

Gap Function Approach to Duality

— discount model vs control model —

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Abstract

This paper discusses two pairs of quadratic optimization problem (primal) and its dual. In particular, we deal with the pair of problems which are called *discount model* and *control model*. For each model, duality is shown through the *gap function*. The method is based upon a complementary identity. Moreover, complete solutions are given through characteristic equations.

1 Introduction

Recently, in [6–14], S.Iwamoto, Y.Kimura, T.Fujita and A.Kira show that a duality for paired optimization problems through several methods. In particular, in [13], we have a method through gap function to show a duality between a primal problem and its dual. As a historical background, see Bellman and others [1–5], [15] for dynamic optimization.

In this paper, we discuss a method through *gap function* to show a duality for pairs of quadratic optimization problems, which are called *discount model* and *control model*. Section 2 considers a pair of n -variable minimization (primal) problem and maximization (dual) problem, which is called discount model. Then we define a gap function and discuss duality and optimal solution (point and value). In section 3, we consider a pair of quadratic optimization problems, which is called control model. As in section 2, we discuss a duality through a gap function and optimal solution for the pair.

Throughout the paper let n be a natural number and $c (\in R^1)$ be a constant. c denotes an *initial state* at time 0 of a dynamic system.

2 Discount model

In this section let ρ be a *positive* constant. We consider a pair of n -variable optimization problems :

$$\begin{array}{ll} \text{P}_n & \begin{array}{l} \text{minimize} \quad \sum_{k=1}^n \rho^{k-1} [(x_{k-1} - x_k)^2 + x_k^2] \\ \text{subject to} \quad \text{(i) } x \in R^n, \quad \text{(ii) } x_0 = c \end{array} \end{array}$$

$$\begin{aligned} & \text{Maximize } 2x_0\mu_1 - \sum_{k=1}^{n-1} \rho^{k-1} [\mu_k^2 + (\mu_k - \rho\mu_{k+1})^2] - 2\rho^{n-1}\mu_n^2 \\ & \text{D}_n \quad \text{subject to } \quad \text{(i)} \quad \mu \in R^n, \quad \text{(ii)} \quad x_0 = c. \end{aligned}$$

Let $f, g : R^n \rightarrow R^1$ be the respective objective functions of P_n, D_n :

$$\begin{aligned} f(x) &= \sum_{k=1}^n \rho^{k-1} [(x_{k-1} - x_k)^2 + x_k^2] \\ g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \rho\mu_{k+1})^2] - 2\rho^{n-1}\mu_n^2. \end{aligned}$$

Note that $f(x)$ is convex and $g(\mu)$ is concave. Then it holds that

$$f(x) \geq g(\mu) \quad (x, \mu) \in R^n \times R^n. \quad (1)$$

The sign of equality holds iff a linear system of $2n$ -equation on $2n$ -variable

$$\begin{aligned} & c - x_1 = \mu_1 \quad x_1 = \mu_1 - \rho\mu_2 \\ \text{EC}(\rho) \quad & x_{k-1} - x_k = \mu_k \quad x_k = \mu_k - \rho\mu_{k+1} \quad 2 \leq k \leq n-1 \\ & x_{n-1} - x_n = \mu_n \quad x_n = \mu_n \end{aligned}$$

holds. $\text{EC}(\rho)$ is called an *equality condition* between P_n and D_n . Thus both problems are called *dual* of each other.

The equality condition $\text{EC}(\rho)$ yields a pair of linear systems of n -equation on n -variable:

$$\begin{aligned} & \gamma x_1 - \rho x_2 = c \\ (\text{EQ}_x) \quad & -x_{k-1} + \gamma x_k - \rho x_{k+1} = 0 \quad 2 \leq k \leq n-1 \\ & -x_{n-1} + 2x_n = 0, \\ & 2\mu_1 - \rho\mu_2 = c \\ (\text{EQ}_\mu) \quad & -\mu_{k-1} + \gamma\mu_k - \rho\mu_{k+1} = 0 \quad 2 \leq k \leq n-1 \\ & -\mu_{n-1} + \gamma\mu_n = 0 \end{aligned}$$

where $\gamma = 2 + \rho$.

2.1 Gap function for discount model

First we present an identity, which takes a fundamental role in analyzing respective pairs of primal and dual. Let $x = \{x_k\}_0^n, \mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$(\text{C}_2) \quad \sum_{k=1}^{n-1} \rho^{k-1} [(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \rho\mu_{k+1})] + \rho^{n-1} [(x_{n-1} - x_n)\mu_n + x_n\mu_n] = c\mu_1$$

holds true. This identity is called *complementary*.

Now we derive both P_n and D_n through gap function. Let us define a *gap function* $h = h(x, \mu)$ between $x \in R^n$ and $\mu \in R^n$ by

$$h(x, \mu) = \sum_{k=1}^{n-1} \rho^{k-1} [(x_{k-1} - x_k - \mu_k)^2 + \{x_k - (\mu_k - \rho\mu_{k+1})\}^2] \\ + \rho^{n-1} [(x_{n-1} - x_n - \mu_n)^2 + (x_n - \mu_n)^2]. \quad (2)$$

Thus $h(x, \mu)$ denotes a *total difference* between x and μ . It turns out that the quadratic function $h = h(x, \mu)$ is convex in (x, μ) .

Lemma 1

- (i) $f(x) - g(\mu) = h(x, \mu) \geq 0 \quad \forall (x, \mu) \in R^n \times R^n$
- (ii) $h(x, \mu) = 0 \implies (x, \mu) \text{ satisfies EC}(\rho).$

Theorem 1 (i) *It holds that*

$$f(x) \geq g(\mu) \quad \text{on } R^n \times R^n.$$

(ii) *It holds that*

$$f(x) = g(\mu) \iff (x, \mu) \text{ satisfies EC}(\rho).$$

Then P_n attains a minimum $f(x)$, while D_n attains a maximum $g(\mu)$.

Hence a solution (x, μ) to $\text{EC}(\rho)$ yields a minimum point x for P_n and a maximum point μ for D_n .

Theorem 2 *Let (x, μ) satisfy $\text{EC}(\rho)$. Then both sides become a common value with five expressions:*

$$(5V_2) \quad f(x) = c(c - x_1) \\ = g(\mu) = \sum_{k=1}^{n-1} \rho^{k-1} [\mu_k^2 + (\mu_k - \rho\mu_{k+1})^2] + 2\rho^{n-1} \mu_n^2 = c\mu_1.$$

The primal P_n has a minimum value

$$m = f(x) = c(c - x_1)$$

at x , while the dual D_n has a maximum value

$$M = g(\mu) = \sum_{k=1}^{n-1} \rho^{k-1} [\mu_k^2 + (\mu_k - \rho\mu_{k+1})^2] + 2\rho^{n-1} \mu_n^2 = c\mu_1$$

at μ .

2.2 Characteristic equation for discount model

Now let us solve the pair of linear systems (EQ_x) and (EQ_μ). We introduce a second-order linear difference equation

$$\rho x_{n+2} - \gamma x_{n+1} + x_n = 0, \quad x_1 = 1, \quad x_0 = 0. \quad (3)$$

Lemma 2 *Eq (3) has a unique solution*

$$x_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} \quad (4)$$

where $\alpha (<) \beta$ are two positive solutions

$$\alpha = \frac{\gamma - \sqrt{D}}{2\rho}, \quad \beta = \frac{\gamma + \sqrt{D}}{2\rho}; \quad D = \rho^2 + 4 \quad (> 4) \quad (5)$$

to the associated characteristic equation

$$(CE) \quad \rho t^2 - \gamma t + 1 = 0. \quad (6)$$

Definition 1 *Let us define the sequence $\{K_n\}$ by*

$$K_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}. \quad (7)$$

We call $\{K_n\}$ a *Kibonacci* sequence¹[13]. Thus $\{K_n\}$ satisfies a second-order linear difference equation

$$\rho K_{n+1} = \gamma K_n - K_{n-1}, \quad K_1 = 1, \quad K_0 = 0. \quad (8)$$

This has a unique solution (7).

Lemma 3 *The system (EQ_x) has a unique solution*

$$x_k = \frac{c}{\rho^k} \cdot \frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n} \quad 0 \leq k \leq n$$

, while the system (EQ_μ) has a unique solution

$$\mu_k = \frac{c}{\rho^{k-1}} \cdot \frac{K_{n+1-k}}{2K_n - K_{n-1}} \quad 1 \leq k \leq n.$$

Theorem 3 *The equality condition EC(ρ) has a unique solution (x, μ) ;*

$$x_k = \frac{c}{\rho^k} \cdot \frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n}$$

$$\mu_k = \frac{c}{\rho^{k-1}} \cdot \frac{K_{n+1-k}}{2K_n - K_{n-1}}.$$

Hence the gap function h attains the zero minimum at (x, μ) .

¹Strictly speaking, ρ -Kibonacci sequence.

3 Control model

In this section let $b(\in R^1)$ be a constant. We consider a pair of n -variable optimization problems :

$$\begin{aligned}
 & \text{P}_n \quad \begin{aligned} & \text{minimize} \quad \sum_{k=1}^n [(x_{k-1} - bx_k)^2 + x_k^2] \\ & \text{subject to} \quad \text{(i) } x \in R^n, \quad \text{(ii) } x_0 = c \end{aligned} \\
 & \text{D}_n \quad \begin{aligned} & \text{Maximize} \quad 2x_0\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (b\mu_k - \mu_{k+1})^2] - (1 + b^2)\mu_n^2 \\ & \text{subject to} \quad \text{(i) } \mu \in R^n, \quad \text{(ii) } x_0 = c. \end{aligned}
 \end{aligned}$$

Let $f, g : R^n \rightarrow R^1$ be the respective objective functions of P_n, D_n :

$$\begin{aligned}
 f(x) &= \sum_{k=1}^n [(x_{k-1} - bx_k)^2 + x_k^2] \\
 g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (b\mu_k - \mu_{k+1})^2] - (1 + b^2)\mu_n^2.
 \end{aligned}$$

Note that $f(x)$ is convex and $g(\mu)$ is concave. Then it holds that

$$f(x) \geq g(\mu) \quad (x, \mu) \in R^n \times R^n. \quad (9)$$

The sign of equality holds iff a linear system of $2n$ -equation on $2n$ -variable

$$\begin{aligned}
 & c - bx_1 = \mu_1 \quad x_1 = b\mu_1 - \mu_2 \\
 \text{EC}(b) \quad & x_{k-1} - bx_k = \mu_k \quad x_k = b\mu_k - \mu_{k+1} \quad 2 \leq k \leq n-1 \\
 & x_{n-1} - bx_n = \mu_n \quad x_n = b\mu_n
 \end{aligned}$$

holds. $\text{EC}(b)$ is called an *equality condition* between P_n and D_n . Thus both problems are called *dual* of each other.

The equality condition $\text{EC}(b)$ yields a pair of linear systems of n -equation on n -variable:

$$\begin{aligned}
 & \gamma x_1 - bx_2 = bc \\
 (\text{EQ}_x) \quad & -bx_{k-1} + \gamma x_k - bx_{k+1} = 0 \quad 2 \leq k \leq n-1 \\
 & -bx_{n-1} + \gamma x_n = 0, \\
 & \xi \mu_1 - b\mu_2 = c \\
 (\text{EQ}_\mu) \quad & -b\mu_{k-1} + \gamma \mu_k - b\mu_{k+1} = 0 \quad 2 \leq k \leq n-1 \\
 & -b\mu_{n-1} + \gamma \mu_n = 0
 \end{aligned}$$

where $\gamma = 2 + b^2$, $\xi = 1 + b^2$.

3.1 Gap function for control model

Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then a complementary identity

$$(C_3) \quad \sum_{k=1}^{n-1} [(x_{k-1} - bx_k)\mu_k + x_k(b\mu_k - \mu_{k+1})] + (x_{n-1} - bx_n)\mu_n + x_n \cdot b\mu_n = c\mu_1$$

holds true. Let us define a *gap function* $h = h(x, \mu)$ by

$$h(x, \mu) = \sum_{k=1}^{n-1} [(x_{k-1} - bx_k - \mu_k)^2 + \{x_k - (b\mu_k - \mu_{k+1})\}^2] + [(x_{n-1} - bx_n - \mu_n)^2 + (x_n - b\mu_n)^2]. \quad (10)$$

Thus $h(x, \mu)$ denotes a *total difference*. It turns out that the quadratic function $h = h(x, \mu)$ is convex.

Lemma 4

- (i) $f(x) - g(\mu) = h(x, \mu) \geq 0 \quad \forall (x, \mu) \in R^n \times R^n$
- (ii) $h(x, \mu) = 0 \implies (x, \mu) \text{ satisfies EC}(b)$.

Theorem 4 (i) *It holds that*

$$f(x) \geq g(\mu) \quad \text{on } R^n \times R^n.$$

(ii) *It holds that*

$$f(x) = g(\mu) \iff (x, \mu) \text{ satisfies EC}(b).$$

Then P_n attains a minimum $f(x)$, while D_n attains a maximum $g(\mu)$.

Hence a solution (x, μ) to $\text{EC}(b)$ yields a minimum point x for P_n and a maximum point μ for D_n .

Theorem 5 *Let (x, μ) satisfy $\text{EC}(b)$. Then both sides become a common value with five expressions:*

$$\begin{aligned} (5V_3) \quad f(x) &= c(c - bx_1) \\ &= g(\mu) = \sum_{k=1}^{n-1} [\mu_k^2 + (b\mu_k - \mu_{k+1})^2] + (1 + b^2)\mu_n^2 = c\mu_1. \end{aligned}$$

The primal P_n has a minimum value

$$m = f(x) = c(c - bx_1)$$

at x , while the dual D_n has a maximum value

$$M = g(\mu) = \sum_{k=1}^{n-1} [\mu_k^2 + (b\mu_k - \mu_{k+1})^2] + (1 + b^2)\mu_n^2 = c\mu_1$$

at μ .

3.2 Characteristic equation for control model

We introduce a second-order linear difference equation

$$bx_{n+2} - \gamma x_{n+1} + bx_n = 0, \quad x_1 = 1, \quad x_0 = 0. \quad (11)$$

Lemma 5 *Eq (11) has a unique solution*

$$x_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} \quad (12)$$

where $\alpha (<) \beta$ are two positive solutions

$$\alpha = \frac{\gamma - \sqrt{D}}{2b}, \quad \beta = \frac{\gamma + \sqrt{D}}{2b}; \quad D = b^4 + 4 \quad (> 4) \quad (13)$$

to the associated characteristic equation

$$(CE) \quad bt^2 - \gamma t + b = 0. \quad (14)$$

Now let us define Kibonacci sequence²[13] $\{K_n\}$ by

$$K_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}.$$

Then the sequence $\{K_n\}$ satisfies a second-order linear difference equation

$$bK_{n+1} = \gamma K_n - bK_{n-1}, \quad K_1 = 1, \quad K_0 = 0. \quad (15)$$

Lemma 6 *The system (EQ_x) has a unique solution*

$$x_k = c \frac{bK_{n+1-k} - K_{n-k}}{bK_{n+1} - K_n} \quad 0 \leq k \leq n$$

, while the system (EQ_μ) has a unique solution

$$\mu_k = c \frac{K_{n+1-k}}{\xi K_n - bK_{n-1}} \quad 1 \leq k \leq n.$$

Theorem 6 *The equality condition $EC(b)$ has a unique solution (x, μ) ;*

$$x_k = c \frac{bK_{n+1-k} - K_{n-k}}{bK_{n+1} - K_n}$$

$$\mu_k = c \frac{K_{n+1-k}}{\xi K_n - bK_{n-1}}.$$

Hence the gap function h attains the zero minimum at (x, μ) .

²Strictly speaking, b -Kibonacci sequence.

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