

Introduction to Resolution of Singularities

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We will present an elementary proof of a canonical resolution of singularities in characteristic zero (at least in the hypersurface case) including detailed examples illustrating some (elementary, but important) applications and the constructive features of the “local to global” argument. The proof is by introduction of a discrete local invariant whose maximum locus determines a smooth centre of blowing up, leading to desingularization.

Lecture 1

Blow ups, Desingularization Theorems and examples.

Lecture 2

Proof of Weak Desingularization (in all details). From local to global: local properties of an invariant that imply a global desingularization.

Lecture 3

Constructive definition of the invariant for desingularization and an example illustrating the construction.

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PART 1. BRIEF HISTORY OF DESINGULARIZATION
AND THE MAIN FEATURES (OF MY WORK WITH BIERSTONE)
History.

The problem of resolution of singularities appeared in the middle of the nineteenth century, although in the 1-dimensional case it existed earlier in the guise of finding **good parametrization** of curves. At first, the problem was only considered for $\mathbb{R}$ and $\mathbb{C}$.

$\dim X = 1$: Puiseux, Kronecker, Halphen, XIX century, I. Newton, XVII century.

RIEMANN, XIX century.

$\dim X = 2$, $\text{char } k = 0$: Beppo Levi 1897, Chisini 1921, Albanese 1924, R.J. Walker – Jung 1935, Zariski 1939, 1942.

Local Uniformization Theorem, any dimension, $\text{char } k = 0$: Zariski, 1940.

$\dim X = 3$, $\text{char } k = 0$: Zariski, 1944.

DESINGULARIZATION HISTORY (very brief):

(A) By BLOW UPS:

NEWTON ... ZARISKII ... HIRONAKA
in $\text{char } k = 0$ ('60 - '90)

BLOW UPS INTRODUCED BY:
MAX NOETHER ... ZARISKII

(B) By PROJECTIONS, ALTERATIONS:

ALBANESE ... de JONG
 $\text{char } k \neq 0$ (from '96 - '97)

DRAWBACK: POOR CONTROL OF THE
DESINGULARIZATION MAP.

(C) By NASH BLOW UPS:

NASH ... HIRONAKA (SPIVAKOVSKII)
in $\dim = 2$

FROM AMS FEATURED REVIEW By J. Lipman:

Hironaka's theorem on the existence of resolutions of singularities for any algebraic or analytic variety V over a field of characteristic zero is an outstanding achievement of twentieth-century mathematics, by virtue of the depth both of its proof and of its applications. ...The history of the problem of existence of resolutions goes back more than a century And the history is ongoing ... novel approaches to global desingularization have just been developed by A. J. de Jong et al ..., leading to a new generation of short, but non-constructive, proofs. Hironaka's proof is lengthy, difficult, and non-constructive. Influential as the proof has been, few people can have checked it through entirely, even after some subsequent enhancements of the machinery Simplified, more algorithmic proofs are important not only for imparting better understanding of what is really involved in this great theorem, but also for their potential value in unearthing basic features of singularities and their classification. The challenge of finding more straightforward algorithmic approaches was taken up by Zariski, Abhyankar, and others, and successfully met only in the past decade by Bierstone, Milman, and Villamayor.

THE MAIN FEATURES (BIERSTONE-MILMAN, '88-'97)

1. CANONICAL DESINGULARIZATION
2. LOCAL PROPERTIES (OF AN INVARIANT)
 $\Rightarrow$ GLOBAL RESOLUTION.
3. IN THE HYPERSURFACE CASE INDUCTION
 DOES NOT INVOLVE PASSING TO
 CODIMENSION > 1 .

IN CHARACTERISTIC ZERO (THOUGH
 THE REDUCTION TO THE HYPERSURFACE
 CASE WORKS EVEN IN NONZERO CHARACT.)

4. APPLIES (IN THE LANGUAGE OF LOCALLY
 RINGED SPACES) TO SPACES S' THAT
 ARE LOCALLY $S' \hookrightarrow U$ SMOOTH
 ETALE "COORDINATE CHARTS", WHICH WE
 DEFINE BELOW. (WE ALSO REQUIRE THAT
 $\mathcal{O}_{S'}$ - COHERENT, $|S'|$ - locally

NOETHERIAN AND ONE MORE TECHNICAL
PROPERTY — "PRIVILEGED NBD. PROPERTY".)

ELEMENTS OF $\mathcal{O}(U)$ CALLED REGULAR
FUNCTIONS, E.G. POLYNOMIALS, ANALYTIC,
QUASIANALYTIC, ...

DEFINITION U is a smooth etale "COORDINATE
CHART" IFF EXIST $x_1, \dots, x_n \in \mathcal{O}(U)$, WHICH
WE CALL "COORDINATES", AND A ("TAYLOR")
HOMOMORPHISM $T: \mathcal{O}_U \rightarrow \mathcal{O}_U[[X]]$ INTO THE
RING OF FORMAL POWER SERIES EXPANSIONS IN

$X = (X_1, \dots, X_n)$ SUCH THAT LETTING

$$\sum_{\alpha} f_{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \stackrel{\text{def}}{=} T f \quad \text{WE HAVE:}$$

$$\textcircled{1} \quad f_0 = f \quad ; \quad \textcircled{2} \quad T x_j = x_j + X_j \quad \forall j ;$$

$$\textcircled{3} \quad \text{DENOTE } D_{\alpha} f \stackrel{\text{def}}{=} f_{\alpha} \quad \text{AND FOR A POWER}$$

SERIES EXPANSION $F(X)$ LET $D_{\alpha} F(X)$

$$\text{BE DEFINED BY } \sum_{\alpha} D_{\alpha} F(X) \cdot Y^{\alpha} = F(X+Y)$$

THEN $T \circ D_\alpha = D_\alpha \circ T \quad \forall \alpha;$

(4.) LET $T_a: \mathcal{O}_a \rightarrow \mathbb{F}_a[[X]]$ BE DEFINED BY EVALUATION OF COEFFICIENTS AT $a \in \mathcal{U}$, i.e.
 $T_a f = \sum_{\alpha} f_{\alpha}(a) \cdot X^{\alpha}$, WHERE $\mathbb{F}_a \stackrel{\text{def}}{=} \mathcal{O}_a / m_a$,
 AND $\hat{T}_a: \hat{\mathcal{O}}_a \rightarrow \mathbb{F}_a[[X]]$ BE THE INDUCED MAP OF COMPLETIONS. THEN $\hat{T}_a$ IS AN ISOMORPHISM.

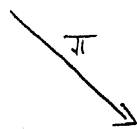
An illustrating example,

e.g. in k -algebraic case

(OR SCHEMES OF FINITE TYPE):

$$\mathcal{U} = \{(x, y) : P(x, y) = 0\} \subseteq \mathbb{A}_{(x, y)}^N$$

etale



$P = (P_1, \dots, P_{N-n})$ polynomials

and π is etale, i.e.

$$\det \frac{\partial P}{\partial y} \neq 0 \quad \text{ON } \mathcal{U}$$

$x_1, \dots, x_n$ - "COORDINATES"

AND (VIA IMPLICIT DIFFERENTIATION)

it is easy to define a homomorphism T

PART 2. BLOW UP, examples.

BLOW UP ALONG $I \subset \mathcal{O}_U$

$$I = (f_1, \dots, f_k) \subset \mathcal{O}(U), U \text{ smooth}$$

$$U - V(f_1, \dots, f_k) \xrightarrow{[f]} \mathbb{P}^{k-1}$$

$$x \mapsto [f_1(x) : \dots : f_k(x)]$$

$$Bl_I U := \text{closure Graph}[f] \subset U \times \mathbb{P}^{k-1}$$

- ① $Bl_I U$ DOES NOT DEPEND ON THE CHOICE OF GENERATORS OF I
-

- ② IF $I = I_C$, WHERE $C \hookrightarrow M$

$$Bl_C M := Bl_I M$$

- ③ EVEN FOR SMOOTH M (UNLESS C SMOOTH $\hookrightarrow M$)

$Bl_C M$ MAY HAVE SINGULARITIES

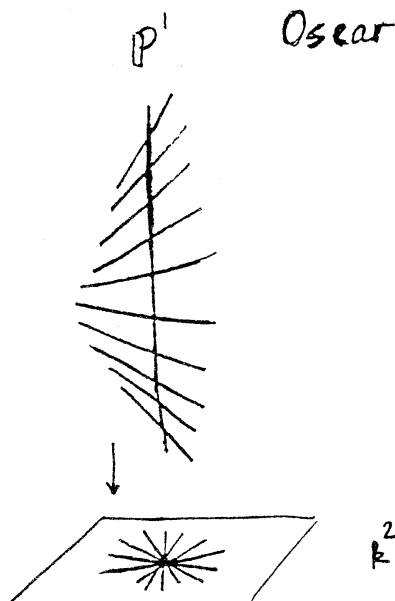
EXAMPLE:

e.g. $M = \mathbb{A}^2$ $C = \{0\}$ $Bl_0 \mathbb{A}^2$

Blowing up.

MAX NOETHER
OSCAR ZARISKI

The plane at a point:



0 is removed and replaced by $\mathbf{P}^1$ parametrizing all the lines in k^2 passing through 0. If $k = \mathbf{R}$, we get the Möbius band.

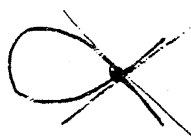
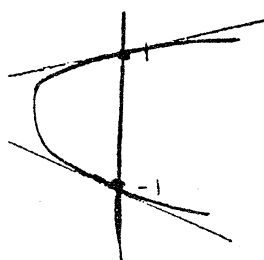
WE LIFT EVERY LINE THROUGH THE ORIGIN WITH THE SLOPE $z = \frac{y}{x}$ (y -axis HAS SLOPE ∞) TO THE HEIGHT z . UNION OF ALL THESE LINES, SAY M' , IS THE BLOW UP OF THE PLANE AT 0.

π induces an isomorphism $M' \setminus \pi^{-1}(0) \simeq k^2 \setminus \{0\}$.

Example 1

Let x, y be coordinates on k^2 . Algebraically, M' is glued together from two coordinate charts, with coordinates $(x, \frac{y}{x})$ and $(y, \frac{x}{y})$, respectively.

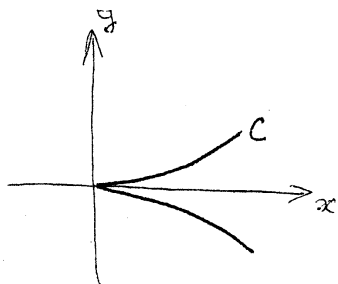
Example. Resolution of singularities of the curve C defined by $y^2 - x^2 - x^3 = 0$.



$$x' = x, \quad y' = \frac{y}{x}.$$

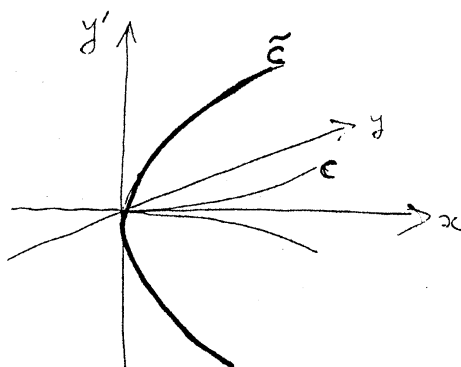
$$y^2 - x^2 - x^3 = (x'y')^2 - x'^2 - x'^3 = x'^2(y'^2 - 1 - x').$$

Example 2. Resolution of singularities of the curve C defined by $y^2 - x^3 = 0$.



$$x' = x, \quad y' = \frac{y}{x}.$$

$$y^2 - x^3 = (x'y')^2 - x'^3 = x'^2(y'^2 - x').$$



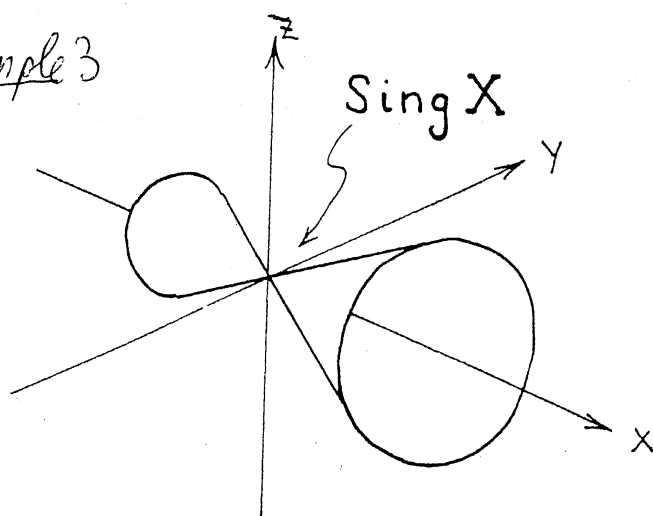
i.e. $\tilde{C} : x' = y'^2$

OR IN (x, y, y') -space

$$x = y'^2 \quad y = y'^3$$

NOT NORMAL CROSSING
(YET ...)

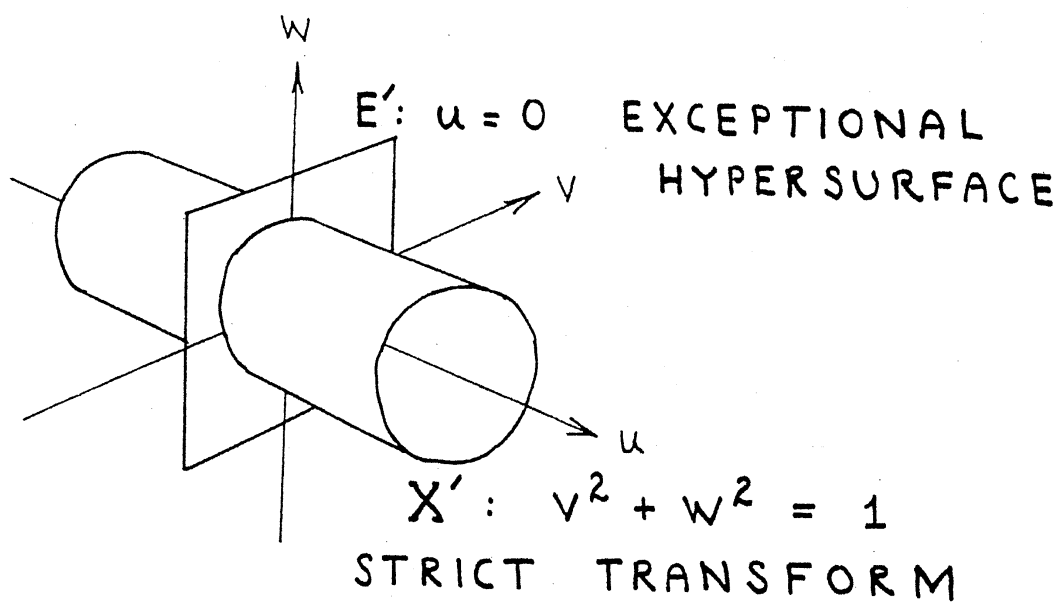
Example 3



$$X : x^2 - y^2 - z^2 = 0$$

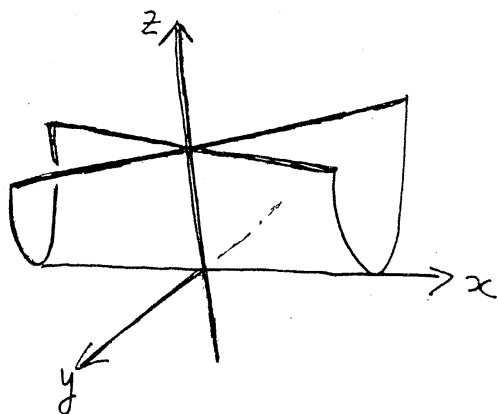
$$(x, \frac{y}{x}, \frac{z}{x}) = (u, v, w) \quad \text{THEN } G : \begin{matrix} x=u \\ y=u \cdot v \\ z=u \cdot w \end{matrix}$$

$$\sigma^{-1}(X) : u^2(1 - v^2 - w^2) = 0$$



Example 4

WHITNEY UMBRELLA



$$X: y^2 - z \cdot x^2 = 0$$

BLOW UP OF 0 DOES NOT DESINGULARIZE:

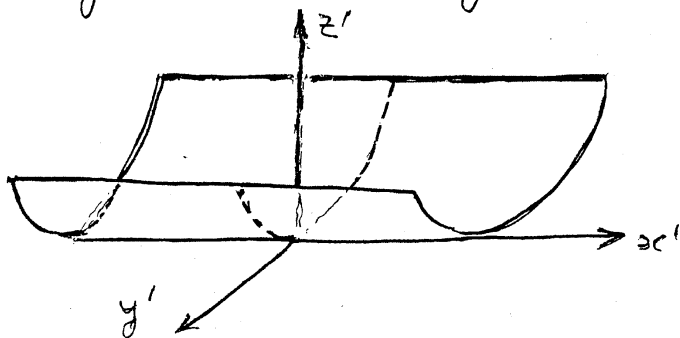
in $(\frac{x}{z}, \frac{y}{z}, z) =: (x_1, y_1, z_1)$ COORDINATES

$$y^2 - z \cdot x^2 = z_1^2 \cdot y_1^2 - z_1 \cdot z_1^2 \cdot x_1^2 = z_1^2 \cdot (y_1^2 - z_1 x_1^2)$$

BUT BLOW UP OF z-AXIS DOES:

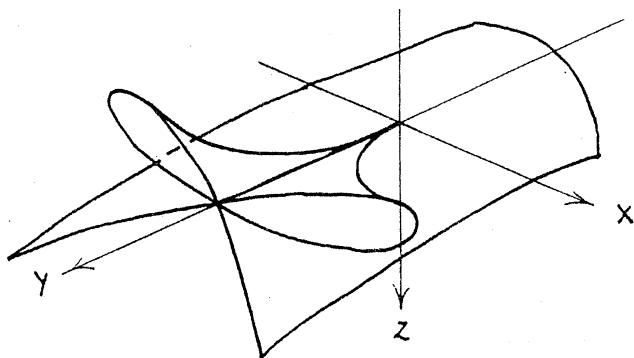
in $(x, \frac{y}{x}, z) =: (x', y', z')$ COORDINATES

$$y^2 - z \cdot x^2 = x'^2 \cdot y'^2 - z' \cdot x'^2 = x'^2 \cdot (y'^2 - z')$$



Example 5

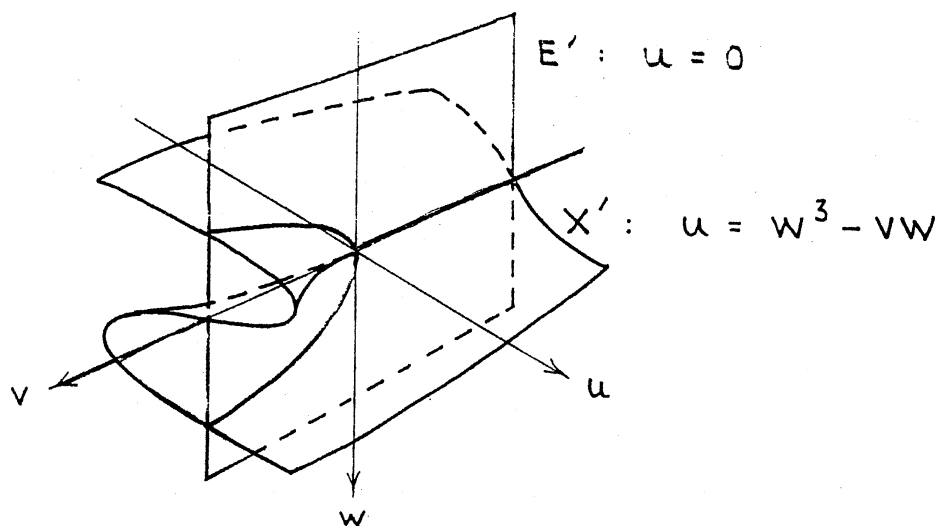
$$X: z^3 - x^2 y z - x^4 = 0$$



$$\begin{array}{l} \uparrow \\ \sigma: \end{array} \quad \begin{array}{l} x = u \\ y = v \\ z = uw \end{array}$$

BLOWING - UP
(IN LOCAL COORDINATES)

$$\sigma^{-1}(X): u^3(w^3 - vw - u) = 0$$



PART 3 EQUIMULTIPLE BLOWUPS.

BLOW UP ALONG SMOOTH C :

$$M \xleftarrow{\sigma} M' = \text{Bl}_C M$$

$$\begin{array}{ccc} U & & U \\ \downarrow & \xleftarrow{\sigma^{-1}} & \downarrow \\ U & & U' = \text{Bl}_{U \cap C} U \\ \text{coord. chart} & & \end{array}$$

$$U \cap C = \{x_1 = \dots = x_m = 0\}$$

$$U' = \{(x, \bar{z}) \text{ s.t. } [x_1 : \dots : x_m] = \bar{z}\} \subset U \times \mathbb{P}^{m-1}$$

$$\boxed{x_i \cdot \bar{z}_j = x_j \cdot \bar{z}_i} \quad \sigma \searrow \quad \downarrow \pi$$

U

$$U' = \bigcup_{1 \leq j \leq m} U_j \quad \text{each } U_j = \{\bar{z}_j \neq 0\}$$

① each U_j coord. chart with coord.

$$y_i = \frac{x_i}{x_j} = \frac{\bar{z}_i}{\bar{z}_j} \quad \text{if } i \neq j, 1 \leq i \leq m$$

$$y_s = x_s \quad \text{if } s = j \text{ or } m < s \leq n$$

② $U_j = U' - \{x_j = 0\}'$

③ in coord. on U_j $\{x_i = 0\}' = \{y_i = 0\}$
 $i \neq j$

BLOW UP ALONG SMOOTH C :

$$U \cap C = \{x_1 = \dots = x_m = 0\}$$

$$U' = \bigcup_{1 \leq j \leq m} U_j$$

$$\begin{cases} U_j \text{ coord. charts} \\ y = (y_1, \dots, y_n) \end{cases}$$

$$\begin{aligned} \sigma_{U_j} : \quad & x_j = y_j \\ & x_i = y_j \cdot y_i \quad i \neq j, 1 \leq i \leq m \\ & x_s = y_s \quad m < s \leq n \end{aligned}$$

IF $X = V(f)$ THEN $X' = V(f')$
 WHERE $(f') := (y_{\text{exc}})^{-d} \cdot (f \circ \sigma)$
 ON U_j $y_{\text{exc}} = y_j$, $\{y_{\text{exc}} = 0\} := \sigma^{-1}(C)$

$$d = \text{maximal integer} = \text{ord}_C(f) \\ \text{such that } f' \text{ is regular}$$

In general, $\text{codim } X > 1$:

IF $X \subset M$ THEN $X' := \overline{\sigma^{-1}(X - C)}$
 AND $I_{X'} = (f' \text{ s.t. } f \in I_X) \subset \mathcal{O}_{M'}$

THE EFFECT OF AN EQUIMULTIPLE BLOW UP:

HYPERSURFACE $f(x) = 0$
 $d = \text{ord}_a f$

$$f(x) = c_0(\tilde{x}) + \dots + c_{d-1}(\tilde{x}) x_n^{d-1} + c_d(x) x_n^d$$

$\nwarrow$ $\nearrow$
 $(x_1, \dots, x_{n-1})$ 0 , BY $\nearrow$ ASSUME $\equiv 1$
 COMPLETING
 d 'TH POWER

EQUIMULTIPLE LOCUS

$$S_d := \{x : \text{ord}_x f = d\}$$

$$= \{x : x_n = 0, \text{ord}_{\tilde{x}} c_q \geq d - q\}$$

$$\min_q \text{ord}_a c_q / (d - q)$$

EFFECT OF A BLOWING-UP

σ , CENTRE $C \subset S_d$

$$C = Z_I := \{x_n = 0, \quad x_i = 0, \quad i \in I\}$$

WHERE $I \subset \{1, \dots, n-1\}$

EFFECT ... IN CHART $U_i', i \in I$:

$$\begin{aligned} \sigma|_{U_i'} : \quad x_j &= y_i y_j & j \in I \cup \{n\} \setminus \{i\} \\ x_j &= y_j & \text{OTHER } j \end{aligned}$$

$$i \in I : \quad f(\sigma(y)) = y_i^d f'(y),$$

$$f'(y) =$$

$$c'_0(\tilde{y}) + \dots + c'_{d-2}(\tilde{y}) y_n^{d-2} + y_n^d$$

$$c'_q(\tilde{y}) = y_i^{-(d-q)} c_q(\tilde{\sigma}(\tilde{y}))$$

EFFECT ... IN CHART $U'_n - \bigcup_{i \neq n} U'_i$

in U'_n coordinates $y = (y_1, \dots, y_n)$

$$\sigma|_{U'_n} : \begin{aligned} x_n &= y_n \\ x_j &= y_n y_j, \quad j \in I \\ x &= y \quad \neq I \cup \{n\} \end{aligned}$$

$$W := U'_n - \bigcup_{i \in I} U'_i = \{y_i = 0 \quad \forall i \in I\}$$

$$U \cap C = \{x_n = 0, x_j = 0 \quad \forall j \in I\}$$

$$(*) \quad \boxed{\text{ord}_C C_k \geq d - k > 0} \quad k = 1, 2, \dots, d-2$$

$$f' = y_n^{-d} \cdot (f \circ \sigma) = (C_d \circ \sigma)(y) + \sum_{0 \leq k < d} C'_k(y)$$

$\hookrightarrow \neq 0 \text{ on } W$

WHERE

$$C'_k(y) := y_n^{-(d-k)} \cdot (C_k \circ \sigma)(y)$$

$\hookrightarrow = 0 \text{ on } W \quad \text{due to } (*)$

Hence, $\boxed{f' \neq 0 \text{ on } U'_n - \bigcup_{i \in I} U'_i}$

$$C_d(x) \neq 1$$

SUMMARY:

$$\tilde{x} = (x_1, \dots, x_{n-1}) \quad d = \text{ord}_0 f$$

$$f = C_0(\tilde{x}) + \dots + C_{d-1}(\tilde{x}) x_n^{d-1} + C_d(x) \cdot x_n^d$$

$$\frac{\partial^{d-1} f}{\partial x_n^{d-1}} \approx x_n \Rightarrow C_{d-1} \equiv 0, C_d(x) \neq 0$$

$$C = \mathbb{Z}_r$$

$$\sigma|_{\mathcal{U}_i} : \begin{array}{ll} x_j = y_i \cdot y_j & j \in I \cup \{n\} - \{i\} \\ x_j = y_j & \text{OTHER } j \end{array}$$

$$f'(y) = y_i^{-d} \cdot (f \circ \sigma)(y) \quad \text{in } \mathcal{U}_i$$

$$f' = C'_0(\tilde{y}) + \dots + C'_{d-2}(\tilde{y}) y_n^{d-2} + C'_d(y) \cdot y_n^d$$

$$C'_q := y_i^{-(d-q)} \cdot (C_q \circ \sigma)$$

THEN:

$$\text{NEAR } \sigma^{-1}(C) = \{y_i = 0\}$$

$$\frac{1}{d!} \frac{\partial^{d-1} f'}{\partial y_n^{d-1}} = y_n \cdot C'_d(y) + y_n^2 \cdot y_i \cdot (\dots) \approx y_n$$

$$\boxed{\text{ord}_y f' \leq d} \quad \text{AND} \quad \boxed{\text{ord}_y f' = d} \Leftrightarrow$$

$$\boxed{y_n = 0 \quad \text{AND} \quad \text{ord}_y C'_q \geq d - q, q = 0, 1, \dots}$$

$$\text{NOTE: } \{y_n = 0\} = \{x_n = 0\}'$$

PART 4 "WEAK" DESINGULARIZATION THEOREM. (PROOF IN ALL DETAILS.) WE WILL NEED THE FOLLOWING:

DEFINITION OF $\varphi \stackrel{\text{def}}{=} \psi \times \text{id}$ FOR " $\psi: N' \rightarrow N$ " THAT OCCUR IN THE INDUCTIVE STEP OF PROOF BELOW:

④ FOR A COORD. CHART U AND $N \stackrel{\text{def}}{=} \{x: g(x)=0\}$ WITH $\frac{\partial g}{\partial x_n} \neq 0$ ON U AND A SMOOTH $C \stackrel{\text{def}}{=} \{x \in N: x_i = 0, i \in I\} =: Z_I, I \subset \{1, \dots, n-1\}$, AND $\psi \stackrel{\text{def}}{=} \sigma: N' \stackrel{\text{def}}{=} \text{Bl}_C N \rightarrow N$ A BLOW UP MAP LET $\varphi \stackrel{\text{def}}{=} \psi \times \text{id}: U' \rightarrow U$ DENOTE THE BLOW UP MAP $U' \stackrel{\text{def}}{=} \text{Bl}_{\tilde{C}} U \rightarrow U$ WITH $\tilde{C} \stackrel{\text{def}}{=} \{x \in U: x_i = 0, i \in I\}$.

⑤ FOR A COORD. CHART U AND $N \stackrel{\text{def}}{=} \{x: g(x)=0\}$ WITH $\frac{\partial g}{\partial x_n} \neq 0$ ON U AND FOR AN OPEN FINITE COVERING $N' = \bigcup_{i \in I} N'_i$ WITH EACH $N'_i \stackrel{\text{def}}{=} N' - \{x_i = 0\}'$ AND $\psi \stackrel{\text{def}}{=} \coprod_{i \in I} N'_i \rightarrow \bigcup_{i \in I} N'_i$, A "COVERING" MAP, LET $\varphi \stackrel{\text{def}}{=} \psi \times \text{id}: M' \rightarrow M$, WHERE $M \stackrel{\text{def}}{=} U$, DENOTE THE "COVERING" MAP $M' \stackrel{\text{def}}{=} \coprod_{i \in I} U'_i \rightarrow \bigcup_{i \in I} U'_i = U' \stackrel{\text{def}}{=} \text{Bl}_{\tilde{C}} U$, WHERE EACH $U'_i := U' - \{x_i = 0\}'$. NOTE: $\sigma: U'_i \rightarrow U$ IN COORDINATES IS $\sigma: \begin{cases} x_j = y_i \cdot y_j & \text{FOR } j \in I - \{i\} \\ x_j = y_j & \text{OTHERWISE} \end{cases}$ AND $N'_i = \{x \in U'_i: [y_i^{-1}(g \circ \sigma)(y)] = 0\} \hookrightarrow U'_i$.

[L.H.E.S. (1988), see §4] — WITH BIERSTONE — [cf. J. of AMS §3 (1989)]

"WEAK" DESING. THM.: FOR $(f(x))$ ON SMOOTH

M EXISTS $\varphi: M' \rightarrow M$ A COMPOSITE OF BLOW UPS (ALONG SMOOTH CENTERS) AND OF "COVERINGS" (i.e.

maps $\coprod_j U_j \rightarrow \bigcup_j U_j$, OPEN $U_j \dots$) SUCH THAT
 $(J_\varphi(x)) \cdot (f \circ \varphi)(x) = (x_1^{\alpha_1} \dots x_n^{\alpha_n})$ locally (THE SO CALLED N.E.R.)
PROOF IN COMPLETE DETAILS BELOW.

DEFINITION: $[\text{SING } V(f)] \# a$ IFF $(f(x)) = (g(x))^d$
 $\text{ord}_a f = d$ "near a ", $\text{ord}_a g = 1$

SET UP AND OTHER BASIC NOTIONS:

IN "YEAR" $x \in M_j \xrightarrow{G_j} \dots \rightarrow M_{i+1} \xrightarrow{G_{i+1}} M_i \rightarrow \dots \rightarrow M_1 \rightarrow M_0 = M$
"ERA" OF HIGHER $d = \dots$

E_j - EXC. HYPERSURFACES ON M_j — ALL SMOOTH

$$(f \circ \varphi_j)(x) = \prod_{H \in E_j} \ell_H(x)^{m_H} \cdot f_0(x)$$

$$X_j \stackrel{\text{def}}{=} \{x: f_0(x) = 0\}, \quad \Sigma = \sum_j \stackrel{\text{def}}{=} \text{SING } X_j$$

$$d \stackrel{\text{def}}{=} \max \{ \text{ord}_x f_0 : x \in \Sigma \}$$

$$S_d \stackrel{\text{def}}{=} \{x: \text{ord}_x f_0 = d\} \quad \underline{\text{PROP.}} \quad a \in \sum \cap S_d \Rightarrow S_d \subseteq \sum$$

$$S(x) \stackrel{\text{def}}{=} \# \{ H \in E_j : H \ni x \text{ AND "COMES" FROM "ERA" OF HIGHER } d \}$$

PROOF OF PROP.: CHOOSE COORD. S.T.H. $\frac{\partial^d f_0}{\partial x_n^d}(a) \neq 0$ THEN

$$N \stackrel{\text{def}}{=} \{x: \frac{\partial^{d+1} f_0}{\partial x_n^{d+1}}(x) = 0\} \supseteq S_d = \{x \in N: \text{ord}_x c_k \geq d-k, 0 \leq k < d\}$$

WHERE $c_k \stackrel{\text{def}}{=} \frac{\partial^k f_0}{\partial x_n^k} \Big|_N$. FOR $x \in S_d$: $x \in \sum$ IFF SOME $c_k \neq 0$! DONE

$$E'(a) \stackrel{\text{def}}{=} \{H \ni a : H \text{ "COMES" FROM "ERA" HIGHER } d\}$$

PROOF OF "W.D." THM. INDUCTION ON $n = \dim M$ AND

$$(d, s) \quad , \quad s \stackrel{\text{def}}{=} \max \{s(x) : x \in S_d \cap \Sigma\} \quad , \quad \text{NAMESLY:}$$

(A) IN THE "YEAR" $i+1$ (WHEN d - JUST DECREASED) CHOOSE
 $a \in S_d \cap \Sigma$ AND COORD. CHART: $\frac{\partial^d f_0}{\partial x_n^d} \neq 0$, $\frac{\partial \ell_H}{\partial x_n} \neq 0$, $\forall H \ni a$
 $N \stackrel{\text{def}}{=} \{x : \frac{\partial^{d-1} f}{\partial x_n^{d-1}}(x) = 0\}$; $c_k \stackrel{\text{def}}{=} \frac{\partial^k f}{\partial x_n^k} \Big|_N$; $b_H \stackrel{\text{def}}{=} \ell_H \Big|_N$, $\forall H \in E'(a)$
 $A \stackrel{\text{def}}{=} \Pi$ "all" $c_k^{d!/d-k}$ "all" $b_H^{d!}$ "all" their differences $\neq 0$

$$A \in \mathcal{O}(N), \dim N = n-1 < n \Rightarrow \exists \psi: N' \rightarrow N \dots (A \circ \psi) \text{ N.C.R.}$$

$$\boxed{\psi \stackrel{\text{def}}{=} " \psi \times id ": M' \rightarrow M} \quad "a_1 \mapsto a_0 \stackrel{\text{def}}{=} a" \quad \text{THEN}$$

LEMMA 1 $\left\{ \begin{array}{l} \text{EACH } c_k^{d!/d-k} \approx \tilde{x}^{\Omega(k)}, b_H^{d!} \approx \tilde{x}^{\Omega_H} \quad \tilde{x} = (x_1, \dots, x_{n-1}) \\ \text{Exp} \stackrel{\text{def}}{=} \{\dots \Omega(k), \dots \Omega_H\} \leq \mathbb{N}^{n-1} \end{array} \right.$ **IS TOTALLY ORDERED!**
PROOF. DIRECT AND EASY.

$$\Rightarrow \exists \min \text{Exp} =: \Omega \quad \Rightarrow S_d \cap \left[\bigcap_{H \in E'(a)} H \right] =$$

$$\begin{aligned} &= \{x \in N : \text{ord}_x c_k \geq d-k \quad \forall k, \text{ord}_x b_H \geq 1 \quad \forall H \in E'(a)\} = \\ &= \{x \in N : \text{ord } \tilde{x}^\Omega \geq d!\} = \bigcup_I Z_I \quad \text{EACH } Z_I \stackrel{\text{def}}{=} \\ &= \{x \in N : x_i = 0 \quad \forall i \in I\} \quad \text{AND } I \text{ minimal } \subseteq \{1, \dots, n-1\} \end{aligned}$$

$$\text{SUCH THAT } \sum_{i \in I} \Omega_i \geq d! \Leftrightarrow 0 \leq \sum_{i \in I} \Omega_i - d! < \Omega_j \quad \forall j \in I.$$

LEMMA 2
CONSIDER BLOW UP WITH $C = Z_I$ (any): $u' \xrightarrow{\sigma} u$

$$u' = \bigcup_{j \in I \cup \{n\}} u'_j, \quad u'_n \stackrel{\text{def}}{=} u' - N' \quad \text{THEN } \forall x \in u', C \in S_d \Rightarrow \text{ord}_x f'_0 \leq d$$

IF $\text{ord}_x f'_0 = d \Rightarrow x \in N' \cap u'_j$, some $j \in I$ $\sigma|_{u'_j}: \begin{cases} x_i = y_j \cdot y_i & i \in I \cup \{n\} \\ x_i = y_i & \text{OTHERWISE} \end{cases}$

PROOF. FOLLOWS FROM "THE EFFECT OF EQUIMULTIPLE"

BLOW UP" CALCULATION.

Also, (THE "EFFECT OF EQUIMULTIPLE BLOW UP" CALCULATION), $c'_k = y_j^{-(d-k)} \cdot (c_k \circ \sigma)$, $b'_H = y_j^{-1} \cdot (b_H \circ \sigma)$.
HENCE,

$\star$ AGAIN IS VALID SINCE $y_j^{-d!} \cdot (\tilde{x}^{\Omega} \circ \sigma) = \tilde{y}^{\Omega'}$ WITH
 $\Omega'_i = \Omega_i$ FOR $i \neq j$ AND $\Omega'_j = \sum_{i \in I} \Omega_i - d!$ ($\forall \Omega \in \mathbb{N}^{n-1}$).

FOR $\Omega = \min[\text{Exp at } a]$ AS ABOVE $\Omega'_j < \Omega_j$ ($j \in I$) AND

$\Omega' = \min[\text{Exp at } a']$. BUT $0 \leq \sum_i \Omega'_i < \sum_i \Omega_i$ (AND

THESE ARE FROM $\mathbb{N}$) AND IF $S(a') = S$ THEN $\sum_i \Omega'_i \geq d!$

WE CONTINUE "THESE" BLOW UPS (UNLESS THE RESPECTIVE $\sum_j = \emptyset$) UNTIL (d, s) decreases

(NEED NO MORE THEN $\sum_i \Omega_i$ BLOW UPS, WHERE Ω IS

AS IN $\star$ AT a). IF d DECREASES WE START

WITH INDUCTION ON $n = \dim M$ STEP ALL OVER...

© OTHERWISE (i.e. EITHER $\sum_j = \emptyset$ OR S IS SMALLER, $\sum_j \neq \emptyset$)

WE CONTINUE THESE BLOW UPS (WHICH WE MAY

SINCE ALTHOUGH FOR SOME $H \in E^1(a)$, $H' \nsubseteq a'$, i.e.

$\Omega_{H'} = 0$, $\star$ REMAINS VALID WITH $H \in E^1(a')$, i.e.

SUCH THAT $\Omega_{H'} \neq 0$, AND "NEW" $\Omega = \min[\text{Exp at } a'] \neq 0$

UNTIL EITHER d DECREASES OR (IF $\sum_j = \emptyset$) $S(x) \leq 1 \forall x$
AND $S(x) = 0 \forall x \in X_j$. DONE. END OF PROOF OF THEM