.

Let us assume that on $\text{supp } a$ the following holds:

(C1) there is $C > 0$ such that estimate

$$|\det \partial_y \partial_\xi \phi(y, \xi)| \geq C$$

holds for all $(y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$;

(C2) there are constants C_α, C_β such that estimates

$$|\partial_y^\alpha \partial_\xi^\beta \phi(y, \xi)| \leq C_\alpha, \quad |\partial_y^\beta \partial_\xi^\alpha \phi(y, \xi)| \leq C_\beta$$

hold for all $(y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $1 \leq |\alpha|, |\beta| \leq 2n + 2$.

Note that condition (C1) is a global version of the well-known “local graph condition”, which is necessary even for local L^2 -bounds for Fourier integral operators with symbols in $S_{1,0}^0$. Microlocal versions of conditions (C1) and (C2) exist too, see [RS2] for details.

The difference with previous results described before is that in (C2) now we take only mixed derivatives. Thus, (C1), (C2) are satisfied in more applications. For example, they are satisfied in our main application to the smoothing estimates, where $\phi(y, \xi) = y \cdot \psi(\xi)$, ψ is homogeneous of order one for large ξ and $|\det D\psi(\xi)| \geq C > 0$.

Consider first operators of the form.

$$(8.1) \quad Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(y, \xi))} a(x, \xi) u(y) d\xi dy$$

Theorem 8.1. *Let $\phi(y, \xi)$ satisfy conditions (C1), (C2). Let $a(x, \xi)$ satisfy one of the following conditions:*

- (1) [Calderón-Vaillancourt type] $\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, $\{0, 1\}^n$.
- (2) [Cordes type] $\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, $|\alpha|, |\beta| \leq [n/2] + 1$.
- (3) [Cordes type] $\exists \lambda, \lambda' > n/2 : (1 - \Delta_x)^{\lambda/2} (1 - \Delta_\xi)^{\lambda'/2} a(x, \xi) \in L^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$.
- (4) [Coifman-Meyer type] $\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, $|\alpha| \leq [n/2] + 1, \beta \in \{0, 1\}^n$.
- (5) [Coifman-Meyer type] there exists $2 \leq p < \infty$ such that $\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \in L^p(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, $|\alpha| \leq [n(1/2 - 1/p)] + 1, |\beta| \leq 2n$.

Then operator T in (8.1) is $L^2(\mathbb{R}^n)$ -bounded.

In brackets we put names of authors of results with similar types of assumptions for pseudo-differential operators, see [RS2] for details. In fact, all of these statements follow from the result with symbols in Besov spaces that we describe now, see [RS2] for details and proofs. Let $\bar{s} = (s_1, \dots, s_N)$, $\bar{s}' = (s'_1, \dots, s'_{N'})$, $\bar{n} = (n_1, \dots, n_N)$, $\bar{n}' = (n'_1, \dots, n'_{N'})$, $n = n_1 + \dots + n_N = n'_1 + \dots + n'_{N'}$, be splittings of $\mathbb{R}_x^n$ and $\mathbb{R}_\xi^n$, respectively. For $1 < p, q \leq \infty$, we say that $f \in B_{p,q}^{(\bar{s}, \bar{s}')}(\mathbb{R}^{2n})$ if $f \in \mathcal{S}'(\mathbb{R}^{2n})$ if and

$$\left\{ \sum_{\bar{j}, \bar{k} \geq 0} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |2^{\bar{j} \cdot \bar{s} + \bar{k} \cdot \bar{s}'} \mathcal{F}^{-1} \Phi_{j,k} \mathcal{F} f(x, \xi)|^p dx d\xi \right)^{q/p} \right\}^{1/q} < \infty,$$

where

$$\Phi_{j,k}(y, \eta) = \Theta_{j_1}(y_1) \cdots \Theta_{j_N}(y_N) \Theta_{k_1}(\eta_1) \cdots \Theta_{k_{N'}}(\eta_{N'})$$

is such that $\text{supp } \Theta_i \subset \{z : 2^{i-1} \leq |z| \leq 2^{i+1}\}$ is the dyadic decomposition with $\sum_{i=0}^\infty \Theta_i(z) = 1$.

Theorem 8.2. *Let $\phi(y, \xi)$ satisfy (C1), (C2), and $2 \leq p \leq \infty$. Then*

$$\|Tu\|_{L^2(\mathbb{R}^n)} \leq C \|a(x, \xi)\|_{B_{p,1}^{(1/2-1/p)(\bar{n}, \bar{n}')}} \|u\|_{L^2(\mathbb{R}^n)},$$

uniformly in $u \in S(\mathbb{R}^n)$ and a .

Note that we get the boundedness in the class $S_{0,0}^0$ if we take $p = \infty$. We also get Theorem 8.1 by taking different appropriate choices of $p, \bar{n}, \bar{n}'$.

The results are slightly different for amplitudes independent of x , for operators in the form

$$(8.2) \quad Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(y, \xi))} a(y, \xi) u(y) d\xi dy,$$

or for “adjoint” operators

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - y \cdot \xi)} a(x, \xi) u(y) d\xi dy = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)} a(x, \xi) \hat{u}(\xi) d\xi.$$

Theorem 8.3. *Assume that*

$$|\partial_y^\alpha \partial_\xi^\beta a(y, \xi)| \leq C_{\alpha\beta},$$

for $|\alpha|, |\beta| \leq 2n + 1$. Also assume (C1), (C2), i.e. that on $\text{supp } a(y, \xi)$,

$$|\det \partial_y \partial_\xi \phi(y, \xi)| \geq C > 0$$

and that mixed derivatives are bounded

$$|\partial_y^\alpha \partial_\xi \phi(y, \xi)| \leq C_\alpha, \quad |\partial_y \partial_\xi^\beta \phi(y, \xi)| \leq C_\beta$$

for $|\alpha|, |\beta| \leq 2n + 2$. Then operator T in (8.2) is $L^2(\mathbb{R}^n)$ -bounded, and satisfies

$$\|T\|_{L^2 \rightarrow L^2} \leq C \sup_{|\alpha|, |\beta| \leq 2n+1} \|\partial_y^\alpha \partial_\xi^\beta a(y, \xi)\|_{L^\infty(\mathbb{R}_y^n \times \mathbb{R}_\xi^n)}.$$

Now we will use a slightly different, but equivalent (after taking adjoints) representation of T , and allow amplitudes to depend on all variables.

Theorem 8.4. *Let T be defined by*

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - y \cdot \xi)} a(x, y, \xi) u(y) dy d\xi.$$

Let the phase $\phi(x, \xi) \in C^\infty$ for some positive constants and all $|\alpha|, |\beta| \geq 1$ satisfy

$$|\det \partial_x \partial_\xi \phi(x, \xi)| \geq C_0 > 0,$$

$$\exists x_\beta : |\partial_\xi^\beta \phi(x_\beta, \xi)| \leq C_\beta, \quad |\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)| \leq C_{\alpha\beta}.$$

Let amplitude $a = a(x, y, \xi) \in C^\infty$ for some $m \in \mathbb{R}$ satisfy

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^m \langle y \rangle^{-m-|\beta|},$$

for all α, β, γ and all $x, y, \xi \in \mathbb{R}^n$. Then T is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Moreover, numbers of derivatives required for this theorem are finite and precise numbers are given in [RS2].

We also note that from the assumptions for phase functions ϕ in this theorem, for some $C_1, C_2 > 0$ we obtain the non-degeneracy estimates

$$C_1 \langle y \rangle \leq \langle \partial_\xi \varphi(y, \xi) \rangle \leq C_2 \langle y \rangle, \quad C_1 \langle \xi \rangle \leq \langle \partial_y \varphi(y, \xi) \rangle \leq C_2 \langle \xi \rangle.$$

Now we will discuss estimates in weighted spaces. For $m \in \mathbb{R}$, we use the notation

$$\langle x \rangle^m = (1 + |x|^2)^{m/2}$$

and let $L_m^2(\mathbb{R}^n)$ be the set of functions f such that

$$\|f\|_{L_m^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\langle x \rangle^m f(x)|^2 dx \right)^{1/2} < \infty.$$

We will say that $f \in H^{s_1, s_2}(\mathbb{R}^n)$ if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\Pi_{s_1, s_2} f \in L^2(\mathbb{R}^n)$, where Π_{s_1, s_2} is a pseudo-differential operator with symbol $\pi_{s_1, s_2}(x, \xi) = \langle x \rangle^{s_1} \langle \xi \rangle^{s_2}$.

Theorem 8.5. *Let operator T be defined by*

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - y \cdot \xi)} a(x, y, \xi) u(y) dy d\xi$$

Let the phase $\phi = \phi(x, \xi) \in C^\infty$ for all $|\alpha|, |\beta| \geq 1$ satisfy

$$|\det \partial_x \partial_\xi \phi(x, \xi)| \geq C_0 > 0, \quad |\partial_x^\alpha \phi(x, \xi)| \leq C_\alpha \langle \xi \rangle, \quad |\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)| \leq C_{\alpha\beta}.$$

Assume also one of the following:

(1) *For all α, β , and γ ,*

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1} \langle y \rangle^{m_2 - |\beta|} \langle \xi \rangle^{m_3},$$

and for all $|\beta| \geq 1$,

$$|\partial_\xi^\beta \phi(x, \xi)| \leq C_\beta \langle x \rangle.$$

(2) *For all α, β , and γ ,*

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1 - |\alpha|} \langle y \rangle^{m_2} \langle \xi \rangle^{m_3},$$

and for all α and $|\beta| \geq 1$,

$$|\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{1 - |\alpha|}.$$

Then T is bounded from $H^{s_1, s_2}(\mathbb{R}^n)$ to $H^{s_1 - m_1 - m_2, s_2 - m_3}(\mathbb{R}^n)$, for all $s_1, s_2 \in \mathbb{R}^n$.

Some calculus of these operators can be constructed already under quite weak assumptions on phase and amplitude. Let operator T be globally defined by

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - y \cdot \xi)} a(x, y, \xi) u(y) dy d\xi,$$

with a smooth amplitude $a(x, y, \xi)$, satisfying

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1} \langle y \rangle^{m_2} \langle \xi \rangle^{m_3}$$

for all α, β, γ . Pseudo-differential operators are covered by

$$Pu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} p(x, \xi) u(y) dy d\xi.$$

Let us now give a brief summary of composition theorems, for details of which we refer to [RSc]:

- Assume that $|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{t_1} \langle \xi \rangle^{t_2 - |\beta|}$. Then the amplitude c of $T \circ P$ satisfies

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma c(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1} \langle y \rangle^{m_2 + t_1} \langle \xi \rangle^{m_3 + t_2},$$

with no conditions on the phase.

- Same result for $P \circ T$ but now with some conditions on the phase:

$$C_1 \langle \xi \rangle \leq \langle \partial_x \varphi(x, \xi) \rangle \leq C_2 \langle \xi \rangle, \quad |\partial_x^\alpha \phi(x, \xi)| \leq C_\alpha \langle \xi \rangle, \quad |\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)| \leq C_{\alpha\beta}.$$

- We have asymptotic formulae. For example, the amplitude of $T \circ P$ has an expansion improving in ξ :

$$c(x, z, \xi) \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \partial_y^\alpha [a(x, y, \xi) \partial_\xi^\alpha p(y, \xi)]|_{y=z}.$$

If a and p have additional decay properties, similar property can be extracted for c . For example, if a, p are SG-symbols, so is c .

If the amplitude $a(x, y, \xi)$ has SG-decay properties in y , the amplitude of the composition TP can be made dependent on two variables only:

Theorem 8.6. *Let operator T be defined by*

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - y \cdot \xi)} a(x, y, \xi) u(y) dy d\xi.$$

Let the phase $\phi = \phi(x, \xi) \in C^\infty$ satisfy partial non-degeneracy and boundedness

$$C_1 \langle x \rangle \leq \langle \nabla_\xi \phi(x, \xi) \rangle \leq C_2 \langle x \rangle, \quad x, \xi \in \mathbb{R}^n,$$

and be such that for all $|\alpha|, |\beta| \geq 1$ we have

$$|\partial_\xi^\beta \phi(x, \xi)| \leq C_\beta \langle x \rangle, \quad |\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)| \leq C_{\alpha\beta}.$$

Let $a = a(x, y, \xi) \in C^\infty$ satisfy

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1} \langle y \rangle^{m_2 - |\beta|} \langle \xi \rangle^{m_3},$$

for all α, β, γ , and all $x, y, \xi \in \mathbb{R}^n$. Let $p = p(x, \xi) \in C^\infty$ for all α, β satisfy

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{t_1 - |\alpha|} \langle \xi \rangle^{t_2}, \quad x, \xi \in \mathbb{R}^n.$$

Then the composition $B = T \circ P(x, D)$ is an operator of the form

$$Bu(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)} c(x, \xi) \widehat{u}(\xi) d\xi$$

with amplitude $c(x, \xi)$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta c(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{m_1+m_2+t_1} \langle \xi \rangle^{m_3+t_2},$$

for all α, β, γ , and all $x, z, \xi \in \mathbb{R}^n$. Moreover, we have the asymptotic expansion, improving in x (due to ∂_x^α) and y (due to ∂_y^β):

$$c(x, \xi) \sim \sum_{\alpha, \beta} \frac{i^{-(|\alpha|+|\beta|)}}{\alpha! \beta!} \partial_x^\alpha p(\nabla_\xi \phi(x, \xi), \xi) \partial_\eta^{\alpha+\beta} [e^{i\Psi(\eta, \xi, x)} \partial_y^\beta a(x, \nabla_\xi \phi(x, \xi), \eta)]|_{\eta=\xi},$$

where $\Psi(\eta, \xi, x) = \phi(x, \xi) - \phi(x, \eta) + (\eta - \xi) \cdot \nabla_\xi \phi(x, \xi)$.

If we apply this Theorem with symbol $p \equiv 1$, we can simplify the amplitude.

Corollary 8.7. *Let T be the operator as in Theorem 8.6. Then T can be written in the form*

$$Tu(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)} c(x, \xi) \widehat{u}(\xi) d\xi,$$

with amplitude $c(x, \xi)$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta c(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{m_1+m_2} \langle \xi \rangle^{m_3},$$

for all α, β, γ , and all $x, z, \xi \in \mathbb{R}^n$. Moreover,

$$c(x, \xi) \sim \sum_{\beta} \frac{i^{-|\beta|}}{\beta!} \partial_\eta^\beta [e^{i\Psi(\eta, \xi, x)} \partial_y^\beta a(x, \nabla_\xi \phi(x, \xi), \eta)]|_{\eta=\xi},$$

where Ψ is as in the previous theorem.

We can apply this corollary to pseudo-differential operators to obtain a “normal form” for generalized SG pseudo-differential operators with decay in only one of the variables.

Corollary 8.8. *Let T be a pseudo-differential operator of the form*

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) dy d\xi$$

with amplitude $a = a(x, y, \xi) \in C^\infty$ satisfying

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1} \langle y \rangle^{m_2-|\beta|} \langle \xi \rangle^{m_3},$$

for all α, β, γ and all $x, y, \xi \in \mathbb{R}^n$. Then T can be written in the form

$$Tu(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} c(x, \xi) \widehat{u}(\xi) d\xi,$$

with

$$|\partial_x^\alpha \partial_\xi^\beta c(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{m_1+m_2} \langle \xi \rangle^{m_3}, \quad \forall \alpha, \beta, \quad x, \xi \in \mathbb{R}^n.$$

Moreover, we have the asymptotic expansion $c(x, \xi) \sim \sum_\beta \frac{i^{-|\beta|}}{\beta!} \partial_\xi^\beta a(x, y, \xi)|_{y=x}$.

Note that from all asymptotic expansions it is clear that if a has additional decay with respect to some variables, so does the new amplitude. For example, if $a \in SG$, so does c , i.e. $a \in SG^{m_1, m_2, m_3}$ implies that $c \in SG^{m_1+m_2, m_3}$.

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