

ON THE INSTABILITY OF THE UNIFORM ROTATION OF A BODY WITH LIQUID INSIDE

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1. Introduction

The present work deals with the stability of a uniformly rotating body with a cavity filled with an incompressible viscous fluid. Various statements of the problem of the motion of a rigid body with a cavity filled with fluid have been discussed in the literature [1-5]. In the present work we shall follow [4-5].

Given a rigid body G with a cavity Ω entirely filled with an incompressible viscous liquid. Let O be the center of mass of the entire system "body+liquid", $Ox_1x_2x_3$ be an orthogonal system of coordinates rigidly connected with the body. We assume that the undisturbed motion of the system relative to the point O be a uniform rotation of whole system with constant angular velocity $\omega_0\mathbf{e}_3$ around the axis Ox_3 . We shall examine the perturbed motion assuming that its deviations from the unperturbed one are small. Then in the rotating system of coordinates $Ox_1x_2x_3$ the linearization of the Navier-Stokes equations and the equations of moments with respect to the point O can be written as follows [4-5]

$$(1) \quad \frac{\partial \mathbf{u}}{\partial t} + 2\omega_0\mathbf{e}_3 \times \mathbf{u} + \frac{d\mathbf{w}}{dt} \times \mathbf{r} = -\frac{1}{\rho_l}\nabla p + \nu\Delta\mathbf{u} + \mathbf{f}$$

$$(2) \quad \operatorname{div} \mathbf{u} = 0$$

$$(3) \quad \mathbf{J} \cdot \frac{d\mathbf{w}}{dt} + \omega_0\mathbf{w} \times \mathbf{J} \cdot \mathbf{e}_3 + \omega_0\mathbf{e}_3 \times \mathbf{J} \cdot \mathbf{w} + \rho_l \int_{\Omega} \mathbf{r} \times \frac{\partial \mathbf{u}}{\partial t} d\Omega + \omega_0\mathbf{e}_3 \times \left(\rho_l \int_{\Omega} \mathbf{r} \times \mathbf{u} d\Omega \right) = \mathbf{M}$$

where

$$\mathbf{J} \cdot \mathbf{w} = \rho_l \int_{\Omega} \mathbf{r} \times (\mathbf{w} \times \mathbf{r}) d\Omega + \rho_b \int_G \mathbf{r} \times (\mathbf{w} \times \mathbf{r}) dG$$

is the inertia tensor of the entire system with respect to the point O . Here t denotes time; $\mathbf{r} = \overrightarrow{(x_1, x_2, x_3)}$; ρ_l, ρ_b are the constant densities of the liquid and the body; ν is the kinematic viscosity of the liquid; $\mathbf{u}$ is a relative velocity of the liquid; $\mathbf{w}$ is a relative

angular velocity of the body; p is a relative pressure; $\mathbf{M}(t)$ is a central moment of exterior forces with respect to O ; $\mathbf{f}(t, x)$ is a field of external force.

To complete the system (1)-(3) we consider the following boundary and initial conditions

$$(4) \quad \mathbf{u}|_{\partial\Omega} = 0$$

$$(5) \quad \mathbf{u}(0, x) = \mathbf{u}_0(x), \quad \mathbf{w}(0) = \mathbf{w}_0$$

As in [5] the closure in the norm of $\mathbf{L}_2(\Omega)$ of the set of all smooth solenoidal fields $\mathbf{u}$, satisfying the condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, will be referred to as the Hilbert space $\mathbf{J}_0(\Omega)$. Then equations (1),(2),(4) can be rewritten in the operator form [4-5]

$$(6) \quad \frac{d\mathbf{u}}{dt} + \mathbf{P}_0 \left(\frac{d\mathbf{w}}{dt} \times \mathbf{r} \right) + \nu \mathbf{A}\mathbf{u} + 2iw_0 \mathbf{T}\mathbf{u} = \mathbf{P}_0 \mathbf{f}$$

where $\mathbf{P}_0$ is the operator of the orthogonal projection of space $\mathbf{L}_2(\Omega)$ onto $\mathbf{J}_0(\Omega)$,

$$(7) \quad \mathbf{T}\mathbf{u} = i\mathbf{P}_0(\mathbf{u} \times \mathbf{e}_3), \quad \mathbf{T} = \mathbf{T}^*, \quad \sigma(\mathbf{T}) = [-1, 1]$$

and $\mathbf{A}$ is the Friedrichs extension [5] of the following symmetric operator

$$\mathbf{A}_0 = -\mathbf{P}_0 \Delta, \quad D(\mathbf{A}_0) = \{\mathbf{u} \in \mathbf{W}_2^2(\Omega) \mid \operatorname{div} \mathbf{u} = 0; \mathbf{u}|_{\partial\Omega} = 0\}$$

The operator $\mathbf{A}$ is usually called Stokes operator [5].

Following [5] we denote

$$(8) \quad \mathbf{v}(t) = (\mathbf{u}(t, x), \mathbf{w}(t)) \in \mathbf{H} = \mathbf{J}_0(\Omega) \oplus \mathbb{R}^3$$

Then in the Hilbert space $\mathbf{H}$ system (1)-(5) can be written in the form of abstract Cauchy problem [5]

$$(9) \quad \frac{d}{dt} \mathbf{N}\mathbf{v} + (\mathbf{C} + \mathbf{B})\mathbf{v} = \mathbf{g}(t), \quad \mathbf{g}(t) = ((\mathbf{P}_0 \mathbf{f})(t), \mathbf{M}(t))$$

$$(10) \quad \mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{v}_0 = (\mathbf{u}_0(x), \mathbf{w}_0)$$

where

$$\begin{aligned} \mathbf{N}\mathbf{v} &= \left(\mathbf{u} + \mathbf{P}_0(\mathbf{w} \times \mathbf{r}), \rho_l \int_{\Omega} \mathbf{r} \times \mathbf{u} d\Omega + \mathbf{J} \cdot \mathbf{w} \right) \\ \mathbf{C} &= \operatorname{diag}(\nu \mathbf{A}, \mathbf{I}) \\ \mathbf{B}\mathbf{v} &= \begin{pmatrix} 2iw_0 \mathbf{T}\mathbf{u} \\ w_0 \mathbf{e}_3 \times \left(\rho_l \int_{\Omega} \mathbf{r} \times \mathbf{u} d\Omega \right) + w_0 (\mathbf{e}_3 \times \mathbf{J} \cdot \mathbf{w} + \mathbf{w} \times \mathbf{J} \cdot \mathbf{e}_3) - \mathbf{w} \end{pmatrix} \end{aligned}$$

It is shown in [5] that if $\mathbf{g}(t)$ satisfies Hölder condition then for any $\mathbf{v}_0 \in \mathbf{H}$ there exists unique solution $\mathbf{v}(t) \in \mathbf{H}$ of the problem (9),(10).

Consider the spectral problem corresponding to the operator equation (9). Let $\mathbf{v}(t) = \exp(-\lambda t)\mathbf{V}$, $\mathbf{V} \in \mathbf{H}$; $\mathbf{g}(t) \equiv 0$. Then $\mathbf{V}$ should satisfy

$$(11) \quad (\mathbf{C} + \mathbf{B})\mathbf{V} = \lambda \mathbf{N}\mathbf{V} \quad , \quad \mathbf{V} \in \mathbf{H}$$

Using the properties of the operators $\mathbf{C}, \mathbf{B}, \mathbf{N}$ it has been proved in [5] that:

- (1) Problem (11) has discrete spectrum $\{\lambda_j\}_{j=1}^{\infty}$. Each eigenvalue λ_j , $j \in \mathbb{N}$ has finite multiplicity.
- (2) There exist positive constants $C_1, C_2 > 0$ such that

$$(12) \quad \operatorname{Re} \lambda_j \geq -C_1 \quad ; \quad |\operatorname{Im} \lambda_j| \leq C_2 \quad , \quad j \in \mathbb{N}$$

- (3) The sequence of eigenfunctions and associated functions corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ is complete in $\mathbf{H}$.

In the present paper we shall consider the following problem: under what conditions there exists an eigenvalue λ_{j_0} such that

$$(13) \quad \operatorname{Re} \lambda_{j_0} < 0$$

If there exists λ_{j_0} satisfying (13) then using results of [6] it can be shown that the uniform rotation with the fixed angular velocity $w_0 \mathbf{e}_3$ of the body G with the fluid-filled cavity Ω is unstable.

2. Symmetry assumption

We assume that the boundary $\partial\Omega$ is smooth enough and domains G, Ω are invariant with respect to the turn to the angle $\frac{\pi}{2}$ about the axis Ox_3 . Exactly speaking, it means that the following symmetry condition holds:

$$(S) \quad \begin{aligned} (x_1, x_2, x_3) \in \Omega &\iff (-x_2, x_1, x_3) \in \Omega \\ (x_1, x_2, x_3) \in G &\iff (-x_2, x_1, x_3) \in G \end{aligned}$$

Consider the inertia tensor $\mathbf{J}$. From (S) it follows that for any $\mathbf{w} = (w_1, w_2, w_3)$

$$(14) \quad \mathbf{J} \cdot \mathbf{w} = (a_0 w_1, a_0 w_2, b_0 w_3) = \operatorname{diag}(a_0, a_0, b_0) \cdot \mathbf{w}$$

where

$$(15) \quad a_0 = \rho_l \int_{\Omega} (x_1^2 + x_2^2) dx_1 dx_2 dx_3 + \rho_b \int_G (x_1^2 + x_2^2) dx_1 dx_2 dx_3$$

$$(16) \quad b_0 = 2\rho_l \int_{\Omega} x_3^2 dx_1 dx_2 dx_3 + 2\rho_b \int_G x_3^2 dx_1 dx_2 dx_3$$

Denote

$$(17) \quad c_0 = b_0 - a_0 = \rho_l \int_{\Omega} (x_1^2 - x_3^2) dx_1 dx_2 dx_3 + \rho_b \int_G (x_1^2 - x_3^2) dx_1 dx_2 dx_3$$

It is easy to check that if $\lambda = 0$ then $(\mathbf{u}, \mathbf{w}) \equiv \alpha(0, \mathbf{e}_3)$ satisfies (11) for any $\alpha \in \mathbb{C}$. Therefore $\lambda = 0$ is eigenvalue of the spectral problem (11). Let $\lambda \neq 0$. Then (11) can be rewritten as follows [7]:

$$(18) \quad \nu' \mathbf{A} \mathbf{u} + 2i \mathbf{T} \mathbf{u} + m \gamma \left[\frac{\gamma + i}{\gamma + ik} \mathbf{P}_1 \mathbf{u} + \frac{\gamma - i}{\gamma - ik} \mathbf{P}_2 \mathbf{u} + d \mathbf{P}_3 \mathbf{u} \right] = \gamma \mathbf{u}$$

where

$$(19) \quad \gamma = \frac{\lambda}{\omega_0}, \quad k = -\frac{c_0}{a_0}, \quad \nu' = \frac{\nu}{w_0}, \quad m = \frac{\rho_l \|\mathbf{c}_1\|_{\mathbf{L}_2(\Omega)}^2}{2a_0}, \quad d = \frac{a_0 \|\mathbf{c}_3\|_{\mathbf{L}_2(\Omega)}^2}{b_0 \|\mathbf{c}_1\|_{\mathbf{L}_2(\Omega)}^2}$$

$$(20) \quad \mathbf{c}_1 = \mathbf{P}_0((1, i, 0) \times \mathbf{r}), \quad \mathbf{c}_2 = \mathbf{P}_0((1, -i, 0) \times \mathbf{r}), \quad \mathbf{c}_3 = \mathbf{P}_0((0, 0, 1) \times \mathbf{r})$$

$$(21) \quad \mathbf{P}_j \mathbf{u} = \frac{(\mathbf{u}, \mathbf{c}_j)_{\mathbf{L}_2(\Omega)}}{\|\mathbf{c}_j\|_{\mathbf{L}_2(\Omega)}} \cdot \mathbf{c}_j, \quad j = 1, 2, 3, \quad \mathbf{u} \in \mathbf{J}_0(\Omega)$$

Lemma 1 [7]. $(\mathbf{c}_k, \mathbf{c}_n)_{\mathbf{L}_2(\Omega)} = 0$, $k, n \in \{1, 2, 3\}$, $k \neq n$; $\|\mathbf{c}_1\|_{\mathbf{L}_2(\Omega)} = \|\mathbf{c}_2\|_{\mathbf{L}_2(\Omega)}$.

It is easy to check that the following inequalities hold [7]

$$(22) \quad -1 < k < 1, \quad 0 < m < 1, \quad 0 < md < 1$$

Since $\omega_0 > 0$ then the spectral problem (11) has an eigenvalue $\lambda \in \mathbb{C}^- = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$ if and only if (18) has an eigenvalue $\gamma \in \mathbb{C}^-$.

3. Necessary condition

Let $\gamma \in \mathbb{C}^-$ be an eigenvalue of (69), $\mathbf{u}_\gamma$ be a corresponding eigenfunction. Then (18) implies

$$\begin{aligned} & \nu' (\mathbf{A} \mathbf{u}_\gamma, \mathbf{u}_\gamma)_{\mathbf{L}_2} + 2i (\mathbf{T} \mathbf{u}_\gamma, \mathbf{u}_\gamma)_{\mathbf{L}_2} + m(1 - k) \left[\frac{\gamma k + i \gamma^2}{\gamma^2 + k^2} \|\mathbf{P}_1 \mathbf{u}_\gamma\|_{\mathbf{L}_2}^2 + \frac{\gamma k - i \gamma^2}{\gamma^2 + k^2} \|\mathbf{P}_2 \mathbf{u}_\gamma\|_{\mathbf{L}_2}^2 \right] \\ & = \gamma \left(\|\mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 - m \|\mathbf{P}_1 \mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 - m \|\mathbf{P}_2 \mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 - md \|\mathbf{P}_3 \mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 \right) \end{aligned}$$

Since $\mathbf{A} \gg 0$, $\mathbf{T} = \mathbf{T}^*$, $\nu' > 0$ then using (22) and Lemma 1 we obtain

$$\begin{aligned} & \nu' (\mathbf{A} \mathbf{u}_\gamma, \mathbf{u}_\gamma)_{\mathbf{L}_2} + m(1 - k) k \operatorname{Re} \gamma \left[\|\mathbf{P}_1 \mathbf{u}_\gamma\|_{\mathbf{L}_2}^2 \frac{1}{|\gamma + ik|^2} + \|\mathbf{P}_2 \mathbf{u}_\gamma\|_{\mathbf{L}_2}^2 \frac{1}{|\gamma - ik|^2} \right] \\ & = \operatorname{Re} \gamma \left(\|\mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 - m \|\mathbf{P}_1 \mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 - m \|\mathbf{P}_2 \mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 - md \|\mathbf{P}_3 \mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 \right) \\ & \leq \operatorname{Re} \gamma (1 - \max(m, md)) \|\mathbf{u}_\gamma\|_{\mathbf{L}_2(\Omega)}^2 < 0 \end{aligned}$$

As far as $k \in (-1, 1)$, $m > 0$ then condition $\operatorname{Re} \gamma < 0$ implies

$$(23) \quad k > 0$$

which is a necessary condition for the existence an eigenvalue $\gamma \in \mathbb{C}^-$. Henceforth we shall assume $k \in (0, 1)$.

4. Instability criterion

We denote

$$\mathbf{D}(\gamma, k) = \left[\frac{\gamma + i}{\gamma + ik} \mathbf{P}_1 + \frac{\gamma - i}{\gamma - ik} \mathbf{P}_2 + d \mathbf{P}_3 \right]$$

three-dimensional operator in $\mathbf{J}_0(\Omega)$. Then (18) can be written as follows:

$$(24) \quad \nu' \mathbf{A} \mathbf{u} + 2i \mathbf{T} \mathbf{u} + m \gamma \mathbf{D}(\gamma, k) \mathbf{u} = \gamma \mathbf{u}$$

Using properties of $\mathbf{A}$ and applying usual arguments [5] it is easy to show that the operator $\nu' \mathbf{A} + 2i \mathbf{T}$ has discrete spectrum $\{\lambda_j(\nu')\}_{j=1}^{\infty}$. Since $\mathbf{A}^* = \mathbf{A} \gg 0$, $\mathbf{T} = \mathbf{T}^*$ and $\mathbf{A}^{-1}$ is compact then $\operatorname{Re} \lambda_j(\nu') \geq \nu' \lambda_1(\mathbf{A})$. Therefore for any $\nu' > 0$, $\gamma \in \mathbb{C}^-$ there exists

$$\mathbf{R}(\gamma, \nu') = (\nu' \mathbf{A} + 2i \mathbf{T} - \gamma \mathbf{I})^{-1}$$

Thus for $\gamma \in \mathbb{C}^-$ equation (24) can be written in the form

$$(25) \quad \mathbf{u} + m \gamma \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k) \mathbf{u} = 0$$

Applying the orthogonal projector $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3$ we can rewrite (25) as follows:

$$(26) \quad \mathbf{P} \mathbf{u} + m \gamma \mathbf{P} \circ \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k) \circ \mathbf{P} \mathbf{u} = 0$$

$$(27) \quad (\mathbf{I} - \mathbf{P}) \mathbf{u} + m \gamma (\mathbf{I} - \mathbf{P}) \circ \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k) \circ \mathbf{P} \mathbf{u} = 0$$

It is obvious that the system (26), (27) is solvable if and only if the equation (27) is solvable. Since $\mathbf{P} + m \gamma \mathbf{P} \circ \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k) \circ \mathbf{P}$ is a linear three-dimensional operator mapping $\mathbf{P} \mathbf{J}_0(\Omega)$ into $\mathbf{P} \mathbf{J}_0(\Omega)$ then (27) is solvable if and only if

$$(28) \quad \det \left\| (\mathbf{c}_j, \mathbf{c}_n)_{\mathbf{L}_2(\Omega)} + m \gamma (\mathbf{P} \circ \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k) \mathbf{c}_j, \mathbf{c}_n)_{\mathbf{L}_2(\Omega)} \right\|_{j,n=1,2,3} = 0$$

We denote for $\nu > 0, \gamma \in \mathbb{C}^-$

$$(29) \quad b_{jn}(\gamma, \nu) = (\mathbf{R}(\gamma, \nu) \mathbf{c}_j, \mathbf{c}_n)_{\mathbf{L}_2(\Omega)}, \quad j, n = 1, 2, 3$$

Lemma 2 [7]. For any $\gamma \in \mathbb{C}^-$, $\nu > 0$, $j, n \in \{1, 2, 3\}$, $j \neq n$

$$(30) \quad b_{jn}(\gamma, \nu) = 0$$

From (29), (30) it follows that equation (28) can be rewritten as follows:

$$(31) \quad 0 = \left[\|\mathbf{c}_1\|_{\mathbf{L}_2(\Omega)}^2 + m \gamma \frac{\gamma + i}{\gamma + ik} b_{11}(\gamma, \nu') \right] \cdot \left[\|\mathbf{c}_2\|_{\mathbf{L}_2(\Omega)}^2 + m \gamma \frac{\gamma - i}{\gamma - ik} b_{22}(\gamma, \nu') \right] \cdot \left[\|\mathbf{c}_3\|_{\mathbf{L}_2(\Omega)}^2 + m d \gamma b_{33}(\gamma, \nu') \right]$$

We denote for any $\nu > 0$, $m, k \in (0, 1)$, $\gamma \in \mathbb{C}^-$

$$(32) \quad \begin{cases} f_1(\gamma, \nu, m, k) = \|\mathbf{c}_1\|_{L_2(\Omega)}^2 + m\gamma \frac{\gamma+i}{\gamma+ik} b_{11}(\gamma, \nu) \\ f_2(\gamma, \nu, m, k) = \|\mathbf{c}_2\|_{L_2(\Omega)}^2 + m\gamma \frac{\gamma-i}{\gamma-ik} b_{22}(\gamma, \nu) \\ f_3(\gamma, \nu, m, k) = \|\mathbf{c}_3\|_{L_2(\Omega)}^2 + m\gamma b_{33}(\gamma, \nu) \end{cases}$$

Functions $f_j(\gamma, \nu, m, k)$, $j = 1, 2, 3$ are analytic in $\gamma \in \mathbb{C}^-$ for any $\nu > 0$, $m, k \in (0, 1)$. Since

$$\lim_{\substack{|\gamma| \rightarrow \infty \\ \gamma \in \mathbb{C}^-}} |\gamma| \cdot \|\mathbf{R}(\gamma, \nu)\| = 1$$

then for any $\nu > 0$, $m \in (0, 1)$ there exists $r(\nu, m) > 0$ such that

$$(33) \quad f_j(\gamma, \nu, m, k) \neq 0 \quad , \quad \gamma \in \mathbb{C}^- \quad , \quad |\gamma| \geq r(\nu, m)$$

for any $j = 1, 2, 3$, $k \in (0, 1)$.

Thus we obtain the following instability criterion:

spectral problem (18) has an eigenvalue $\gamma \in \mathbb{C}^-$ if and only if

$$f_1(\gamma, \nu', m, k) \cdot f_2(\gamma, \nu', m, k) \cdot f_3(\gamma, \nu', md, k) = 0$$

All the eigenvalues $\gamma \in \mathbb{C}^-$ are situated inside the half-disk

$$\{z \in \mathbb{C}^- \mid |z| \leq \max(r(\nu', m), r(\nu', md))\}$$

Acknowledgement. I would like to express my thanks to Professor T. Nishida for bringing this problem to my attention and for helpful discussions. I also wish to thank Japan Society for the Promotion of Science for the opportunity to visit Japan and to conduct my researches in Kyoto University.

REFERENCES

1. Chernous'ko F.L., *Motion of a rigid body with cavities containing a viscous fluid*, Moscow, 1968. (Russian)
2. Stewattson K. and Roberts P.H., *On the motion of a liquid in a spheroidal cavity of a precessing rigid body*, Journal of Fluid Mechanics **17** (1963), no. 1, 1-20.
3. Greenspan H.P. and Howard L.N., *On a time-dependent motion of a rotating fluid*, Journal of Fluid Mechanics **17** (1963), no. 3, 385-404.
4. Ngo Zui Kan, *On the rotation of a rigid body with a cavity filled with viscous liquid*, U.S.S.R. Computational Mathematics and Mathematical Physics **11** (1971), no. 6, 161-172.
5. Kopachevsky N.D., Krein S.G. and Ngo Zui Kan, *Operator methods in linear hydrodynamics: Evolution and spectral problems*, Nauka, Moscow, 1989. (Russian)
6. Yudovich V.I., *The linearization method in hydrodynamical stability theory*, Translations of Math. Monographs, **74**, 1989.
7. A.A.Lyashenko, *On the instability of a rotating body with a cavity filled with viscous liquid*, To appear in Japan Journal of Industrial and Applied Mathematics.