

Inertial Sets for Phase Transition Models Induced by the Variational Principles

岐阜大・工 伊藤昭夫 (AKIO ITO)

0. Introduction

We consider one-dimensional non-isothermal phase separation model with constraints in the following form, denoted by (PSC):={ (0.1)-(0.6) }:

$$e := \theta + \lambda(w) \quad \text{in } Q := \Omega \times (0, +\infty), \quad (0.1)$$

$$e_t - (\alpha(\theta))_{xx} + \nu\theta = f(x) \quad \text{in } Q, \quad (0.2)$$

$$w_t - \{-\kappa w_{xx} + g(w) + \beta(w) - \alpha(\theta)\lambda'(w)\}_{xx} = 0 \quad \text{in } Q, \quad (0.3)$$

$$\pm[\alpha(\theta)]_x(\pm L, t) + n_0\alpha(\theta(\pm L, t)) = h_{\pm} \quad \text{for } t > 0, \quad (0.4)$$

$$w_x(\pm L, t) = w_{xxx}(\pm, t) = 0 \quad \text{for } t > 0, \quad (0.5)$$

$$\theta(x, 0) = \theta_0(x), \quad w(x, 0) = w_0(x) \quad \text{in } \Omega. \quad (0.6)$$

Here, $\Omega := (-L, L)$ with a given finite number $L > 0$; α and β are non-decreasing and smooth functions; λ and g are sufficiently smooth functions; λ' is the derivative of λ ; ν, κ and n_0 are positive constants; $f, h_{\pm}, \theta_0$ and w_0 are given data.

Physically, this model describes the non-isothermal phase separation phenomena of the binary alloys composed by two components A and B. The original model with $\nu = 0$ was introduced by Penrose and Fife [13] and in it θ represents the absolute temperature and w the conserved order parameter. Actually, we see from the kinetic equation (0.3) and the boundary conditions (0.5) of w that

$$\frac{d}{dt} \int_{-L}^L w(t, x) dx = 0 \quad \text{for any } t > 0,$$

that is,

$$\int_{-L}^L w(x, t) dx = \int_{-L}^L w_0(x) dx =: m_0 \quad \text{for any } t \geq 0.$$

Roughly speaking, in our model the mass quantity is conserved. From this point of view, throughout this paper it is convenient to introduce a new function v by

the relation $v := w - m_0$ and consider this function v instead of w . Here, you note that the fact

$$\int_{-L}^L v(x, t) dx = 0 \quad \text{for any } t \geq 0.$$

The typical examples of α and β are

$$\alpha(\theta) := -\frac{1}{\theta} \quad \text{for any } \theta > 0$$

and

$$\beta(w) := k_0 \log \frac{1+w}{1-w} \quad \text{for any } w \in (-1, 1) \quad \text{with some constant } k_0 > 0.$$

Since the domain of β is restricted in the interval $(-1, 1)$, this model is a kind of the phase separation models with constraints. For these models, there have already been some works which guarantees the global existence and uniqueness of solutions (cf. [2], [9], [14]). But, in these papers they assumed that λ is convex and this assumption is essential.

Recently, in [12] we discussed the weak well-posedness (i.e. (global) existence, uniqueness and weakly continuous dependence upon the data of the solution) without the assumption that λ is convex for the case $\nu \geq 0$ and in [7] we constructed the global attractor for the case $\nu > 0$.

But, it is not sufficient to discuss the asymptotic behavior as $t \rightarrow +\infty$ because we have at least two questions for the global attractor. One is to investigate the structure of the global attractor. The other is to give the estimate of the speed under which any solution is attracted to the global attractor. In order to give the answers to these questions we use the notion of inertial set (sometimes it is called the exponential attractor), which was established by Eden, Foias, Nicolaenko and Temam in [3], for the semigroup associated with our system. In consequence, we proved in this paper that the global attractor has a finite fractal dimension and the inertial set uniformly attracts all solutions starting from some compact set.

Notation. We fix a positive number L , and put $\Omega := (-L, L)$. For simplicity we use the following notation:

(1) In $H := L^2(\Omega)$, the usual inner product is denoted by $(\cdot, \cdot)_H$ and the norm by $|\cdot|_H$.

(2) $V := H^1(\Omega)$ is the Hilbert space with the inner product $(\cdot, \cdot)_V$ given by

$$(v, z)_V := (v_x, z_x)_H + n_0 \{v(L)z(L) + v(-L)z(-L)\} \quad \text{for any } v, z \in V$$

and the norm $|\cdot|_V := (\cdot, \cdot)_V^{\frac{1}{2}}$. The dual space of V is denoted by V^* , and the duality pair between V^* and V is denoted by $\langle \cdot, \cdot \rangle_{V^*, V}$. Furthermore, the duality mapping $F : V \rightarrow V^*$ is defined by

$$\langle Fv, z \rangle_{V^*, V} = (v, z)_V \quad \text{for any } v, z \in V.$$

- (3) V^* is the Hilbert space equipped with the inner product $(\cdot, \cdot)_{V^*}$ given by

$$(v, z)_{V^*} := \langle v, F^{-1}z \rangle_{V^*, V} (= \langle z, F^{-1}v \rangle_{V^*, V}) \quad \text{for any } v, z \in V^*.$$

The corresponding norm $|v|_{V^*}$ is given by $|F^{-1}v|_V$.

- (4) H_0 is the subspace of H defined by

$$H_0 := \left\{ z \in H; \int_{\Omega} z(x) dx = 0 \right\}.$$

Then, H_0 is the Hilbert space by succeeding to the inner product of H , that is, the inner product $(\cdot, \cdot)_{H_0}$ in H_0 is given by

$$(v, z)_{H_0} := (v, z)_H \quad \text{for any } v, z \in H_0.$$

Moreover, we define a projection operator π_0 from H onto H_0 by

$$\pi_0[z](x) := z(x) - \frac{1}{2L} \int_{\Omega} z(y) dy \quad \text{for any } x \in \Omega.$$

- (5) $H^1(\Omega)$, $H^2(\Omega)$ and $H^3(\Omega)$ are the usual Sobolev spaces; especially, we distinguish $H^1(\Omega)$ from V because of the difference of the inner products throughout this paper.
- (6) $V_0 := H_0 \cap H^1(\Omega)$ is the Hilbert space with the norm $|\cdot|_{V_0}$ and the inner product $(\cdot, \cdot)_{V_0}$ given by

$$(v, z)_{V_0} := (v_x, z_x)_H \quad \text{for any } v, z \in V_0.$$

The dual space of V_0 is denoted by V_0^* , and the duality pair between V_0^* and V_0 is denoted by $\langle \cdot, \cdot \rangle_{V_0^*, V_0}$. Furthermore, the duality mapping $F_0 : V_0 \rightarrow V_0^*$ is defined by

$$\langle F_0 v, z \rangle_{V_0^*, V_0} = (v, z)_{V_0} \quad \text{for any } v, z \in V_0.$$

(7) V_0^* is the Hilbert space equipped with inner product $(\cdot, \cdot)_{V_0^*}$ given by

$$(v, z)_{V_0^*} := \langle v, F_0^{-1} z \rangle_{V_0^*, V_0} (= \langle z, F_0^{-1} v \rangle_{V_0^*, V_0}) \quad \text{for any } v, z \in V_0^*.$$

The corresponding norm $|v|_{V_0^*}$ is also given by $|F_0^{-1} v|_{V_0}$.

(8) $\mathcal{H} := V^* \times V_0^*$, which is the Hilbert space with the inner product

$$(U, \bar{U})_{\mathcal{H}} := (e, \bar{e})_{V^*} + (v, \bar{v})_{V_0^*} \quad \text{for any } U := [e, v], \bar{U} := [\bar{e}, \bar{v}] \in \mathcal{H}.$$

(9) $\mathcal{E} := H \times V_0$, which is the Hilbert space with the inner product

$$(U, \bar{U})_{\mathcal{E}} := (e, \bar{e})_H + (v, \bar{v})_{V_0} \quad \text{for any } U := [e, v], \bar{U} := [\bar{e}, \bar{v}] \in \mathcal{E}.$$

(10) By Δ_N we mean the Laplacian Δ with homogeneous Neumann boundary condition, namely, $\Delta v = v_{xx}$ in Ω with $v_x(\pm L) = 0$; $-\Delta_N$ is the maximal monotone operator in H_0 with the domain $D(-\Delta_N) := \{v \in H^2(\Omega) \cap H_0; v_x(\pm L) = 0\}$.

1. Known results

In this section, let us recall some results established in [7, 12].

Throughout this paper we consider our system under the following assumptions:

(A1) α is a strictly increasing function of C^2 -class from $(0, +\infty)$ onto $(-\infty, 0)$ such that

$$|\alpha(r)| \geq \frac{c_0}{r} \quad \text{for any } r > 0$$

for some suitable positive constant c_0 and

$$\lim_{r \uparrow +\infty} \alpha(r) = 0, \quad \lim_{r \downarrow 0} \alpha(r) = -\infty.$$

(A2) β is a non-decreasing function of C^2 -class from $D(\beta) := (-1, 1)$ onto R such that

$$\lim_{r \downarrow -1} \beta(r) = -\infty, \quad \lim_{r \uparrow 1} \beta(r) = +\infty.$$

We fix a non-negative primitive $\hat{\beta}$ of β : note $(-1, 1) \subset D(\hat{\beta}) \subset [-1, 1]$.

(A3) λ is a C^3 -function on R with compact support.

(A4) g is a C^2 -function on R with compact support; we fix a primitive $\hat{g}$ of g such that $\hat{g} \geq 0$ on R .

(A5) $m_0 \in (-1, 1)$, $\nu > 0$, $\kappa > 0$ and $n_0 > 0$.

(A6) $f \in H$ and $h_{\pm}$ are negative constants.

(A7) $\theta_0 \in H$, $v_0 \in V_0$.

Now, we define a solution to (PSC):={ (0.1)-(0.5) } in a weak variational sense.

Definition 1.1. Let $0 < T < +\infty$, $m_0 \in (-1, 1)$ and define $f^* \in V^*$ by

$$\langle f^*, z \rangle_{V^*, V} := (f, z)_H + h_+ z(L) + h_- z(-L) \quad \text{for any } z \in V.$$

Moreover, we define a new function v by $v := w - m_0$. Then, we call a couple of functions $[e, v]$ a solution to (PSC) on $[0, T]$ if the following properties (i)-(iv) are satisfied:

(i) $e := \theta + \lambda(v + m_0) \in W^{1,2}(0, T; V^*) \cap L^\infty(0, T; H) (\subset C_w([0, T]; H))$.

(ii) $v \in W^{1,2}(0, T; V_0^*) \cap L^\infty(0, T; V_0) (\subset C_w([0, T]; V_0))$.

(iii) $\alpha(\theta) \in L^2(0, T; V)$ and

$$e_t(t) + F\alpha(\theta(t)) + \nu\theta(t) = f^* \quad \text{in } V^* \text{ for a.e. } t \in (0, T).$$

(iv) $\beta(v + m_0) \in L^2(0, T; H)$ and

$$\begin{aligned} F_0^{-1}v_t(t) - \kappa\Delta_N v(t) + \pi_0[g(v(t) + m_0) + \beta(v(t) + m_0) - \alpha(\theta(t))\lambda'(v(t) + m_0)] \\ = 0 \quad \text{in } H_0 \quad \text{for a.e. } t \in (0, T). \end{aligned}$$

Given initial data $e_0 \in H$ and $v_0 \in V_0$, $[e, v]$ is called a solution to the Cauchy problem (PSC; e_0, v_0):={ (0.1)-(0.6) } on $[0, T]$ if it is a solution to (PSC) on $[0, T]$ with initial data $e(0) = e_0$ and $v(0) = v_0$.

Moreover, $[e, v]$ is called a global solution to (PSC) if it is a solution to (PSC) on $[0, T]$ for any finite time $T > 0$.

Under these situations, we relate the results in [7, 12]. To do so, first of all, we introduce a functional Φ on $H \times H_0$ in the following way:

$$\Phi(e, v) := j(e - \lambda(v + m_0)) + \Psi(v) \quad \text{for any } [e, v] \in H \times H_0,$$

where

$$j(z) := \begin{cases} \int_{\Omega} \hat{\alpha}(z(x)) dx, & \hat{\alpha}(z) \in L^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\Psi(v) := \begin{cases} \frac{\kappa}{2}|v|_{V_0}^2 + \int_{\Omega} \hat{g}(v(x) + m_0)dx + \int_{\Omega} \hat{\beta}(v(x) + m_0)dx, \\ \quad \text{if } v \in V_0 \text{ with } \hat{\beta}(v + m_0) \in L^1(\Omega), \\ +\infty, \quad \text{otherwise;} \end{cases}$$

note that Ψ is non-negative on H_0 by (A2) and (A4). Now, we define a subset D of $\mathcal{E}$ by

$$D := \{[e, v] \in \mathcal{E}; \Phi(e, v) < +\infty\}.$$

Then, according to the results of [7, 12] we see that for each $m_0 \in (-1, 1)$ and $[e_0, v_0] \in D$ the Cauchy problem (PSC; e_0, v_0) has one and only one global solution $[e, v]$. Furthermore for any two initial data $[e_{0i}, v_{0i}] \in D$ the global solutions $[e_i, v_i]$ to (PSC; e_{0i}, v_{0i}) ($i = 1, 2$) satisfy

$$\begin{aligned} & |e_2(t) - e_1(t)|_{V^*}^2 + |v_2(t) - v_1(t)|_{V_0^*}^2 + C_1 \int_s^t |v_2(\tau) - v_1(\tau)|_{V_0}^2 d\tau \\ & \leq \exp \left(C_2 \int_s^t (1 + |\alpha(\theta_1(\tau))|_V^2 + |\alpha(\theta_2(\tau))|_V^2) d\tau \right) \\ & \quad \times (|e_2(s) - e_1(s)|_{V^*}^2 + |v_2(s) - v_1(s)|_{V_0^*}^2) \\ & \quad \text{for any } s, t \text{ with } 0 \leq s \leq t < +\infty. \end{aligned} \quad (1.1)$$

for some suitable positive constants C_i ($i = 1, 2$), which are independent of initial data in D .

Hence, we can define a dynamical system $\{S(t)\} := \{S(t); t \geq 0\}$ on D associated with (PSC) by for each $[e_0, v_0] \in D$, $[e(t), v(t)] = S(t)[e_0, v_0]$ is a global solution to (PSC; e_0, v_0).

Moreover, we have already obtained the following properties (S1)-(S6) as well as the above facts:

(S1) $S(0) = I$ on D .

(S2) $S(t+s) = S(t)S(s)$ for any $t, s \geq 0$.

(S3) D is positively invariant under $\{S(t)\}_{t \geq 0}$, namely, $S(t)D \subset D$ for any $t \geq 0$.

(S4) If $[e_{0n}, v_{0n}] \in D$, $[e_{0n}, v_{0n}] \rightarrow [e, v]$ in $\mathcal{H}$ and $\{\Phi(e_{0n}, v_{0n})\}$ is bounded, then $S(\cdot)[e_{0n}, v_{0n}] \rightarrow S(\cdot)[e_0, v_0]$ in $C([0, T]; \mathcal{H})$ for every $0 < T < +\infty$. Moreover, if $e_{0n} \rightarrow e_0$ weakly in H , then $S(\cdot)[e_{0n}, v_{0n}] \rightarrow S(\cdot)[e_0, v_0]$ in $C_w([0, T]; \mathcal{E}) \cap C_w([0, T]; H \times D(-\Delta_N))$ for every $0 < \delta < T < +\infty$.

Before stating the statement (S5) and (S6), we have to prepare a functional J with some properties. For each $\eta > 0$ let us consider a functional J_η on D which is defined by

$$J_\eta(e, v) := \Phi(e, v) - \langle e, \alpha(\theta_0) \rangle_{V^*, V} + \eta |e|_H^2 + C_3(\eta) \quad \text{for any } [e, v] \in D,$$

where a pair $[\theta_0, \alpha(\theta_0)] \in H \times V$ is a unique pair satisfying

$$(\alpha(\theta_0), z)_V + \nu(\theta_0, z)_H = \langle f^*, z \rangle_{V^*, V}$$

and $C_3(\eta)$ are chosen, depending only on η , so that

$$J_\eta(e, v) \geq \frac{\eta}{2} |e - \lambda(v + m_0)|_H^2 \quad \text{for any } [e, v] \in D.$$

This is a Lyapunov-like functional for our system. Actually, the following inequality of Gronwall's type holds: there exist $\eta_1 > 0$ and $N_0 > 0$, which are independent of the initial data $[e_0, v_0] \in D$, such that

$$\frac{d}{dt} J(e(t), v(t)) + \eta_1 J(e(t), v(t)) \leq N_0 \quad \text{for a.e. } t \geq 0, \quad (1.2)$$

where $J := J_{\eta_1}$ and $[e(t), v(t)] = S(t)[e_0, v_0]$ for any $[e_0, v_0] \in D$; for the proof of (1.2) we leave to the paper [7] and it is omitted in this paper. But, we emphasized that we used the positiveness of ν to prove (1.2), namely, $\nu(> 0)$ plays an important role to obtain the above inequality.

Now, we state (S5) and (S6):

(S5) (Global estimate) For each finite time $T > 0$ and bounded subset $B(\subset \mathcal{E})$ with $\sup_{[e, v] \in B} J(e, v) < +\infty$ there exists a positive constant $T(B, T)$, depends upon B and T , such that

$$\begin{aligned} & |t^{\frac{1}{2}} v_t|_{L^\infty(0, T; V_0^*)} + |t^{\frac{1}{2}} v_t|_{L^2(0, T; V_0)} + |t^{\frac{1}{2}} \alpha(\theta)|_{L^\infty(0, T; V)} + |t^{\frac{1}{2}} v|_{L^\infty(0, T; H^2(\Omega))} \\ & + |t^{\frac{1}{2}} \beta(v + m_0)|_{L^\infty(0, T; H)} \leq M(B, T) \end{aligned}$$

for any solutions $[e(\cdot), v(\cdot)]$ with initial datum $[e_0, v_0] \in D$.

(S6) [7; Lemma 4.2] (Existence of an absorbing set) There exists a subset B_0 of D satisfying the following properties (i)-(iii):

- (i) B_0 is weakly compact in $\mathcal{E}$ and $\sup_{[e, v] \in B_0} J(e, v) < +\infty$.
- (ii) B_0 is arcwise connected in the weak topology of $\mathcal{E}$.

- (iii) For each subset B of D with $\sup_{[e,v] \in B} J(e,v) < +\infty$ there exists a finite time $t_B > 0$ such that

$$S(t)B \subset B_0 \quad \text{for any } t \geq t_B.$$

As a result of (S1)-(S6) we have the following theorem.

Theorem 1.1. (cf. [7; Theorem 3.1]; Existence of a global attractor) *Assume that (A1)-(A6) hold. Then the set*

$$\mathcal{A} := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B_0}^{V^* \times V_0}$$

satisfies (i)-(iii) below, where $\overline{X}^{V^ \times V_0}$ denotes the closure of X in $V^* \times V_0$:*

- (i) $\mathcal{A}$ is compact and connected in the weak topology of $H \times (H^2(\Omega) \cap H_0)$.
- (ii) $\mathcal{A}$ is invariant under $\{S(t)\}$, namely, $S(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$.
- (iii) for each subset $B (\subset D)$ with $\sup_{[e,v] \in B} J(e,v) < +\infty$

$$\lim_{t \rightarrow +\infty} \text{dist}_{V^* \times V_0}(S(t)B, \mathcal{A}) = 0,$$

where for any subsets X, Y of $V^ \times V_0$*

$$\text{dist}_{V^* \times V_0}(X, Y) := \sup_{x \in X} \{ \inf_{y \in Y} |x - y|_{V^* \times V_0} \}.$$

Throughout this paper, we call $\mathcal{A}$ a global attractor for the dynamical system $\{S(t)\}$ on D associated with (PSC).

2. Main Theorem

We consider our system (PSC) under the same assumptions and use the same notation as in the previous section.

Before stating our main theorem in this paper, we introduce some notions, which are important to investigate the large-time behavior of solutions to (PSC).

Definition 2.1. Let X be compact in $\mathcal{H}$ and $\mathcal{M}$ is a subset of X . Then, $\mathcal{M}$ is called an inertial set in X for $\{S(t)\}$, if $\mathcal{M}$ has the following properties (IS1)-(IS4):

(IS1) $\mathcal{A} \subset \mathcal{M} \subset X$.

(IS2) $\mathcal{M}$ has a finite fractal dimension.

(IS3) $\mathcal{M}$ is positively invariant under $\{S(t)\}$, that is,

$$S(t)\mathcal{M} \subset \mathcal{M} \quad \text{for any } t \geq 0.$$

(IS4) There exist positive constants c_1 and c_2 such that

$$\text{dist}_{\mathcal{H}}(S(t)X, \mathcal{M}) \leq c_1 e^{-c_2 t} \quad \text{for any } t \geq 0.$$

Remark 2.1. From (IS1) and (IS2), we see that the fractal dimension of $\mathcal{A}$ is also finite. Moreover, by using the fact that the Hausdorff dimension is less than or equal to the fractal dimension it follows that the Hausdorff dimension of $\mathcal{A}$ is finite, too.

Definition 2.2. Let T be a Lipschitz continuous mapping on X with respect to the strong topology of $\mathcal{H}$. Then, we call that T has a squeezing property on X with respect to the strong topology of $\mathcal{H}$, if there is an orthogonal projection P , with finite rank, such that

$$|TU_2 - TU_1|_{\mathcal{H}} \leq \frac{1}{8}|U_2 - U_1|_{\mathcal{H}}$$

holds for any pair of $U_1, U_2 \in X$ satisfying

$$|P(TU_2 - TU_1)|_{\mathcal{H}} \leq |(I - P)(TU_2 - TU_1)|_{\mathcal{H}}.$$

Our main theorem is follows.

Theorem 2.1. *There exist a compact subset $\mathcal{X}$ of $\mathcal{H}$ and a finite time t^* such that $S^* := S(t^*)$ has a squeezing property on $\mathcal{X}$ as well as the Lipschitz continuity on $\mathcal{X}$ with respect to the strong topology of $\mathcal{H}$.*

And by applying the results of Eden, Foias, Nicolaenko and Temam (cf. [3]) we get the following corollary to Theorem 2.1.

Corollary to Theorem 2.1. *There exists an inertial set $\mathcal{M}$ in $\mathcal{X}$ for $\{S(t)\}$ and the fractal dimension of $\mathcal{M}$ is dominated by the number*

$$N_* \max \left\{ 1, \frac{\log(16\text{Lip}(S^*)) + 1}{\log 2} \right\},$$

where $\text{Lip}(S^*)$ is a Lipschitz constant of S^* and N_* is the rank of the orthogonal projection $P := P^*$ appearing in the squeezing property of S^* .

3. Proof of Theorem 2.1

In this section, we give some lemmas, which are tools to prove Theorem 2.1. But, we will not write their proofs and they are written in [5] in detail.

As the first lemma, we give the global uniform estimates of global solutions starting from the absorbing set B_0 given in (S6).

Lemma 3.1. *For any global solution $[e(\cdot), v(\cdot)] := [\theta(\cdot) + \lambda(v(\cdot) + m_0), v(\cdot)]$ with initial datum $[e_0, v_0] \in B_0$, the following estimates hold:*

- (i) *There exists a positive constant R_0 , depending upon the absorbing set B_0 , such that*

$$|v_t(t)|_{V_0^*} + |v(t)|_{H^2(\Omega)} + |\alpha(\theta(t))|_V + |\beta(v(t) + m_0)|_H \leq R_0$$

for any $t \geq t_{B_0} + 1$

and

$$\sup_{t \geq t_{B_0} + 1} |v_t|_{L^2(t, t+3; V_0)} \leq R_0,$$

where t_{B_0} is a finite time satisfying

$$S(t)B_0 \subset B_0 \quad \text{for any } t \geq t_{B_0}.$$

- (ii) (cf. [6; Lemma 3.1]) *There exist positive and finite constants θ_* and θ^* and a finite time $t_1 (> t_{B_0} + 1)$ such that*

$$\theta_* \leq \theta := e - \lambda(v + m_0) \leq \theta^* \quad \text{on } [-L, L] \times [t_1, +\infty).$$

- (iii) *There exists a positive constant ε_0 such that*

$$-1 + \varepsilon \leq v + m_0 \leq 1 - \varepsilon_0 \quad \text{on } [-L, L] \times [t_1, +\infty),$$

where t_1 is the same number as in (ii).

It is easy from the global estimate (S5) to prove (i). And the proves of (ii) and (iii) are quite similar to those of Lemma 3.1 in [6]. We will omit them in this paper.

Remark 3.1. From Lemma 3.1 without loss of generality we may assume that α is a bi-Lipschitz strictly increasing function in C^2 -class with $\alpha'' \in L_{loc}^\infty(R)$ and β is a non-decreasing continuous function on $[-1, 1]$ as well as continuous on R in C^2 -class with compact support, respectively, as long as we consider the solutions to (PSC) on $[t_1, +\infty)$ with the initial data in B_0 . Moreover, we see that any solution $[e(\cdot), v(\cdot)]$ to (PSC) with initial datum $[e_0, v_0] \in B_0$ has the following regularities:

$$e \in W_{loc}^{1,2}([t_1, +\infty); H) \cap L^\infty([t_1, +\infty); V), \quad \alpha(\theta) \in L_{loc}^2([t_1, +\infty); H^2(\Omega)),$$

$$v \in L^\infty([t_1, +\infty); H^3(\Omega)).$$

From now we assume that α and β satisfy the properties in Remark 3.1, respectively.

In the next lemma, we will give some global uniform estimate with respect to θ_t and v_t . And this lemma plays a quite important role to prove Theorem 2.1.

Lemma 3.2. *Let $[e(\cdot), v(\cdot)]$ be any solution to (PSC) with initial datum $[e_0, v_0] \in B_0$. Then, there exists a positive constant R_3 such that for each $s \geq t_1$ and $T > 0$*

$$\begin{aligned} & \sup_{s \leq t \leq s+T} \{(t-s)|\theta_t(t)|_H^2\} + \sup_{s \leq t \leq s+T} \{(t-s)|v_t(t)|_{V_0}^2\} \\ & + \int_s^{s+T} (t-s)|(\alpha(\theta))_t(t)|_V^2 dt + \int_s^{s+T} (t-s)|v_{tt}(t)|_{V_0^*}^2 dt \leq R_3 \\ & \text{for any } [e_0, v_0] \in B_0. \end{aligned}$$

Proof. To prove this lemma we consider the following system: for each $\mu \in (0, 1)$, $s \in [t_1, +\infty)$ and $T > 0$

$$e_t^{\mu,s} - (\alpha(\theta^{\mu,s}))_{xx} + \nu\theta^{\mu,s} = f(x) \quad \text{in } Q_{s,T} := (-L, L) \times (s, s+T), \quad (3.1)$$

$$\begin{aligned} & v_t^{\mu,s} - \{\mu v_t^{\mu,s} - \kappa v_{xx}^{\mu,s} + g(v^{\mu,s} + m_0) + \beta(v^{\mu,s} + m_0) \\ & - \alpha(\theta^{\mu,s})\lambda'(v^{\mu,s} + m_0)\}_{xx} = 0 \end{aligned} \quad (3.2)$$

$$\text{in } Q_{s,T},$$

$$\pm(\alpha(\theta^{\mu,s}))_x(\pm L, t) + n_0\alpha(\theta^{\mu,s})(\pm L, t) = h_\pm \quad \text{for any } t \in (s, s+T), \quad (3.3)$$

$$v_x^{\mu,s}(\pm L, t) = v_{xx}^{\mu,s}(\pm L, t) = 0 \quad \text{for any } t \in (s, s+T), \quad (3.4)$$

$$e^{\mu,s}(s) = e(s) \quad v^{\mu,s}(s) = v(s), \quad (3.5)$$

where $[e(s), w(s)]$ is any solution to (PSC) at time $t = s$ with initial datum in B_0 . For this system we have already known the following results (cf. [9, 12]):

(1) The above system has one and only one solution $[e^{\mu,s}(\cdot), v^{\mu,s}(\cdot)]$ on $[s, s+T]$ satisfying the following properties:

- (i) $e^{\mu,s} \in W^{1,2}(s, s+T; H) \cap L^\infty(s, s+T; V)$.
- (ii) $v^{\mu,s} \in L^\infty(s, s+T; H^2(\Omega))$, $v_t^{\mu,s} \in C([s, s+T]; H_0)$,
 $v_t^{\mu,s} \in L^\infty(s, s+T; V_0)$, $v_{tt}^{\mu,s} \in L^2(s, s+T; H_0)$.

(iii) $\alpha(\theta^{\mu,s}) \in L^\infty(s, s+T; V)$ and

$$e_t^{\mu,s}(t) + F\alpha(\theta^{\mu,s}(t)) + \nu\theta^{\mu,s}(t) = f^* \quad \text{in } V^* \text{ for a.e. } t \in [s, s+T].$$

(iv) $\beta(v^{\mu,s} + m_0) \in L^\infty(s, s+T; H)$ and

$$\begin{aligned} & (F_0^{-1} + \mu I)v_t^{\mu,s}(t) - \kappa\Delta_N v^{\mu,s}(t) \\ & + \pi_0[g(v^{\mu,s}(t) + m_0) + \beta(v^{\mu,s}(t) + m_0) - \alpha(\theta^{\mu,s}(t))\lambda'(v^{\mu,s}(t) + m_0)] = 0 \\ & \text{in } H_0 \quad \text{for a.e. } t \in [s, s+T]. \end{aligned}$$

(2) For each $T > 0$ there exists a positive constant $R_1 := R_1(T)$ such that

$$\begin{aligned} & |e^{\mu,s}|_{W^{1,2}(s,s+T;V^*)} + |v^{\mu,s}|_{W^{1,2}(s,s+T;V_0^*)} + \mu^{\frac{1}{2}}|v^{\mu,s}|_{W^{1,2}(s,s+T;H_0)} \\ & + |j(\theta^{\mu,s})|_{L^\infty(s,s+T)} + |v^{\mu,s}|_{L^\infty(0,T;V_0)} + |\alpha(\theta^{\mu,s})|_{L^2(s,s+T;V)} \\ & + |v^{\mu,s}|_{L^2(s,s+T;H)} + |\beta(v^{\mu,s} + m_0)|_{L^2(s,s+T;H)} \leq R_1 \end{aligned}$$

for any $\mu \in (0, 1]$, $s \geq t_1$ and $[e_0, v_0] \in B_0$.

(3) For each $T > 0$ there exists a positive constant $R_2 := R_2(T)$ such that

$$\begin{aligned} & |\theta_t^{\mu,s}|_{L^2(s,s+T;H)} + \sup_{s \leq t \leq s+T} |\alpha(\theta^{\mu,s}(t))|_V + \sup_{s \leq t \leq s+T} |v_t^{\mu,s}(t)|_{V_0^*} \\ & + \mu^{\frac{1}{2}} \sup_{s \leq t \leq s+T} |v_t^{\mu,s}(t)|_{H_0} + |v_t^{\mu,s}|_{L^2(s,s+T;V_0)} \leq R_2 \end{aligned}$$

for any $\mu \in (0, 1]$, $s \geq t_1$ and $[e_0, v_0] \in B_0$.

From these estimates, we note that there exist positive constants R_i ($4 \leq i \leq 6$) such that

$$\begin{aligned} & R_4 \leq \alpha'(\theta^{\mu,s}) \leq R_5 \quad \text{on } [-L, L] \times [s, s+T], \\ & -R_6 \leq \theta^{\mu,s} \leq R_6 \quad \text{on } [-L, L] \times [s, s+T] \end{aligned}$$

for any $\mu \in (0, 1]$, $s \geq t_1$ and $[e_0, v_0] \in B_0$. And we put

$$R_7 := \max_{|r| \leq R_6} |\alpha''(r)| + \max_{|r| \leq R_6} |\alpha(r)|.$$

Now, we use the above fact and calculate $(d/dt)(3.1) \times (\alpha(\theta^{\mu,s}))_t(t)$ in $H \times H$ to obtain

$$\frac{d}{dt} \int_{-L}^L \frac{\alpha'(\theta^{\mu,s}(x, t))|\theta_t^{\mu,s}(x, t)|^2}{2} dx + (1 - \varepsilon)|(\alpha(\theta^{\mu,s}))_t(t)|_V^2$$

$$\begin{aligned}
& (\lambda'(v^{\mu,s}(t) + m_0)v_{tt}^{\mu,s}(t), (\alpha(\theta^{\mu,s}))_t(t))_H \\
& \leq \frac{c_0 R_7^2}{2\varepsilon R_4^4} \left(\int_{-L}^L \alpha'(\theta^{\mu,s}(x, t)) |\theta_t^{\mu,s}(x, t)|^2 dx \right) \\
& \quad \times \left(\int_{-L}^L \frac{\alpha'(\theta^{\mu,s}(x, t)) |\theta_t^{\mu,s}(x, t)|^2}{2} dx \right) \\
& \quad + c_1^2 L(\lambda') R_5 |v_t^{\mu,s}(t)|_{V_0}^2 \int_{-L}^L |\theta_t^{\mu,s}(x, t)|^2 dx \\
& \quad \text{for a.e. } t \in [s, s+T]
\end{aligned} \tag{3.6}$$

for some suitable positive constants c_1 and c_2 .

Secondly, we take the inner product between $(d/dt)(3.2)$ and $v_{tt}^{\mu,s}(t)$ in H_0 to obtain

$$\begin{aligned}
& |v_{tt}^{\mu,s}(t)|_{V_0^*}^2 + \mu |v_{tt}^{\mu,s}(t)|_{H_0}^2 + \frac{d}{dt} \left\{ \frac{\kappa}{2} |v_t^{\mu,s}(t)|_{V_0}^2 \right\} \\
& \leq 3\varepsilon' |v_{tt}^{\mu,s}(t)|_{V_0^*}^2 + R_8 |v_t^{\mu,s}(t)|_{V_0}^2 + (\lambda'(v^{\mu,s}(t) + m_0)v_{tt}^{\mu,s}(t), (\alpha(\theta^{\mu,s}))_t(t))_H \\
& \quad \text{for a.e. } t \in [s, s+T],
\end{aligned} \tag{3.7}$$

where R_8 is a suitable positive constant, which is independent of $\mu \in (0, 1]$, $s \geq t_1$ and $[e_0, v_0] \in B_0$.

Now we choose $\varepsilon = 1/2$, $\varepsilon' = 1/6$ and add (3.6) to (3.7) to obtain

$$\begin{aligned}
& \frac{d}{dt} \left\{ \int_{-L}^L \frac{\alpha'(\theta^{\mu,s}(x, t)) |\theta_t^{\mu,s}(x, t)|^2}{2} dx + \frac{\kappa}{2} |v_t^{\mu,s}(t)|_{V_0}^2 \right\} \\
& \quad + \frac{1}{2} |(\alpha(\theta^{\mu,s}))_t(t)|_V^2 + \frac{1}{2} |v_{tt}^{\mu,s}(t)|_{V_0^*}^2 + \mu |v_{tt}^{\mu,s}(t)|_{H_0}^2 \\
& \leq R_9 (|\theta_t^{\mu,s}(t)|_H^2 + 1) \left(\int_{-L}^L \frac{\alpha'(\theta^{\mu,s}(x, t)) |\theta_t^{\mu,s}(x, t)|^2}{2} dx + \frac{\kappa}{2} |v_t^{\mu,s}(t)|_{V_0}^2 \right) \\
& \quad \text{for a.e. } t \in [s, s+T]
\end{aligned} \tag{3.8}$$

for some suitable constant $R_9 > 0$.

By applying the Gronwall's lemma to the inequality (3.8) $\times (t-s)$ and using (3), we derive that there exists a positive constant R_{10} such that

$$\begin{aligned}
& \sup_{s \leq t \leq s+T} \{(t-s) |\theta_t^{\mu,s}(t)|_H^2\} + \sup_{s \leq t \leq s+T} \{(t-s) |v_t^{\mu,s}(t)|_{V_0}^2\} \\
& \quad + \int_s^{s+T} |(\alpha(\theta^{\mu,s}))_t(t)|_V^2 dt + \int_s^{s+T} (t-s) |v_{tt}^{\mu,s}(t)|_{V_0^*}^2 dt
\end{aligned}$$

$$+\mu \int_s^{s+T} (t-s)|v_{tt}^{\mu,s}(t)|_{H_0}^2 dt \leq R_{10}$$

for any $\mu \in (0, 1]$, $s \geq t_1$ and $[e_0, v_0] \in B_0$.

By letting $\mu \downarrow 0$, we obtain this lemma. $\diamond$

In the next step, we will construct $\mathcal{X}$ and give the linearized system of (PSC) on $\mathcal{X}$.

We define the subset $\mathcal{X}$ of $V^* \times V_0^*$ by

$$\mathcal{X} := \bigcup_{t \geq t_1} S(t)B_0 \subset B_0,$$

where t_1 is the same number in Section 3. Then, it is easy to check that $\mathcal{X}$ satisfies the following lemma.

Lemma 3.3. *$\mathcal{X}$ satisfies the following properties (i)-(iv):*

- (i) *$\mathcal{X}$ is compact and connected in $V^* \times V_0^*$ as well as bounded in $V \times (H_0 \cap H^3(\Omega))$.*
- (ii) *$\mathcal{X}$ is positively invariant for $\{S(t)\}_{t \geq 0}$, namely, $S(t)\mathcal{X} \subset \mathcal{X}$ for all $t \geq 0$.*
- (iii) *$\mathcal{X}$ is an absorbing set for $\{S(t)\}_{t \geq 0}$.*
- (iv) *For any $t \geq 0$, $S(t)$ is Lipschitz on $\mathcal{X}$ with respect to the norm of $\mathcal{H}$.*

Now, let $[e_{0i}, v_{0i}] \in \mathcal{X}$ ($i = 1, 2$) be any two elements and put

$$[e_i(t), v_i(t)] := S(t)[e_{0i}, v_{0i}], \quad \theta_i := e_i - \lambda(v_i + m_0), \quad i = 1, 2,$$

$$e := e_2 - e_1, \quad v := v_2 - v_1, \quad \theta := \theta_2 - \theta_1.$$

Then it is easy to see that the difference equations of $[e, v]$ is described by

$$e_t(t) + F(\alpha(\theta_2(t)) - \alpha(\theta_1(t))) + \nu\theta(t) = 0 \quad \text{in } V^* \quad \text{for a.e. } t \geq 0, \quad (4.1)$$

$$F_0^{-1}v_t(t) - \kappa\Delta_N v(t) + \pi_0[p_2(t) - p_1(t)] = 0 \quad \text{in } H_0 \quad \text{for a.e. } t \geq 0, \quad (4.2)$$

$$e(0) = e_0 := e_{02} - e_{01}, \quad v(0) = v_0 := v_{02} - v_{01}, \quad (4.3)$$

where

$$p_i := g(v_i + m_0) + \beta(v_i + m_0) - \alpha(\theta_i)\lambda'(v_i + m_0) \quad (i = 1, 2).$$

Next, in order to rewrite the above difference equation into the linearized equation we introduce the functions σ_i ($1 \leq i \leq 7$) from R^m into R defined by

$$\begin{aligned}\sigma_1(e_1, e_2, v_1, v_2) &= \int_0^1 \alpha'(e_1 + r(e_2 - e_1) - \lambda(v_1 + r(v_2 - v_1) + m_0)) dr, \\ \sigma_2(e_1, e_2, v_1, v_2) &= \int_0^1 \alpha'(e_1 + r(e_2 - e_1) - \lambda(v_1 + r(v_2 - v_1) + m_0)) \\ &\quad \times \lambda'(v_1 + r(v_2 - v_1) + m_0) dr, \\ \sigma_3(v_1, v_2) &= \int_0^1 \lambda'(v_1 + r(v_2 - v_1) + m_0) dr, \\ \sigma_4(v_1, v_2) &= \int_0^1 g'(v_1 + r(v_2 - v_1) + m_0) dr, \\ \sigma_5(v_1, v_2) &= \int_0^1 \beta'(v_1 + r(v_2 - v_1) + m_0) dr, \\ \sigma_6(e_1, e_2, v_1, v_2) &= \int_0^1 \alpha'(e_1 + r(e_2 - e_1) - \lambda(v_1 + r(v_2 - v_1) + m_0)) \\ &\quad \times (\lambda'(v_1 + r(v_2 - v_1) + m_0))^2 dr \\ &\quad - \int_0^1 \alpha(e_1 + r(e_2 - e_1) - \lambda(v_1 + r(v_2 - v_1) + m_0)) \\ &\quad \times \lambda''(v_1 + r(v_2 - v_1) + m_0) dr\end{aligned}$$

and

$$\sigma_7 := \sigma_4 + \sigma_5 + \sigma_6,$$

where $m = 4$ if $i = 1, 2, 6, 7$ and $m = 2$ if $m = 3, 4, 5$.

Then, it is easily seen that (4.1) and (4.2) can be rewritten in the following form;

$$e_t(t) + F(\sigma_1(t)e(t) - \sigma_2(t)v(t)) + \nu e(t) - \nu \sigma_3(t)v(t) = 0 \quad \text{in } V^* \quad (4.4)$$

for a.e. $t \geq 0$,

$$F_0^{-1}v_t(t) - \kappa \Delta_N v(t) + \pi_0[\sigma_7(t)v(t) - \sigma_2(t)e(t)] = 0 \quad \text{in } H_0 \quad (4.5)$$

for a.e. $t \geq 0$,

where $\sigma_i(t) := \sigma_i(e_1(t), e_2(t), v_1(t), v_2(t))$ ($1 \leq i \leq 7$).

At first, we note that the following lemma hold.

Lemma 3.4. *There exist positive constants M_1 and M_2 such that*

$$\sum_{i=1}^7 |\sigma_i(x, t)| \leq M_1 \quad \text{and} \quad \sigma_1(x, t) \geq M_2, \quad \forall (x, t) \in [-L, L] \times [0, +\infty),$$

where $[e_i(\cdot), v_i(\cdot)]$ ($i = 1, 2$) are solutions to (PSC) with initial data $[e_{0i}, v_{0i}] \in \mathcal{X}$.

Next from Remark 3.1 for each $t \geq 0$ we define an operator $B(t)$ with domain $\mathcal{Y} := D(B(t)) = V \times (D(-\Delta_N) \cap H^3(\Omega))$ and range in $\mathcal{H}$ by

$$\begin{aligned} (B(t)W, \overline{W})_{\mathcal{H}} &:= (F(\sigma_1(t)e - \sigma_2(t)v), \overline{e})_V \\ &\quad + (F_0[-\kappa\Delta_N v + \pi_0[\sigma_7(t)v - \sigma_2(t)e]], \overline{v})_{H_0} \end{aligned}$$

$$\text{for any } W := [e, v] \in \mathcal{Y} \quad \text{and} \quad \overline{W} := [\overline{e}, \overline{v}] \in \mathcal{H}.$$

Here, we note from Remark 3.1 the fact that $\mathcal{X} \subset \mathcal{Y}$. Moreover, by means of $B(t)$, the system (4.5) and (4.6) is equivalent to the following evolution equation:

$$U_t(t) + B(t)U(t) + G(t)U(t) = 0 \quad \text{in } \mathcal{H} \quad \text{for a.e. } t \geq 0, \quad (4.6)$$

where $U(t) := [e(t), v(t)]$ and G is an operator in $\mathcal{H}$ defined by

$$G(t)U := [\nu e - \nu\sigma_3(t)v, 0] \quad \text{for any } U := [e, v] \in \mathcal{H}. \quad (4.7)$$

As to the operators $B(t)$ and $G(t)$ we easily get the following lemmas. Furthermore, the constants M_i ($3 \leq i \leq 8$) in this lemma are independent of any solutions $\{e_i, v_i\}$ ($i = 1, 2$) starting from $\mathcal{X}$.

Lemma 3.5. *The following properties (i)-(vi) are fulfilled:*

(i) *There exists a positive constant M_3 such that*

$$|(B(t)U, U)_{\mathcal{H}}| \leq M_3 |U|_{\mathcal{E}}^2 \quad \text{for any } U \in \mathcal{Y} \text{ and } t \geq 0.$$

(ii) *There exists a positive constant M_4 and M_5 such that*

$$|U|_{\mathcal{E}}^2 \leq M_4 (B(t)U, U)_{\mathcal{H}} + M_5 |v|_{V_0^*} \quad \text{for any } U \in \mathcal{Y} \text{ and } t \geq 0.$$

(iii) *There exists a positive constant M_6 such that*

$$|(G(t)U, U)_{\mathcal{H}}| \leq M_6 |U|_{\mathcal{H}}^2 \quad \text{for any } U \in \mathcal{H} \text{ and } t \geq 0.$$

(iv) *There exists a positive constant M_7 such that*

$$|(B(t)U, G(t)U)_{\mathcal{H}}| \leq M_7 |U|_{\mathcal{E}}^2 \quad \text{for any } U \in \mathcal{Y} \text{ and } t \geq 0.$$

(v) For each $t \geq 0$, we define an operator $B_t(t)$ from $H \times H_0$ into itself by

$$B_t(t)W := [(\sigma_1)_t(t)e - (\sigma_2)_t(t)v, \pi_0[(\sigma_7)_t(t)v - (\sigma_2)_t(t)e]]$$

for any $W := [e, v] \in H \times H_0$.

Then, there exists a positive constant M_8 such that

$$|(B_t(t)W, W)_{H \times H_0}| \leq M_8 \left\{ \sum_{i=1}^2 |(\alpha(\theta_i))_t(t)|_V + \sum_{i=1}^2 |(v_i)_t(t)|_{V_0} \right\} \\ \times (|e|_H^2 + |v|_{H_0}^2)$$

for any $W := [e, v] \in H \times H_0$ and a.e. $t \geq 0$,

where for each $i = 1, 2$ $[e_i(\cdot), v_i(\cdot)] := [\theta_i(\cdot) + \lambda(v_i(\cdot) + m_0), v_i(\cdot)]$ are solutions to (PSC) with initial data $[e_{0i}, v_{0i}] \in \mathcal{X}$.

(vi) Let $Z \in W_{loc}^{1,2}(R_+; \mathcal{H})$ such that $Z(t) \in \mathcal{Y}$ for a.e. $t \geq 0$. Then,

$$\frac{d}{dt}(B(t)Z(t), Z(t))_{\mathcal{H}} = (B_t(t)Z(t), Z(t))_{H \times H_0} + 2(B(t)Z(t), Z_t(t))_{\mathcal{H}}$$

for a.e. $t \geq 0$.

By using the above lemmas, we can actually prove Theorem 2.1, i.e., we can check the existence of a finite time t^* and the squeezing property of $S^* := S(t^*)$.

References

1. A. Damlamian and N. Kenmochi, Evolution equations generated by subdifferentials in the dual space of $H^1(\Omega)$, Discrete and Continuous Dynamical Systems **5** (1999), 269-278.
2. A. Damlamian and N. Kenmochi, Evolution equations associated with non-isothermal phase separation: subdifferential approach, Annali di Matematica pura ed applicata, **CLXXVI** (1999), 167-190.
3. A. Eden, C. Foias, B. Nicolaenko and R. Temam, Inertial sets for dissipative evolution equations. Part 1. Construction and applications, IMA preprint 812, 1991.

4. A. Eden and J. M. Rakotoson, Exponential attractors for a doubly nonlinear equation, *J. Math. Anal. Appl.*, **185** (1994), 321-339.
5. A. Ito and N. Kenmochi, Inertial set for the one-dimensional non-isothermal phase separation model, in preparation.
6. A. Ito and N. Kenmochi, Inertial set for a phase transition model of Penrose-Fife type, *Adv. Math. Sci. Appl.*, **10** (2000), 353-374.
7. A. Ito, N. Kenmochi and M. Kubo, Non-isothermal phase separation models: construction of attractors, in *Proceedings of International Conference on Free Boundary Problems: Theory and Applications*, edited by N. Kenmochi, GAKUTO Inter. Ser. Math. Sci. Appl., Vol. 13, Gakkōtoshō, Tokyo, Japan, 2000.
8. N. Kenmochi, Attractors of semigroups associated with nonlinear systems for diffusive phase separation, *Abstract and Applied Analysis*, **1** (1996), 169-192.
9. N. Kenmochi and M. Niezgódka, Non-linear system for non-isothermal diffusive phase separation, *J. Math. Anal. Appl.*, **188** (1994), 651-679.
10. N. Kenmochi and M. Niezgódka, Viscosity approach to modelling non-isothermal diffusive phase separation, *Japan J. Indust. Appl. Math.*, **13** (1996), 135-169.
11. N. Kenmochi, M. Niezgódka and S. Zheng, Global attractor of a non-isothermal model for phase separation, pp. 129-144 in *Curvature Flows and Related Topics*, edited by A. Damblamian, J. Spruck and A. Visintin, GAKUTO Inter. Ser. Math. Sci. Appl., 5, Gakkōtoshō, Tokyo, 1995.
12. M. Kubo, A. Ito and N. Kenmochi, Non-isothermal phase separation models: weak well-posedness and global estimates, in *Proceedings of International Conference on Free Boundary Problems: Theory and Applications*, edited by N. Kenmochi, GAKUTO Inter. Ser. Math. Sci. Appl., Vol. 14, Gakkōtoshō, Tokyo, Japan, 2000.
13. O. Penrose and P. C. Fife, Thermodynamically consistent models of phase-field type for the kinetics of phase transitions, *Physica D*, **43** (1990), 44-62.
14. K. Shirakawa, Large time behavior for doubly nonlinear systems generated by subdifferentials, to appear in *Adv. Math. Sci. Appl.*