

ON A PERTURBED SYSTEM OF CHEMOTAXIS

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1. Introduction

This is a joint work with Takasi Senba (Miyazaki University) and Takashi Suzuki (Osaka University). We consider the following parabolic-elliptic system describing chemotactic aggregation of the slime molds:

$$(CZ) \quad \begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v), & (x, t) \in \Omega \times (0, T), \\ 0 = \Delta v - \gamma v - \beta |v|^{p-1} v + \alpha u, & (x, t) \in \Omega \times (0, T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial \Omega$ and $T > 0$. Here $u(x, t)$ is the cell density of the cellular slime molds and $v(x, t)$ is the concentration of the chemical substance at place x and time t , respectively. $\chi, \alpha, \beta, \gamma$ are positive constants and $1 < p < \infty$. ν denotes the unit outward normal vector to $\partial \Omega$. The term $F = \nabla u - \chi u \nabla v$ is the flux of u so that effect of diffusion $\nabla \cdot \nabla u$ and that of chemotaxis $\chi \nabla \cdot (u \nabla v)$ are competing for u to vary.

For the problem (CZ), in the case of $\beta = 0$, Nagai [6] has confirmed the conjectures of Childress and Percus [4], which is chemotaxis collapse can occur if a total cell number on $\Omega \subset \mathbf{R}^2$ is larger than a critical number, but not occur for the total cell number on Ω less than it, and he find the critical number is $\frac{8\pi}{\alpha\chi}$. While Senba and Suzuki [11] have made clear the blowup mechanism.

Here we study the blowup solution to the simplest Keller-Segel model (CZ) in the case of $\beta \neq 0$. For the initial function u_0 we suppose

1. $u_0 \geq 0$ and u_0 is not identical to 0 on Ω ,
2. u_0 is smooth on $\bar{\Omega}$.

H. Chen and X.H. Zhong [2] has shown that the system (CZ) has a unique classical positive solution (u, v) , $(x, t) \in \Omega \times [0, T_{\max})$, under the assumptions that $1 < p < +\infty$ for spatial dimension $N = 2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, where $T_{\max} = \sup\{T > 0; (u, v) \text{ exists for } x \in \Omega, t \in [0, T)\}$ denotes the maximal time of existence for the solution of (CZ). And they [3] obtained that if $\|u_0\|_{L^1} = \lambda < \frac{4\pi}{\alpha\chi}$ then $T_{\max} = +\infty$ and $\|u(t)\|_{\infty} < C$. Moreover they showed the critical number is $\frac{8\pi}{\alpha\chi}$, which determines occurrence of blowup in case that u_0 is radially symmetric.

In this paper we show the blowup results of the problem. Henceforth we can assume that $\chi = \alpha = \beta = \gamma = 1$ without loss of generality. The main theorem is as follows. The first theorem justifies the terminology blowup.

Theorem 1. *If $T_{\max} < \infty$, then*

$$(1) \quad \lim_{t \nearrow T_{\max}} \|u(t)\|_{\infty} = \infty.$$

Regarding this, we define the blowup set $\mathcal{B}$ of u

$$(2) \quad \mathcal{B} = \{x_0 \in \bar{\Omega} : \text{there exist } t_k \nearrow T_{\max} \text{ and } x_k \rightarrow x_0 \text{ such that } u(x_k, t_k) \rightarrow \infty \text{ as } k \rightarrow \infty\}$$

and call each $x_0 \in \mathcal{B}$ a blowup point. Condition $T_{\max} < +\infty$ implies $\mathcal{B} \neq \emptyset$, but more importantly, the finiteness of blowup point follows.

Theorem 2. *If $T_{\max} < \infty$, then $\#\mathcal{B}$ is finite.*

Remark 3. Keller and Segel(1970) discussed the initiation of cell aggregation as instability of the spatially homogeneous steady state. As concerned dynamics aspects of solutions, Nanjundiah [8] has posed a conjecture that cell density $u(x, t)$ will blow up in a finite time and form a δ -function singularity. Such a result is established in [11]. We expect a similar blowup mechanism for (CZ) system.

2. Preliminaries

First of all we recall the Gagliard-Nirenberg inequality in two dimensional case;

$$(3) \quad \|w\|_{L^2}^2 \leq K^2(\|w\|_{L^2}^2 + \|w\|_{L^1}^2), \quad w \in W^{1,1}(\Omega),$$

where K is a constant determined by Ω .

In this part we shall show some inequalities (3) for later use. Henceforth, we set $B_R(x_0) = \{x \in \mathbf{R}^2 : |x - x_0| < R\}$. We introduce the cut-off function φ satisfying

$$(4) \quad 0 \leq \varphi \leq 1 \quad \text{in } \mathbf{R}^2, \quad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Given $x_0 \in \Omega$, we have $0 < R' < R$ with $B_{2R}(x_0) \subset \Omega$. Then we take φ satisfying

$$(5) \quad \varphi_{x_0, R', R}(x) = \begin{cases} 1 & (x \in B_{R'}(x_0)) \\ 0 & (x \in \mathbf{R}^2 \setminus B_R(x_0)). \end{cases}$$

Given $x_0 \in \partial\Omega$, we take a smooth conformal mapping $X : B_{2R}(x_0) \cap \bar{\Omega} \mapsto \mathbf{R}^2$ satisfying $x_0 \rightarrow 0$ and

$$\begin{aligned} X(B_{2R}(x_0) \cap \Omega) &\subset \{(x_1, x_2) : x_2 > 0\} \\ X(B_{2R}(x_0) \cap \partial\Omega) &\subset \{(x_1, x_2) : x_2 = 0\} \\ X(B_{R'}(x_0) \cap \Omega) &\subset B_{\frac{1}{2}}(0), \quad X(\Omega \setminus B_R(x_0)) \subset \mathbf{R}^2 \setminus B_1(x_0) \end{aligned}$$

for $0 < R' \ll 1$. Then we have set $\varphi = \zeta(X(x))$. It holds that

$$\frac{\partial}{\partial \nu} \zeta \circ X = \frac{\partial X}{\partial \nu} \cdot (\nabla \zeta \circ X) = 0 \quad \text{on } \partial\Omega,$$

because $\frac{\partial X}{\partial \nu}$ is proportional to $(0, 1)$ on $\partial\Omega$, and such a φ satisfies (4) and (5). Then, $\psi = \left(\varphi_{x_0, R', R}\right)^6$ satisfies

$$(6) \quad \psi(x) = \begin{cases} 1 & (x \in B_{R'}(x_0)) \\ 0 & (x \in \mathbf{R}^2 \setminus B_R(x_0)), \end{cases}$$

$$0 \leq \psi \leq 1 \quad \text{in } \mathbf{R}^2, \quad \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

And it holds that

$$(7) \quad |\nabla \psi| \leq A\psi^{\frac{5}{6}}, \quad |\Delta \psi| \leq B\psi^{\frac{2}{3}},$$

where $A > 0, B > 0$ are constants determined by $0 < R' < 1 \ll 1$.

For the estimate of u , we have the following lemma from Gagliard-Nirenberg inequality and the characteristic of the cut-off function ψ .

Lemma 4. [11] *The following inequalities hold for any $s > 1$, where $C > 0$ is a constant:*

$$(8) \quad \int_{\Omega} u^2 \psi dx \leq 2K^2 \int_{B_R(x_0) \cap \Omega} u dx \int_{\Omega} u^{-1} |\nabla u|^2 \psi dx + K^2 \left(\frac{A^2}{2} + 1 \right) \|u\|_{L^1}^2$$

$$(9) \quad \int_{\Omega} u^2 dx \leq \frac{2K^2}{\log s} \int_{\Omega} (u \log u + e^{-1}) dx \int_{\Omega} u^{-1} |\nabla u|^2 dx + 2K^2 \|u\|_{L^1}^2 + 3s^2 |\Omega|$$

$$(10) \quad \int_{\Omega} u^3 \psi dx \leq \frac{72K^2}{\log s} \int_{B_R(x_0) \cap \Omega} (u \log u + e^{-1}) dx \int_{\Omega} |\nabla u|^2 \psi dx + C \|u\|_{L^1(B_R(x_0) \cap \Omega)}^3 + 10|\Omega|s^3.$$

We can obtain the estimate of v . From $u > 0, v > 0$ and the second equation, we get

$$(11) \quad \|v\|_{L^1} + \|v\|_{L^p}^p = \|u\|_{L^1}.$$

We rewrite the second equation of v

$$(12) \quad -\Delta v + v = h \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega$$

by putting $h = u - |v|^{p-1}v$. From (11) we find

$$(13) \quad \|h\|_{L^1} \leq \|u\|_{L^1} + \|v\|_{L^p}^p \leq 2\|u\|_{L^1}.$$

Then the L^1 estimate (H.Brezis and W.Strauss [1]) to the second equation of (CZ) gives

$$(14) \quad \sup_{0 \leq t \leq t_{\max}} \{ \|v(t)\|_{W^{1,q}(\Omega)} + \|v\|_r \} < C(r) \|u\|_{L^1}$$

for $q \in [1, 2)$ and $r \in [1, \infty)$.

3. Characterization of blowup point

Henceforth, we always assume that $T_{\max} < \infty$ and $\mathcal{B}$ denotes the blowup points. By using the estimate (14) and the first equation of (CZ), we lead to the following lemma.

Lemma 5. $x_0 \in \Omega$ is a blowup point of u if and only if

$$(15) \quad \limsup_{t \nearrow T_{\max}} \int_{B_R(x_0) \cap \Omega} u \log u dx = +\infty$$

for $R > 0$ sufficiently small.

Proof. We start to prove the 'if' part. Let be $0 < R \ll 1$ and $\psi = (\varphi_{x_0, R'}, R)^6$. We assume that (15) holds. Multiplying the first equation of (CZ) by $u\psi$, we have

$$(16) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \psi dx + \int_{\Omega} |\nabla u|^2 \psi dx + \int_{\Omega} u \nabla u \cdot \nabla \psi dx = \int_{\Omega} u \psi \nabla v \cdot \nabla u dx + \int_{\Omega} u^2 \nabla v \cdot \nabla \psi dx.$$

The first integral term of the right hand side in (16) is

$$(17) \quad \int_{\Omega} u \psi \nabla u \cdot \nabla v dx = -\frac{1}{2} \int_{\Omega} u^2 \Delta v \cdot \psi dx - \frac{1}{2} \int_{\Omega} u^2 \nabla v \cdot \nabla \psi dx.$$

By the second equation of (CZ), the equation (17) is treated to get

$$(18) \quad \begin{aligned} \int_{\Omega} u \psi \nabla u \cdot \nabla v dx &= \frac{1}{2} \int_{\Omega} u^2 v \psi dx - \frac{1}{2} \int_{\Omega} u^2 |v|^{p-1} v dx + \frac{1}{2} \int_{\Omega} u^3 \psi dx - \frac{1}{2} \int_{\Omega} u^2 \nabla v \cdot \nabla \psi dx \\ &\leq \frac{1}{2} \int_{\Omega} u^3 \psi dx - \frac{1}{2} \int_{\Omega} u^2 \nabla v \cdot \nabla \psi dx \\ &= \frac{1}{2} \int_{\Omega} u^3 \psi dx + \frac{1}{2} \int_{\Omega} v \nabla(u^2) \psi dx + \frac{1}{2} \int_{\Omega} u^2 v \Delta \psi dx. \end{aligned}$$

On the other hand, about the second integral term of right side in (16) we get

$$(19) \quad \int_{\Omega} u^2 \nabla \psi \cdot \nabla v dx = - \int_{\Omega} v \nabla(u^2) \cdot \nabla \psi dx - \int_{\Omega} u^2 v \Delta \psi dx.$$

Therefore we have

$$(20) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \psi dx + \int_{\Omega} |\nabla u|^2 \psi dx + \int_{\Omega} u \nabla u \cdot \nabla \psi dx = \frac{1}{2} \int_{\Omega} u^3 \psi dx - \frac{1}{2} \int_{\Omega} v \nabla(u^2) \nabla \psi dx - \int_{\Omega} u^2 v \Delta \psi dx.$$

By using Young's inequality and the estimate of ψ , we have

$$(21) \quad \frac{1}{2} \left| \int_{\Omega} u^2 v \Delta \psi dx \right| \leq \frac{1}{3} \int_{\Omega} u^3 \psi dx + \frac{B^3}{6} \|v\|_3^3,$$

$$(22) \quad \left| \int_{\Omega} u \nabla u \cdot \nabla \psi dx \right| \leq \frac{1}{4} \int_{\Omega} |\nabla u|^2 \psi dx + \frac{1}{3} \int_{\Omega} u^3 \psi dx + \frac{4A^6}{3} |\Omega|,$$

and

$$(23) \quad \frac{1}{2} \left| \int_{\Omega} v \nabla u^2 \cdot \nabla \psi dx \right| \leq \frac{1}{4} \int_{\Omega} |\nabla u|^2 \psi dx + \frac{1}{3} \int_{\Omega} u^3 \psi dx + \frac{A^6}{48} \|v\|_6^6.$$

From (20)-(23) we obtain

$$(24) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \psi dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \psi dx \leq \frac{3}{2} \int_{\Omega} u^3 \psi dx + C_1.$$

From (10) of Lemma 4 with $s \ll 1$, we have

$$(25) \quad \frac{d}{dt} \int_{\Omega} u^2 \psi dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \psi dx \leq C_1.$$

This implies

$$(26) \quad \sup_{0 \leq t < T_{\max}} \int_{\Omega} u^2 \psi dx \leq +\infty.$$

In similar way, multiplying the first equation of (CZ) by $u^2 \psi$ and integrating by parts, we have

$$(27) \quad \frac{d}{dt} \int_{\Omega} w^2 \psi dx + \frac{5}{3} \int_{\Omega} |\nabla w|^2 \psi dx \leq 4 \int_{\Omega} w^3 \psi dx + C$$

for $w = u^{\frac{3}{2}}$. In particular, we have

$$\sup_{0 \leq t < T_{\max}} \int_{B_{R'}(x_0) \cap \Omega} w \log w dx \leq +\infty,$$

$$\sup_{0 \leq t < T_{\max}} \|w\|_{L^1(B_{R'}(x_0) \cap \Omega)} \leq +\infty.$$

Therefore, taking $R'' \in (0, R')$, we can apply the argument with u , R , and $\psi = (\varphi_{x_0, R', R})^6$, replacing by w , R' , and $\psi_1 = (\varphi_{x_0, R'', R'})^6$, respectively. Similar to (26) it follows that

$$\sup_{0 \leq t < T_{\max}} \|w\|_{L^{\frac{2}{3}}(B_r(x_0) \cap \Omega)}^{\frac{2}{3}} = \sup_{0 \leq t < T_{\max}} \|u\|_{L^3(B_r(x_0) \cap \Omega)} < +\infty$$

for any $r \in (0, R)$, because $R' \in (0, R)$ and $R'' \in (0, R')$ are arbitrary. From second equation of (CZ) this implies $\sup_{0 \leq t < T_{\max}} \|v\|_{W^{2,3}(B_r(x_0) \cap \Omega)} < +\infty$ for $r' \in (0, r)$. Therefore

$$(28) \quad \sup_{0 \leq t < T_{\max}} \|v\|_{C^1(B_r(x_0) \cap \Omega)} < +\infty$$

holds for any $r \in (0, R)$. Repeating the argument once more, we have

$$(29) \quad \sup_{0 \leq t < T_{\max}} \|u\|_{L^4(B_r(x_0) \cap \Omega)} < +\infty.$$

Next we take $r' \in (0, R)$ and put $\psi_1 = (\varphi_{x_0, r', r})^6$. For $p \geq 1$ we multiply the first equation of (CZ) by $u^p \psi_1^{p+1}$ and get

$$\frac{1}{p+1} \frac{d}{dt} \int_{\Omega} (u \psi_1)^{p+1} dx = - \int_{\Omega} \nabla(u^p \psi_1^{p+1}) \cdot \nabla u dx + \int_{\Omega} u \nabla(u^p \psi_1^{p+1}) \cdot \nabla v dx = -I + II.$$

Here we have

$$I = \int_{\Omega} \left(p u^{p-1} \psi_1^{p+1} \nabla u + u^p \nabla \psi_1^{p+1} \right) \cdot \nabla u dx$$

$$\geq \frac{2}{p+1} \int_{\Omega} \left| \nabla(u \psi_1)^{\frac{p+1}{2}} \right|^2 dx - \frac{A^2(p+1)}{2} \|u_0\|_{L^1(\Omega)}^{\frac{1}{3}} \left(\int_{\Omega} (u \psi_1)^{1+\frac{2}{3}p} dx \right)^{\frac{2}{3}}.$$

On the other hand, estimate (28) means that

$$L \equiv \sup_{0 \leq t < T_{\max}} \|\nabla v\|_{L^\infty(B_r(x_0) \cap \Omega)} < +\infty.$$

We obtain

$$II \leq \frac{1}{p+1} \int_{\Omega} \left| \nabla(u \psi_1)^{\frac{p+1}{2}} \right|^2 dx + 4L^2(p+1) \int_{\Omega} (u \psi_1)^{p+1} dx + LA(p+1) \|u_0\|_{L^1(\Omega)}^{\frac{1}{6}} \left(\int_{\Omega} (u \psi_1)^{1+\frac{6}{5}p} dx \right)^{\frac{5}{6}}.$$

It holds that

$$(30) \quad \frac{d}{dt} \int_{\Omega} (u_1)^{p+1} dx \leq - \int_{\Omega} \left| \nabla(u_1)^{\frac{p+1}{2}} \right|^2 dx$$

$$+ C_3(p+1)^2 \int_{\Omega} u_1^{p+1} dx + C_3(p+1)^2 \left(\left(\int_{\Omega} u_1^{1+\frac{3}{2}p} \right)^{\frac{2}{3}} + \left(\int_{\Omega} u_1^{1+\frac{6}{5}p} \right)^{\frac{5}{6}} \right),$$

where $u_1 = u \psi_1$. Here, $C_3 > 0$ is independent of $p \geq 1$ and we can apply an iteration scheme of Moser's type. As a result, we have

$$(31) \quad \sup_{0 \leq t < T_{\max}} \|u_1\|_{L^\infty(\Omega)} \leq C \max \left\{ \left(\sup_{0 \leq t < T_{\max}} \|u_1\|_{L^4(\Omega)}^4 + 1 \right)^{\frac{1}{4}}, d \right\}$$

where $d = \|u_0\|_{L^\infty(\Omega)} + 1$. By using (29), we obtain

$$(32) \quad \sup_{0 \leq t < T_{\max}} \|u_1\|_{L^\infty(\Omega)} = \sup_{0 \leq t < T_{\max}} \|u\psi_1\|_{L^\infty(\Omega)} < +\infty,$$

or $\limsup_{t \nearrow T_{\max}} \|u\|_{L^\infty(B_{r'}(x_0) \cap \Omega)} < +\infty$. This means $x_0 \notin \mathcal{B}$.

On the other hand, the "only if" part is clear, because $x_0 \notin \mathcal{B}$ implies (15) for $0 < R \ll 1$ by the definition. The proof of Lemma 5 is complete. $\square$

4. Proof of Theorem 1

The global version of Lemma 5 is expressed as follows:

$$(33) \quad \limsup_{t \nearrow T} \int_{\Omega} u \log u dx < +\infty$$

implies

$$(34) \quad \limsup_{t \nearrow T} \|u\|_{\infty} < +\infty.$$

In fact, this is proven just by replacing the cut-off function φ with the constant function 1. If (34) follows, then general theory of parabolic equation yields that the solution u is continued after $t = T$. We shall show that (33) follows from

$$(35) \quad \liminf_{t \nearrow T} \int_{\Omega} u \log u dx < +\infty.$$

Then $t_{\max} < \infty$ holds only if

$$(36) \quad \liminf_{t \nearrow T} \int_{\Omega} u \log u dx = +\infty.$$

And in particular relation

$$(37) \quad \lim_{t \nearrow T} \|u\|_{\infty} = +\infty$$

follows.

Next we multiply $\log u$ by the first equation of (CZ). By using the second equation of (CZ), we have

$$(38) \quad \frac{d}{dt} \int_{\Omega} u \log u dx + \int_{\Omega} u^{-1} |\nabla u|^2 dx + \int_{\Omega} uv = \int_{\Omega} u^2 dx.$$

The right hand side is dominated by the second inequality (9) of Lemma 4. It follows that

$$(39) \quad \frac{d}{dt} \int_{\Omega} u \log u dx + \left(1 - \frac{2K^2}{\log s}\right) \int_{\Omega} (u \log u + e^{-1}) dx + \int_{\Omega} u^{-1} |\nabla u|^2 dx \leq C \|u_0\|_{L^1}^2 + 3s^2 |\Omega|.$$

Taking $s = s(t) = \exp \left(2K^2 \int_{\Omega} u \log u + e^{-1} dx \right) > 1$, we obtain

$$(40) \quad \frac{dJ}{dt} \leq C \|u_0\|_{L^1}^2 + 3|\Omega| \exp(4K^2 J),$$

where $J = \int_{\Omega} (u \log u + e^{-1}) dx$. Inequality (40) and $\liminf_{t \nearrow T} J(t) < \infty$ implies $\limsup_{t \nearrow T} J(t) < \infty$ by the comparison theorem for ordinary differential equation. In particular, inequality (36) implies (33). The proof is complete.

5. Proof of Theorem 2

In this part we show the finiteness of blowup points. Given $x_0 \in \bar{\Omega}$, we take $0 < R' \ll 1$ and set $\psi = \left(\psi_{x_0, R', R}\right)^6$. Let $G = G(x, y)$ be the Green's function of the operator $\mathcal{L} + 1$, so that it solves

$$(-\Delta_y + 1)G = \delta(y - x) \quad (y \in \Omega)$$

with $\frac{\partial}{\partial \nu_y} G = 0$ ($y \in \partial\Omega$) for $x \in \Omega$. From the elliptic regularity, it is extended to a smooth function on $\bar{\Omega} \times \bar{\Omega} \setminus \{(x, x) : x \in \bar{\Omega}\}$. Also the symmetry $G(x, y) = G(y, x)$ follows. Here we have the following.

Lemma 6. [11] *The function $\rho(x, y) = \nabla\psi(x)\nabla_x G(x, y) + \nabla\psi(y)\nabla_y G(x, y)$ belongs to $L^\infty(\Omega \times \Omega)$.*

And we have

Lemma 7. [11] *It holds that*

$$(41) \quad \frac{d}{dt} \int_{\Omega} (u \log u) \psi dx + \frac{1}{4} \int_{\Omega} u^{-1} |\nabla u|^2 \psi dx \leq 2 \int_{\Omega} u^2 \psi dx + C_6.$$

We start to prove Theorem 2. There is $\varepsilon_0 > 0$ such that any $x_0 \in \mathcal{B}$ and $0 < R \ll 1$ admit the estimate

$$(42) \quad \limsup_{t \nearrow T_{\max}} \int_{B_R(x_0) \cap \Omega} u dx \geq \varepsilon_0.$$

Take $R' \in (0, R)$ and set $\psi = \left(\varphi_{x_0, R', R}\right)^6$. From (41) and the first inequality (8) of Lemma 4, we have

$$(43) \quad \frac{d}{dt} \int_{\Omega} (u \log u) \psi dx + \frac{1}{4} \left(1 - 16K^2 \int_{B_R(x_0) \cap \Omega} u dx \right) \int_{\Omega} u^{-1} |\nabla u|^2 \psi dx \leq C_7.$$

Therefore, if $\limsup_{t \nearrow T_{\max}} \int_{B_R(x_0) \cap \Omega} u dx < \varepsilon_0 \equiv \frac{1}{16K^2}$, then

$$(44) \quad \limsup_{t \nearrow T_{\max}} \int_{B_{R'}(x_0) \cap \Omega} u \log u dx < \limsup_{t \nearrow T_{\max}} \int_{\Omega} (u \log u) \psi dx < +\infty.$$

This implies $x_0 \notin \mathcal{B}$ by Lemma 5, that is a contradiction.

Next we show that

$$(45) \quad \left| \frac{d}{dt} \int_{\Omega} u \psi dx \right| < B \|u_0\|_{L^1} + \frac{1}{2} \|\rho\|_{L^\infty(\Omega \times \Omega)} \|u_0\|_{L^1}^2.$$

The first equation of (CZ) gives

$$(46) \quad \frac{d}{dt} \int_{\Omega} u \psi dx = \int_{\Omega} u \Delta \psi dx + \int_{\Omega} u \nabla v \cdot \nabla \psi dx.$$

The second integral term of the right side in (46) is

$$(47) \quad \begin{aligned} \int_{\Omega} u \nabla v \cdot \nabla \psi dx &= \int_{\Omega} \int_{\Omega} u(x, t) \nabla \psi(x) \cdot \nabla_x G(x, y) u(y, t) dy dx \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(x, y) u(x, t) u(y, t) dx dy. \end{aligned}$$

From Lemma 6 we have

$$\left| \int_{\Omega} \int_{\Omega} \rho(x, y) u(x, t) u(y, t) dx dy \right| \leq \|\rho\|_{\infty(\Omega \times \Omega)} \|u_0\|_{L^1}^2.$$

Since, it is obvious that $\int_{\Omega} u \Delta \psi dx \leq B \|u_0\|_{L^1}$, we get inequality (45). This means that the value

$$(48) \quad \lim_{t \nearrow T_{\max}} \int_{\Omega} u \psi dx = \int_{\Omega} u_0(x) \psi dx + \int_0^{T_{\max}} \left(\frac{d}{dt} \int_{\Omega} u(\cdot, t) \psi dx \right) dt$$

exists. Because $0 < R \ll 1$ is arbitrary, (4) and inequality (42) are improved as

$$(49) \quad \liminf_{t \nearrow T_{\max}} \int_{B_R(x_0) \cap \Omega} u dx \geq \lim_{t \nearrow T_{\max}} \int_{\Omega} u \psi dx \geq \limsup_{t \nearrow T_{\max}} \int_{B_{R'}(x_0) \cap \Omega} u dx \geq \varepsilon_0.$$

Therefore, by using the L^1 norm preserving $\|u\|_{L^1} = \|u_0\|_{L^1}$ ($0 \leq t \leq T_{\max}$), we conclude

$$(50) \quad \#B \leq \frac{\|u_0\|_{L^1}}{\varepsilon_0} < \infty.$$

The proof of Theorem 2 is complete.

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