

ON THE PAINLEVÉ I HIERARCHY

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Every solution of the first Painlevé equation

$$(I) \quad Z'' = 6Z^2 + 4t$$

($' = d/dt$) is meromorphic in $\mathbb{C}$, that is to say, equation (I) admits the Painlevé property. It is known that the fourth-order equation

$$(I_4) \quad Z^{(4)} = 20ZZ'' + 10(Z')^2 - 40Z^3 + 16t$$

also admits the Painlevé property, which is proved by using Miwa's result concerning the isomonodromic deformation ([1,2,3]). In this note, we show that

(1) there exists a hierarchy of systems of nonlinear equations, from which we can derive (I), (I_4) and

$$(I_6) \quad Z^{(6)} = 28ZZ^{(4)} + 56Z'Z^{(3)} + 42(Z'')^2 - 280(Z^2Z'' + Z(Z')^2 - Z^4) + 64t,$$

(2) all the systems in the hierarchy and the nonlinear equations derived from it such as (I), (I_4) , (I_6) admit the Painlevé property.

1. Results

Consider the following formal power series in ξ :

$$Q(\xi) = \sum_{\nu \geq 1} Z_\nu \xi^\nu,$$

$$R(\xi) = \sum_{\nu \geq 1} U_\nu \xi^\nu,$$

$$F(\xi) = 2\xi^{-1}Q(\xi)(1 + Z_1\xi) + (\xi^{-1}Q(\xi)^2 - R(\xi)^2)(1 - Q(\xi))^{-1} - u_0^2,$$

where u_0, Z_ν, U_ν ($\nu \in \mathbb{N}$) are parameters. Then, $F(\xi)$ is written in the form

$$F(\xi) = \sum_{\nu \geq 0} F_\nu \xi^\nu$$

$$F_0 = 2Z_1 - u_0^2,$$

$$F_\nu = 2Z_{\nu+1} + G_\nu(Z_j, U_k; 1 \leq j \leq \nu, 1 \leq k \leq \nu - 1) \quad (\nu \in \mathbb{N}).$$

Here $G_\nu(Z_j, U_k; \dots)$ denotes a polynomial in Z_j and U_k ($1 \leq j \leq \nu, 1 \leq k \leq \nu - 1$). Let m be a nonnegative integer and t a variable. Then the relations

$$\frac{d}{dt}(u_0 + R(\xi)) \equiv F(\xi) + 2(t - Z_{m+1})\xi^m \pmod{\xi^{m+1}},$$

$$\frac{d}{dt}Q(\xi) \equiv 2R(\xi) \pmod{\xi^{m+1}},$$

define the following systems:

for $m = 0$,

$$(S_1) \quad u'_0 = 2t - u_0^2;$$

for $m \geq 1$,

$$(S_m) \quad \begin{aligned} u'_0 &= 2Z_1 - u_0^2, \\ Z'_\nu &= 2U_\nu, \\ U'_\nu &= 2Z_{\nu+1} + G_\nu(Z_j, U_k; 1 \leq j \leq \nu, 1 \leq k \leq \nu - 1), \\ Z'_m &= 2U_m, \\ U'_m &= 2t + G_m(Z_j, U_k; 1 \leq j \leq m, 1 \leq k \leq m - 1) \end{aligned}$$

($1 \leq \nu \leq m - 1$). Then we have

Theorem 1.1. *Every solution $(u_0(t), Z_\nu(t), U_\nu(t))$ ($1 \leq \nu \leq m$) of (S_m) ($m \geq 0$) is meromorphic in $\mathbb{C}$.*

As an immediate corollary of this theorem, we have

Corollary 1.2. *Every solution of (I_4) or (I_6) is meromorphic in $\mathbb{C}$.*

It is known that, for an arbitrary solution $P(t)$ of (I), every solution of

$$y'' - 2P(t)y = 0$$

is meromorphic in $\mathbb{C}$. Furthermore we have

Corollary 1.3. *Let $P_4(t)$ (resp. $P_6(t)$) be an arbitrary solution of (I_4) (resp. (I_6)). Then every solution of*

$$y'' - 2P_4(t)y = 0 \quad (\text{resp. } y'' - 2P_6(t)y = 0)$$

is meromorphic in $\mathbb{C}$.

2. Outline of the proof of Theorem 1.1

Consider the 2 by 2 matrix linear differential equation

$$(2.1) \quad \frac{d\Xi}{dx} = A(x)\Xi, \quad A(x) = - \sum_{j=0}^{2(m+1)} A_{-j}x^j + A_1x^{-1}.$$

Here $A_{-\nu}$ are given as below:

$$\begin{aligned} A_{-2(m+1)} &= J, \quad A_{-(2m+1)} = -u_0L, \\ A_{-2m} &= v_1K - w_1J, \quad A_{-(2m-1)} = -u_1L, \\ A_{-2(m+1)+2i} &= v_iK - w_iJ, \quad A_{-(2m+1)+2i} = -u_iL \quad (1 \leq i \leq m), \\ A_0 &= s(J + K), \quad A_1 = (I - L)/2 \end{aligned}$$

with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 2.1. *Let $t, u_0, u_1, \dots, u_m, v_1, \dots, v_m$ be arbitrary parameters. System (2.1) admits a formal matrix solution of the form*

$$\begin{aligned} \Xi &= \Xi(x) = Y(x) \exp T(x), \\ T(x) &= -\frac{J}{2m+3}x^{2m+3} - tJx + \frac{I}{2}\log(1/x), \quad Y(x) = \sum_{j \geq 1} Y_j x^{-j}, \end{aligned}$$

if and only if

$$(2.2) \quad \begin{aligned} w_1 &= u_0^2/2, \\ w_\nu &= \frac{1}{2} \left(\sum_{j=1}^{\nu-1} w_j w_{\nu-j} - \sum_{j=1}^{\nu-1} v_j v_{\nu-j} + \sum_{j=1}^{\nu} u_{j-1} u_{\nu-j} \right), \\ s &= t - \frac{1}{2} \left(\sum_{j=1}^m w_j w_{m+1-j} - \sum_{j=1}^m v_j v_{m+1-j} + \sum_{j=1}^{m+1} u_{j-1} u_{m+1-j} \right) \end{aligned}$$

$$(1 \leq \nu \leq m).$$

For the deformation parameter t , the deformation equation with respect to (2.1) is written in the form

$$(2.3) \quad \begin{aligned} dA(x) &= \frac{\partial}{\partial x} \Omega(x, t) + [\Omega(x, t), A(x)], \\ \Omega(x, t) &= \Phi_{-1}(t)x + \Phi_0(t), \end{aligned}$$

where $\Phi_{-1}(t)$ and $\Phi_0(t)$ are 1-forms of t defined by

$$\sum_{k=-\infty}^1 \Phi_{-k}(t)x^k = Y(x)(-xdt)JY(x)^{-1}.$$

Proposition 2.2. *Equation (2.3) is equivalent to*

$$\begin{aligned} u'_{\nu-1} &= 2v_\nu, & v'_\nu &= 2u_\nu + 2u_0w_\nu, & w'_\nu &= 2u_0v_\nu, \\ u'_m &= 2s, & s' &= 1 - 2u_0s \end{aligned}$$

($1 \leq \nu \leq m$), where w_ν, s are the parameters defined by (2.2).

System (2.1) possesses an apparent singularity at $x = 0$, and Miwa's theorem [2] is not applicable. To remove it, we employ the Schlesinger transformation

$$W = \Psi(x)\Xi, \quad \Psi(x) = \begin{pmatrix} 1 & 1 \\ u_0/2 & u_0/2 + x \end{pmatrix}.$$

Then system (2.1) is changed into

$$(2.4) \quad \frac{dW}{dx} = B(x)W, \quad B(x) = - \sum_{j=0}^{2(m+1)} B_{-j}x^j,$$

where

$$\begin{aligned} B_{-2(m+1)} &= J, \\ B_{-(2\nu+1)} &= \begin{pmatrix} -u_{m-\nu} - u_0(v_{m-\nu} + w_{m-\nu}) & 2(v_{m-\nu} + w_{m-\nu}) \\ -u_0^2(v_{m-\nu} + w_{m-\nu})/2 - u_0u_{m-\nu} - v_{m-(\nu-1)} & u_{m-\nu} + u_0(v_{m-\nu} + w_{m-\nu}) \end{pmatrix}, \\ B_{-2\nu} &= \begin{pmatrix} -(v_{m+1-\nu} + w_{m+1-\nu}) & 0 \\ -u_0(v_{m+1-\nu} + w_{m+1-\nu}) - u_{m+1-\nu} & v_{m+1-\nu} + w_{m+1-\nu} \end{pmatrix}, \\ B_{-1} &= \begin{pmatrix} -u_m - u_0(v_m + w_m) & 2(v_m + w_m) \\ -u_0^2(v_m + w_m)/2 - u_0u_m - s & u_m + u_0(v_m + w_m) \end{pmatrix}, \\ B_0 &= \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} \end{aligned}$$

($1 \leq \nu \leq m$), $v_0 = w_0 = 0$. Applying Miwa's theorem to (2.4), we can show that $u_0, Z_\nu = v_\nu + w_\nu$ and $U_\nu = u_\nu + u_0Z_\nu$ are meromorphic in $\mathbf{C}$. Since the isomonodromy property is invariant under the Schlesinger transformation, from (2.2) and Proposition 2.2 we derive the deformation equation with respect to Z_ν, U_ν , which coincides with (S_m) . This completes the proof.

3. Derivation of the corollaries

Eliminating the unknown variables other than Z_1 , from (S_2) and (S_3) we get equations (I_4) and (I_6) , respectively. Thus we have Corollary 1.2.

To show Corollary 1.3, let us consider, for example, system (S_3) . By Corollary 1.2, an arbitrary solution $Z = P_6(t)$ of (I_6) is meromorphic in $\mathbf{C}$, and, around each pole $t = t_0$, it is expanded into one of the following Laurent series:

$$(3.1) \quad (t - t_0)^{-2} + \cdots, \quad 3(t - t_0)^{-2} + \cdots, \quad 6(t - t_0)^{-2}.$$

By Theorem 1.1, every solution of

$$(3.2) \quad u' = 2P_6(t) - u^2,$$

is meromorphic in $\mathbb{C}$, which is the first equation of (S_3) . The transformation $u = y'/y$ takes (3.2) into

$$(3.3) \quad y'' - 2P_6(t)y = 0.$$

Let $y(t)$ be an arbitrary solution of (3.3). It is sufficient to show that an arbitrary pole $t = t_0$ of $P_6(t)$ is at most a pole of $y(t)$. To do this, we note that $u(t) = y'(t)/y(t)$ is written in the form

$$u(t) = c(t - t_0)^{-1} + \dots,$$

around it, where c is an integer equal to one of $-3, -2, -1, 2, 3, 4$. Hence we get an expression of the form

$$y(t) = (t - t_0)^c \sum_{j=0}^{\infty} C_j(t - t_0)^j,$$

from which Corollary 1.3 follows.

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