

Relaxation in the Cauchy problem for Hamilton-Jacobi equations

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1. Introduction. In this note we study a little further the *relaxation* of Hamilton-Jacobi equations developed recently in [4,5]. In [4] we initiated the study of the relaxation of Hamilton-Jacobi equations of eikonal type and in [5] we extended this study to a larger class of Hamilton-Jacobi equations.

Let us recall the relaxation in calculus of variations. In general a non-convex variational problem (P) does not have its minimizer. A natural way to attack such a variational problem is to introduce its relaxed (or convexified) variational problem (RP) which has a minimizer and to regard such a minimizer as a generalized solution of the original problem (P). The main result (or principle) in this direction states that $\min(\text{RP}) = \inf(\text{P})$. That is, any accumulation point of a minimizing sequence of (P) is a minimizer of (RP). This fact or principle is called the relaxation of non-convex variational problems. See [3] for a treatment of the relaxation of non-convex variational problems.

Relaxation of Hamilton-Jacobi equations is the principle which says that the pointwise supremum over a suitable collection of Lipschitz continuous subsolutions in the almost everywhere sense of a non-convex Hamilton-Jacobi equation yields a viscosity solution of the equation with convexified Hamiltonian. See [4,5].

Here we are concerned with the Cauchy problem for Hamilton-Jacobi equations and generalize some results obtained in [5].

2. Main result for the Cauchy Problem. We consider the Cauchy Problem

$$(1) \quad u_t(x, t) + H(x, D_x u(x, t)) = 0 \quad \text{for } (x, t) \in \mathbf{R}^n \times (0, T),$$

$$(2) \quad u|_{t=0} = g,$$

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where H and g are given continuous functions respectively on $\mathbf{R}^{2n}$ and $\mathbf{R}^n$, T is a given positive number or $T = \infty$, $u = u(x, t)$ is the unknown continuous function on $\mathbf{R}^n \times [0, T)$, u_t denotes the t -derivative of u , and $D_x u$ denotes the x -gradient of u .

Let $\widehat{H}$ denote the convex envelope of the function H , that is,

$$\widehat{H}(x, p) = \sup\{l(p) \mid l \text{ affine function, } l(q) \leq H(x, q) \text{ for } q \in \mathbf{R}^n\}.$$

We also consider the convexified Hamilton-Jacobi equation

$$(3) \quad u_t(x, t) + \widehat{H}(x, D_x u(x, t)) = 0 \quad \text{for } (x, t) \in \mathbf{R}^n \times (0, T).$$

We use the notation: for $a \in \mathbf{R}^n$ and $r \geq 0$, $B^n(a, r)$ denotes the n -dimensional closed ball of radius r centered at a . For $\Omega \subset \mathbf{R}^m$, $BUC(\Omega)$ and $UC(\Omega)$ denote the spaces of bounded uniformly continuous functions on Ω and of uniformly continuous functions on Ω , respectively. Furthermore, $Lip(\Omega)$ denotes the space of Lipschitz continuous functions on Ω . Notice that $f \in Lip(\Omega)$ is not assumed to be a bounded function.

Throughout this note we assume:

$$(4) \quad H, \widehat{H} \in BUC(\mathbf{R}^n \times B^n(0, R)) \text{ for all } R > 0.$$

$$(5) \quad \lim_{R \rightarrow \infty} \inf \left\{ \frac{H(x, p)}{|p|} \mid (x, p) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus B^n(0, R)) \right\} > 0.$$

For $R > 0$ we define the function $H_R : \mathbf{R}^{2n} \rightarrow \mathbf{R} \cup \{\infty\}$ by

$$H_R(x, p) = \begin{cases} H(x, p) & \text{if } x \in B^n(0, R), \\ \infty & \text{if } x \notin B^n(0, R), \end{cases}$$

and write $\widehat{H}_R$ for $\widehat{G}$, where $G = H_R$.

$$(6) \quad \text{For each } R > 0 \text{ and } \varepsilon > 0 \text{ there is a constant } \rho \geq R \text{ such that}$$

$$\widehat{H}_\rho(x, p) \leq \widehat{H}(x, p) + \varepsilon \quad \text{for } (x, p) \in \mathbf{R}^n \times B^n(0, R).$$

$$(7) \quad g \in UC(\mathbf{R}^n).$$

Proposition 1. (i) If $u \in USC(\mathbf{R}^n \times [0, T))$ and $v \in LSC(\mathbf{R}^n \times [0, T))$ are a viscosity subsolution and a viscosity supersolution of (3) respectively. Assume that $u(x, 0) \leq v(x, 0)$ for $x \in \mathbf{R}^n$ and that there is a (concave) modulus ω such that for all $(x, t) \in \mathbf{R}^n \times [0, T)$ and $y \in \mathbf{R}^n$,

$$\begin{cases} u(x, t) \leq u(y, 0) + \omega(|x - y| + t), \\ v(x, t) \geq v(y, 0) - \omega(|x - y| + t). \end{cases}$$

Then $u \leq v$ on $\mathbf{R}^n \times [0, T)$. (ii) There is a (unique) viscosity solution $u \in UC(\mathbf{R}^n \times [0, \infty))$ of (3) which satisfies (2). If, in addition, $g \in Lip(\mathbf{R}^n)$, then $u \in Lip(\mathbf{R}^n \times [0, \infty))$.

We remark that the same proposition as above is valid for (1). We omit giving the proof of the above proposition.

Let $\mathcal{V}_T$ denote the set of functions $v \in \text{Lip}(\mathbf{R}^n \times [0, T])$ such that

$$(8) \quad v_t(x, t) + H(x, D_x v(x, t)) \leq 0 \quad \text{a.e. } (x, t) \in \mathbf{R}^n \times (0, T).$$

The following theorem is the main result in this note.

Theorem 2. *Assume that (4)–(7) hold. Let $u \in \text{UC}(\mathbf{R}^n \times [0, T])$ be the unique viscosity solution of (3) satisfying (2). Then, for $(x, t) \in \mathbf{R}^n \times [0, T]$,*

$$(9) \quad u(x, t) = \sup\{v(x, t) \mid v \in \mathcal{V}_T, v|_{t=0} \leq g\}.$$

Remark. In general the above formula does not give a subsolution of

$$u_t(x, t) + H(x, D_x u(x, t)) = 0 \quad \text{a.e. } (x, t) \in \mathbf{R}^n \times (0, \infty).$$

For instance, let $n = 2$ and define $H \in C(\mathbf{R}^2)$ and $g \in \text{UC}(\mathbf{R}^2)$ by $H(p, q) = (|p|^{\frac{1}{2}} + |q|^{\frac{1}{2}})^2$ and $g(x, y) = -|x| - |y|$, respectively. Note that $\hat{H}(p, q) = |p| + |q|$ for $(p, q) \in \mathbf{R}^2$. We set $\rho(x, y, t) = -2t - |x| - |y|$. Then, for instance, by computing $D^\pm \rho(x, y, t)$, we infer that ρ is the viscosity solution of

$$\begin{cases} u_t(x, y, t) + |u_x(x, y, t)| + |u_y(x, y, t)| = 0 & \text{in } \mathbf{R}^2 \times (0, \infty), \\ u(x, y, 0) = g(x, y) & \text{for } (x, y) \in \mathbf{R}^2. \end{cases}$$

On the other hand, since at any point $(x, y, t) \in \mathbf{R}^2 \times (0, \infty)$, where $x, y \neq 0$, we have

$$H(\rho_x(x, y, t), \rho_y(x, y, t)) = 4, \quad \rho_t(x, y, t) = -2,$$

ρ is not a subsolution of

$$u_t(x, y, t) + (|u_x(x, y, t)|^{\frac{1}{2}} + |u_y(x, y, t)|^{\frac{1}{2}})^2 = 0 \quad \text{a.e. } (x, y, t) \in \mathbf{R}^n \times (0, \infty).$$

Theorem 2 is an easy consequence of the following theorem.

Theorem 3. *Assume that (4)–(6) hold. Let $u \in \text{UC}(\mathbf{R}^n \times [0, T])$ be a viscosity subsolution of (3). Then, for all $(x, t) \in \mathbf{R}^n \times [0, T]$,*

$$(10) \quad u(x, t) = \sup\{v(x, t) \mid v \in \mathcal{V}_T, v \leq u \text{ in } \mathbf{R}^n \times [0, T]\}.$$

Conceding Theorem 3 for the moment, we finish the proof of Theorem 2 as follows.

Proof of Theorem 2. We write $w(x, t)$ for the right hand side of (9). By Theorem 3 we find that $u \leq w$ on $\mathbf{R}^n \times [0, T)$. Let $v \in \mathcal{V}_T$ satisfy $v(\cdot, 0) \leq g$ on $\mathbf{R}^n$. Then, since $\hat{H} \leq H$, we have

$$v_t(x, t) + \hat{H}(x, D_x v(x, t)) \leq 0 \quad \text{a.e. } (x, t) \in \mathbf{R}^n \times (0, T).$$

Since $\hat{H}(x, \cdot)$ is convex, v is a viscosity subsolution of (3). By (i) of Proposition 1, we have $v \leq u$ on $\mathbf{R}^n \times (0, T)$, from which we get $w \leq u$ on $\mathbf{R}^n \times (0, T)$. Thus we have $u = w$ on $\mathbf{R}^n \times (0, T)$. $\square$

For our proof of Theorem 3, we need several lemmas. For a proof of the next three lemmas, we refer to [5].

Lemma 4. Let K be a non-empty convex subset of $\mathbf{R}^m$ and set

$$L(\xi) = \sup\{\xi \cdot p \mid p \in K\} \in \mathbf{R} \cup \{\infty\} \quad \text{for all } \xi \in \mathbf{R}^m.$$

Let U be an open subset of $\mathbf{R}^m$ and let $v \in C(\bar{U})$ satisfy

$$D^+v(x) \subset K \quad \text{for all } x \in U.$$

Let $x, y \in \bar{U}$, and assume that the open line segment $l_0(x, y) := \{tx + (1-t)y \mid t \in (0, 1)\} \subset U$. Then

$$u(x) \leq u(y) + L(x - y).$$

In the above lemma and in what follows, for $v \in C(U)$ and $x \in U$, $D^+v(x)$ denotes the superdifferential of v at x .

Lemma 5. Let Ω be an open subset of $\mathbf{R}^m$ and $f_1, \dots, f_N \in \text{Lip}(\Omega)$, with $N \in \mathbf{N}$. Set

$$f(x) = \max\{f_1(x), \dots, f_N(x)\} \quad \text{for } x \in \Omega.$$

Then $f \in \text{Lip}(\Omega)$ and $f, f_1, \dots, f_N$ are almost everywhere differentiable. Moreover for almost every $x \in \Omega$,

$$Df(x) \in \{Df_1(x), \dots, Df_N(x)\},$$

where $Df(x)$ denotes the gradient of f at x .

Lemma 6. Let Z be a non-empty closed subset of $\mathbf{R}^m$. Define $L : \mathbf{R}^m \rightarrow \mathbf{R} \cup \{\infty\}$ by

$$L(\xi) = \sup\{\xi \cdot p \mid p \in Z\}.$$

Let $\bar{\xi} \in \mathbf{R}^m$ be a point where L is differentiable. Then

$$DL(\bar{\xi}) \in Z \cap \partial(\bar{\text{co}} Z)$$

We introduce the notation: for $(x, r) \in \mathbf{R}^n \times \mathbf{R}$ let

$$Z(x, r) := \{(p, q) \in \mathbf{R}^{n+1} \mid q + H(x, p) \leq r\}$$

and $K(x, r) := \overline{\text{co}} Z(x, r)$, the closed convex hull of $Z(x, r)$. We note that

$$K(x, r) = \{(p, q) \in \mathbf{R}^{n+1} \mid q + \widehat{H}(x, p) \leq r\}.$$

For $\delta > 0$, let $\Delta(\delta) := \{(x, y) \in \mathbf{R}^{2n} \mid |x - y| \leq \delta\}$.

Lemma 7. *Assume that (4) holds. For any $R > 0$ and $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for any $(x, y) \in \Delta(\delta)$ and $r \in \mathbf{R}$,*

$$Z_R(x, r) + B^{n+1}(0, \delta) \subset Z_{R+1}(y, r + \varepsilon),$$

where, for $R > 0$, $Z_R(x, r) = Z(x, r) \cap B^{n+1}(0, R)$.

Proof. Fix $\varepsilon > 0$ and $R > 0$. Let ω denote the modulus of continuity of H on $\mathbf{R}^n \times B^n(0, R+1)$.

Fix a constant $\delta \in (0, 1)$ so that $\delta + \omega(2\delta) \leq \varepsilon$. Fix $(\xi, \eta) \in B^{n+1}(0, \delta)$, $(x, y) \in \Delta(\delta)$, $(p, q) \in Z_R(x, 0)$, and $r \in \mathbf{R}$.

Noting that $(p, q) + (\xi, \eta) \in B^{n+1}(0, R+1)$, we observe that

$$q + \eta + H(y, p + \xi) \leq q + H(x, p) + \eta + \omega(|x - y| + |\xi|) \leq r + \delta + \omega(2\delta) \leq r + \varepsilon.$$

Thus we have

$$(p + \xi, q + \eta) \in Z_{R+1}(y, r + \varepsilon),$$

which concludes the proof. $\square$

Lemma 8. *Assume that (4)–(6) hold. For any $R > 0$ and $\varepsilon > 0$ there exists a constant $M \geq R$ such that for any $x \in \mathbf{R}^n$,*

$$K_R(x, 0) \subset \text{co } Z_M(x, \varepsilon),$$

where $K_R(x, r) = K(x, r) \cap B^{n+1}(0, R)$.

Proof. For $R > 0$ and $\varepsilon > 0$ let $\rho \equiv \rho(R, \varepsilon) \geq R$ be the constant from (6). That is, $\rho = \rho(R, \varepsilon)$ is a constant for which

$$\widehat{H}_\rho(x, p) \leq \widehat{H}(x, p) + \varepsilon \quad \text{for } (x, p) \in \mathbf{R}^n \times B^n(0, R).$$

In view of (4), for $R > 0$ let $M_R \geq 0$ be the constant defined by

$$M_R = \sup\{|H(x, p)| \mid (x, p) \in \mathbf{R}^n \times B^n(0, R)\}.$$

Fix $R > 0$, $\varepsilon > 0$, $x \in \mathbf{R}^n$, and $(p, q) \in K_R(x, 0)$. We have

$$\widehat{H}(x, p) + q \leq 0,$$

and hence

$$\widehat{H}_\rho(x, p) + q \leq \varepsilon.$$

Choose sequences $\{\lambda_i\}_{i=1}^m \subset (0, 1]$ and $\{p_i\}_{i=1}^m \subset B^n(0, \rho)$, with $m \in \mathbf{N}$, so that

$$\begin{aligned} \sum_{i=1}^m \lambda_i p_i &= p, & \sum_{i=1}^m \lambda_i &= 1, \\ \sum_{i=1}^m \lambda_i H(x, p_i) + q &\leq 2\varepsilon. \end{aligned}$$

(See the proof of Lemma 10 below.) Setting

$$h = q + \sum_{i=1}^m \lambda_i H(x, p_i), \quad q_i = h - H(x, p_i) \quad \text{for } i = 1, 2, \dots, m,$$

we observe that

$$\begin{aligned} h &\leq 2\varepsilon, & h &\geq -|q| - M_\rho \geq -R - M_\rho, \\ |q_i| &\leq |h| + M_\rho \leq 2\varepsilon + R + 2M_\rho \quad \text{for } i = 1, 2, \dots, m, \end{aligned}$$

and that

$$\begin{aligned} (p_i, q_i) &\in Z(x, h) \subset Z(x, 2\varepsilon) \quad \text{for } i = 1, 2, \dots, m, \\ \sum_{i=1}^m \lambda_i q_i &= h - \sum_{i=1}^m \lambda_i H(x, p_i) = q, \\ \sum_{i=1}^m \lambda_i (p_i, q_i) &= (p, q). \end{aligned}$$

These together show that $(p, q) \in \text{co } Z_M(x, 2\varepsilon)$, with $M = (\rho^2 + (2\varepsilon + R + 2M_\rho)^2)^{1/2}$.
□

Proof of Theorem 3. We write $Q = \mathbf{R}^n \times (0, T)$ and $Q_\delta = \mathbf{R}^n \times (-\delta, T + \delta)$ for $\delta > 0$.

Firstly, without loss of generality we may assume that u is defined and Lipschitz continuous on Q_δ for some constant $\delta > 0$ and that

$$(11) \quad u_t(x, t) + \widehat{H}(x, D_x u(x, t)) \leq 0 \quad \text{in } Q_\delta$$

in the viscosity sense. Indeed, we have

$$(12) \quad u(x, t) = \sup\{v(x, t) \mid v \in \text{Lip}(Q_\delta) \text{ for some } \delta > 0, \\ v \text{ is a viscosity solution of (11), } v \leq u \text{ on } Q\}.$$

To see this, assuming $T < \infty$, we solve the Cauchy problem

$$w_t(x, t) + \hat{H}(x, D_x w(x, t)) \leq 0 \quad \text{in } \mathbf{R}^n \times (T, T+1)$$

with the initial condition

$$(13) \quad w(x, T) = \lim_{t \nearrow T} u(x, t) \quad \text{for } x \in \mathbf{R}^n.$$

In view of (4) and (5), there is a constant $C > 0$ such that $\hat{H}(x, p) \geq -C$ for all $(x, p) \in \mathbf{R}^{2n}$, which shows that u is a viscosity solution of $u_t \leq C$ in $\mathbf{R}^n \times (0, T)$. This monotonicity of the function $u(x, t)$ in t and the uniform continuity of u guarantee that the limit on the right hand side of (13) defines a uniform continuous function on $\mathbf{R}^n$.

By (ii) of Proposition 1, there is a unique viscosity solution $w \in \text{UC}(\mathbf{R}^n \times [T, T+1))$ for which (13) holds. We extend the domain of definition of w to $\mathbf{R}^n \times (0, T+1)$ by setting

$$w(x, t) = u(x, t) \quad \text{for } (x, t) \in \mathbf{R}^n \times (0, T).$$

It is easy to see that $w \in \text{UC}(\mathbf{R}^n \times (0, T+1))$ that w is a viscosity subsolution of

$$w_t(x, t) + \hat{H}(x, D_x w(x, t)) = 0 \quad \text{in } \mathbf{R}^n \times (0, T+1).$$

Now, if $T = \infty$, we define $w \in \text{UC}(\mathbf{R}^n \times [0, \infty))$ by setting $w = u$.

Fix any $\varepsilon > 0$. Since $w \in \text{UC}(\mathbf{R}^n \times (0, T+1))$, there is a constant $\delta \in (0, 1/2)$ such that

$$(14) \quad u(x, t) - 2\varepsilon \leq w(x, t - \delta) - \varepsilon \leq u(x, t) \quad \text{for } (x, t) \in \mathbf{R}^n \times (0, T).$$

It is clear that the function $z(x, t) := w(x, t - \delta) - 2\varepsilon$ is defined and uniformly continuous on Q_δ and is a viscosity solution of (11).

Now, we take the sup-convolution of z in the t -variable. That is, for $\gamma > 0$, we consider the function

$$z^\gamma(x, t) = \sup\{z(x, s) - \frac{1}{2\gamma}(t - s)^2 \mid s \in (-\delta, T + \delta)\} \quad \text{for } (x, t) \in \mathbf{R}^{n+1}.$$

If $\gamma > 0$ is small enough, then z^γ is a viscosity solution of (11) in $Q_{\delta/2}$ and

$$(15) \quad z(x, t) \leq z^\gamma(x, t) \leq z(x, t) + \varepsilon \quad \text{for } (x, t) \in Q_\delta.$$

Note also that, for each $\gamma > 0$, the collection of functions $z^\gamma(x, \cdot)$, with $x \in \mathbf{R}^n$, is equi-Lipschitz continuous on $(-\delta/2, T + \delta/2)$. By virtue of (5), we may choose constants $c_0 > 0$ and $C_1 > 0$ such that

$$\widehat{H}(x, p) \geq c_0|p| - C_1 \quad \text{for } (x, p) \in \mathbf{R}^{2n}.$$

Since z^γ is a viscosity solution of

$$c_0|D_x z^\gamma(x, t)| \leq C_1 + L_\gamma \quad \text{in } Q_{\delta/2},$$

where $L_\gamma > 0$ is a uniform Lipschitz bound of the functions $z^\gamma(x, \cdot)$ on $(-\delta/2, T + \delta/2)$, we see that the functions $z^\gamma(\cdot, t)$ are Lipschitz continuous on $\mathbf{R}^n$, with a Lipschitz bound independent of $t \in (-\delta/2, T + \delta/2)$.

Now, using (14) and (15) and writing $U(x, t)$ for the right hand side of (12), we see that for sufficiently small $\gamma > 0$ and for all $(x, t) \in Q$,

$$u(x, t) \geq z(x, t) + \varepsilon \geq z^\gamma(x, t),$$

and hence,

$$U(x, t) \geq z^\gamma(x, t) \geq z(x, t) \geq u(x, t) - 3\varepsilon,$$

which proves (12).

Henceforth we assume that, for some constant $\delta > 0$, u is a member of $\text{Lip}(Q_\delta)$ and satisfies (11) in the viscosity sense.

Let $R > 0$ be a Lipschitz bound of the function u . Fix any $\varepsilon \in (0, 1)$. Due to Lemma 8, there is a constant $\rho \geq R$ such that for all $x \in \mathbf{R}^n$,

$$K_R(x, 0) \subset \text{co } Z_\rho(x, \varepsilon).$$

In view of Lemma 7, there is a constant $\gamma \in (0, 1)$ such that for any $(x, y) \in \Delta(\gamma)$,

$$Z_\rho(x, \varepsilon) + B^{n+1}(0, \gamma) \subset Z_{\rho+1}(y, 2\varepsilon).$$

$$Z_{\rho+1}(y, 2\varepsilon) \subset Z_{\rho+2}(x, 3\varepsilon).$$

Consequently, for $(x, y) \in \Delta(\gamma)$, we have

$$(16) \quad K_R(x, 0) + B^{n+1}(0, \gamma) \subset \text{co } Z_{\rho+1}(y, 2\varepsilon),$$

$$(17) \quad Z_{\rho+1}(y, 2\varepsilon) \subset Z_{\rho+2}(x, 3\varepsilon).$$

We may assume that $\gamma < \delta$. Let $\mu \in (0, \gamma)$ be a constant to be fixed later. We choose a set $Y_\mu \subset Q_\delta$ so that

$$(18) \quad \#(Y_\mu \cap B^{n+1}(0, r)) < \infty \quad \text{for all } r > 0,$$

$$(19) \quad \bigcup_{(y, s) \in Y_\mu} B^{n+1}((y, s), \mu) \supset Q_\delta.$$

We set

$$L(\xi, \eta; y) = \sup\{\xi \cdot p + \eta q \mid (p, q) \in Z_{\rho+1}(y, 2\varepsilon)\} \quad \text{for } \xi, y \in \mathbf{R}^n, \eta \in \mathbf{R}$$

and

$$v(x, t; y, s) = u(y, s) + L(x - y, t - s; y) \quad \text{for } (x, t) \in \mathbf{R}^{n+1}, (y, s) \in Q_\delta.$$

By Lemma 6, we get for $(x, y) \in \Delta(\gamma)$,

$$(20) \quad D_{\xi, \eta} L(\xi, \eta; y) \in Z_{\rho+1}(y, 2\varepsilon) \subset Z_{\rho+2}(x, 3\varepsilon) \quad \text{a.e. } (\xi, \eta) \in \mathbf{R}^{n+1}.$$

Noting that

$$D^+ u(x, t) \subset K_R(x, 0) \quad \text{for } (x, t) \in Q_\delta,$$

and setting $\tilde{u}(x, t) := u(x, t) + \gamma|(x, t) - (y, s)|$ for $(x, t), (y, s) \in Q_\delta$, we find that for $(x, t), (y, s) \in Q_\delta$, if $0 < |x - y| \leq \gamma$, then

$$D^+ \tilde{u}(x, t) \subset D^+ u(x, t) + B^{n+1}(0, \gamma) \subset \text{co } Z_{\rho+1}(y, 2\varepsilon).$$

Hence, by Lemma 4, we get

$$(21) \quad u(x, t) + \gamma|(x, t) - (y, s)| \leq v(x, t; y, s) \quad \text{for } (x, t), (y, s) \in Q_\delta, \text{ with } |x - y| \leq \delta.$$

Set $\beta = \gamma/5$ and define the function $w : Q_{2\beta} \rightarrow \mathbf{R}$ by

$$w(x, t) = \min\{v(x, t; y, s) \mid (y, s) \in Y_\mu \cap B^{n+1}((x, t), 3\beta)\}.$$

Now, we show that if μ is sufficiently small, then for $(\bar{x}, \bar{t}) \in Q_\beta$ and $(x, t) \in B^{n+1}((\bar{x}, \bar{t}), \beta)$

$$(22) \quad w(x, t) = \min\{v(x, t; y, s) \mid (y, s) \in Y_\mu \cap B^{n+1}((\bar{x}, \bar{t}), 2\beta)\}.$$

To do this, fix $(\bar{x}, \bar{t}) \in Q_\beta$ and $(x, t) \in Y_\mu \cap B^{n+1}((\bar{x}, \bar{t}), 2\beta)$. Noting that $Y_\mu \cap B^{n+1}((x, t), \mu) \neq \emptyset$ and $B^{n+1}((x, t), \mu) \subset B^{n+1}((x, t), 5\beta)$ and choosing a point $(y, s) \in Y_\mu \cap B^{n+1}((x, t), \mu)$, we see that

$$\begin{aligned} w(x, t) &\leq v(x, t; y, s) \leq u(y, s) + (\rho + 1)|(x, t) - (y, s)| \\ &\leq u(x, t) + (R + \rho + 1)|(x, t) - (y, s)|. \end{aligned}$$

Here we have used the fact that the functions $L(\xi, \eta; y)$ of (ξ, η) are Lipschitz continuous functions with $\rho + 1$ as a Lipschitz bound. Fix now $\mu \in (0, \gamma)$ by setting

$$\mu = \frac{1}{2} \min\left\{\gamma, \frac{\gamma\beta}{R + \rho + 1}\right\}$$

and observe that

$$(23) \quad w(x, t) < u(x, t) + \gamma\beta.$$

Fix $(y, s) \in Q_\delta \setminus B^{n+1}((\bar{x}, \bar{t}), 2\beta)$ and note that $|(y, s) - (x, t)| \geq \beta$. Using (21), we have

$$v(x, t; y, s) \geq u(x, t) + \gamma\beta.$$

From this and (23), we conclude that (22) holds.

Next, we observe from (22) that the function w is Lipschitz continuous on $B^{n+1}((\bar{x}, \bar{t}), \beta)$ for all $(\bar{x}, \bar{t}) \in Q_\beta$, with $\rho + 1$ as a Lipschitz bound, which guarantees that $w \in \text{Lip}(Q_\beta)$. Applying Lemma 5 and using (20), we observe that w is almost everywhere differentiable on Q_β and, at any point $(x, t) \in Q_\beta$ where w is differentiable,

$$Dw(x, t) \in \bigcup \{D_{x,t}v(x, t; y, s) \mid (y, s) \in Y_\mu \cap B^{n+1}((\bar{x}, \bar{t}), 2\beta)\} \subset Z_{\rho+2}(x, 3\varepsilon),$$

which yields readily

$$w_t(x, t) + H(x, D_x w(x, t)) \leq 3\varepsilon \quad \text{a.e. } (x, t) \in Q_\beta.$$

Setting

$$z(x, t) = w(x, t) - \gamma\beta - 3\varepsilon t \quad \text{for } (x, t) \in Q_\beta,$$

we have

$$z_t(x, t) + H(x, D_x z(x, t)) \leq 0 \quad \text{a.e. } (x, t) \in Q_\beta.$$

By (23), we have $z(x, t) \leq u(x, t) - 3\varepsilon t$ for $(x, t) \in Q_\beta$ and, by (21), we have $z(x, t) \geq u(x, t) - \gamma\beta - 3\varepsilon t$ for $(x, t) \in Q_\beta$. In the above two inequalities, we may take $\gamma > 0$ as small as we wish. Thus we get

$$u(x, t) = \sup\{z(x, t) \mid z \in \mathcal{V}_T, z \leq u \text{ on } Q\} \quad \text{for } (x, t) \in Q,$$

which completes the proof. $\square$

3. Examples. In this section we consider some examples of Hamiltonians H and examine if H satisfies conditions (4)–(6) or not.

Let $H \in C(\mathbf{R}^{2n})$ be a function of the form

$$H(x, p) = G(x, p)^m + f(x),$$

where $G \in C(\mathbf{R}^{2n})$ satisfies

$$(24) \quad G \in \text{BUC}(\mathbf{R}^n \times B^n(0, R)) \quad \text{for } R > 0,$$

$$(25) \quad G(x, \lambda p) = \lambda G(x, p) \quad \text{for } \lambda \geq 0, (x, p) \in \mathbf{R}^{2n},$$

$$(26) \quad \delta_G := \inf_{\mathbf{R}^n \times \partial B^n(0,1)} G > 0.$$

m is a constant satisfying $m \geq 1$, and $f \in \text{BUC}(\mathbf{R}^n)$.

Proposition 9. *The function H given above satisfies (4)–(6).*

We need the following Lemma.

Lemma 10. *For all $(x, p) \in \mathbf{R}^{2n}$, we have*

$$(27) \quad \widehat{G}(x, p) = \min\{r \in \mathbf{R} \mid p = \sum_{i=1}^k \lambda_i p_i, \lambda_i > 0, \sum_{i=1}^k \lambda_i = 1, G(x, p_i) = r\}.$$

Proof. We fix $x \in \mathbf{R}^n$ and write $G(p)$ for $G(x, p)$ for notational simplicity. By using the separation theorem and Carathéodory's theorem in convex analysis, we see easily that

$$(28) \quad \widehat{G}(p) = \inf\left\{\sum_{i=1}^{n+1} \lambda_i G(p_i) \mid \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, \sum_{i=1}^{n+1} \lambda_i p_i = p\right\} \quad \text{for } p \in \mathbf{R}^n.$$

It is clear from the above representation formula that

$$\begin{aligned} \widehat{G}(\lambda p) &= \lambda \widehat{G}(p) \quad \text{for } (\lambda, p) \in [0, \infty) \times \mathbf{R}^n, \\ G(p) &\geq \widehat{G}(p) \geq \delta_G |p| \quad \text{for } p \in \mathbf{R}^n. \end{aligned}$$

Fix $p \in \mathbf{R}^n$. If $p = 0$, then it is clear that (27) holds. We may thus assume that $p \neq 0$. For any $r > \widehat{G}(p)$, by the above formula, there are $\{\lambda_i\}_{i=1}^{n+1} \subset [0, 1]$ and $\{p_i\}_{i=1}^{n+1} \subset \mathbf{R}^n$ such that

$$r > \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \sum_{i=1}^{n+1} \lambda_i = 1, \quad \sum_{i=1}^{n+1} \lambda_i p_i = p.$$

Set

$$s = \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \mu_i = s^{-1} G(p_i).$$

Notice that $s \geq \widehat{G}(p) > 0$ by (28). By rearranging the order in i if necessary, we may assume that

$$\lambda_i \mu_i > 0 \quad \text{for } i \leq k, \quad \lambda_i \mu_i = 0 \quad \text{for } i > k$$

for some $k \in \{1, \dots, n+1\}$. Note that if $i > k$ and $\lambda_i > 0$, then $p_i = 0$. We now have

$$\begin{aligned} \sum_{i=1}^k \lambda_i \mu_i &= s^{-1} \sum_{i=1}^{n+1} \lambda_i G(p_i) = 1, \\ \sum_{i=1}^k \lambda_i \mu_i (\mu_i^{-1} p_i) &= \sum_{i=1}^k \lambda_i p_i = \sum_{i=1}^{n+1} \lambda_i p_i = p, \\ G(\mu_i^{-1} p_i) &= s G(p_i)^{-1} G(p_i) = s \quad \text{for } i = 1, \dots, k. \end{aligned}$$

Hence we get

$$\widehat{G}(p) \geq \inf\{s \in \mathbf{R} \mid \lambda_i > 0, G(p_i) = s, \sum_{i=1}^k \lambda_i p_i = p, k \leq n+1\}.$$

Since the set $\{q \in \mathbf{R}^n \mid G(q) \leq \widehat{G}(p) + 1\}$ is a compact set, it is not hard to see that the infimum on the right hand side of the above inequality is actually attained. That is, we have

$$\widehat{G}(p) \geq \min\{s \in \mathbf{R} \mid \lambda_i > 0, G(p_i) = s, \sum_{i=1}^k \lambda_i p_i = p, k \leq n+1\}.$$

The opposite inequality is obvious. The proof is now complete. $\square$

Proof of Proposition 9. First we observe that

$$(29) \quad \widehat{H}(x, p) = \widehat{G}(x, p)^m + f(x) \quad \text{for } (x, p) \in \mathbf{R}^{2n}.$$

Indeed, since the function:

$$p \mapsto \widehat{G}(x, p)^m + f(x)$$

is convex on $\mathbf{R}^n$ for every $x \in \mathbf{R}^n$ and

$$\widehat{G}(x, p)^m + f(x) \leq H(x, p) \quad \text{for } (x, p) \in \mathbf{R}^{2n},$$

we see that

$$\widehat{G}(x, p)^m + f(x) \leq \widehat{H}(x, p) \quad \text{for } (x, p) \in \mathbf{R}^{2n}.$$

On the other hand, by Lemma 10, for $(x, p) \in \mathbf{R}^{2n}$ we have

$$\begin{aligned} \widehat{G}(x, p)^m &= \min\{r^m \in \mathbf{R} \mid k \leq n+1, \lambda_i > 0, G(x, p_i) = r, \sum_{i=1}^k \lambda_i = 1, \sum_{i=1}^k \lambda_i p_i = p\} \\ &\geq \inf\left\{\sum_{i=1}^k \lambda_i G(x, p_i)^m \mid k \in \mathbf{N}, \lambda_i > 0, \sum_{i=1}^k \lambda_i = 1, \sum_{i=1}^k \lambda_i p_i = p\right\}. \end{aligned}$$

Hence, by the formula

$$\widehat{H}(x, p) = \inf\left\{\sum_{i=1}^k \lambda_i H(x, p_i) \mid k \in \mathbf{N}, \lambda_i > 0, \sum_{i=1}^k \lambda_i = 1, \sum_{i=1}^k \lambda_i p_i = p\right\},$$

we have

$$\widehat{G}(x, p)^m + f(x) \geq \widehat{H}(x, p).$$

Thus we have shown (29).

To show that H satisfies (4), we just need to prove that

$$\widehat{G} \in \text{BUC}(\mathbf{R}^n \times B^n(0, R)) \quad \text{for } R > 0.$$

Fix $R > 0$, set

$$\rho_1 = \sup_{\mathbf{R}^n \times B^n(0, R)} G,$$

and, in view of (26), choose $\rho_2 > 0$ so that

$$\inf_{\mathbf{R}^n \times (\mathbf{R}^n \setminus B^n(0, \rho_2))} G > \rho_1.$$

Then, by Lemma 10, we have

$$\begin{aligned} \widehat{G}(x, p) &= \min \left\{ \sum_{i=1}^k \lambda_i G(x, p_i) \mid \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1, G(x, p_i) \leq \rho_1, \sum_{i=1}^k \lambda_i p_i = p \right\} \\ &= \min \left\{ \sum_{i=1}^k \lambda_i G(x, p_i) \mid \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1, p_i \in B^n(0, \rho_2), \sum_{i=1}^k \lambda_i p_i = p \right\} \\ &\quad \text{for } (x, p) \in \mathbf{R}^n \times B^n(0, R). \end{aligned}$$

This shows that the collection of functions:

$$x \mapsto \widehat{G}(x, p),$$

with $p \in B^n(0, R)$, is equi-continuous on $\mathbf{R}^n$. On the other hand,

$$\{\widehat{G}(x, \cdot) \mid x \in \mathbf{R}^n\}$$

is a uniformly bounded collection of convex functions on $B^n(0, R)$. Consequently, this collection is equi-Lipschitz continuous on $B^n(0, R)$. Thus we see that $\widehat{G} \in \text{BUC}(\mathbf{R}^n \times B^n(0, R))$ for all $R > 0$.

By assumptions (25) and (26), H clearly satisfies (5).

To show (6), fix $R > 0$ and choose $\rho_2 > 0$ as above. Then, by Lemma 10, we get

$$\begin{aligned} \widehat{G}(x, p)^m &= \min \left\{ \sum_{i=1}^k \lambda_i G(x, p_i)^m \mid k \in \mathbf{N}, \lambda_i \geq 0, G(x, p_i) = \widehat{G}(x, p), \right. \\ &\quad \left. \sum_{i=1}^k \lambda_i = 1, \sum_{i=1}^k \lambda_i p_i = p \right\} \\ &= \min \left\{ \sum_{i=1}^k \lambda_i G(x, p_i)^m \mid k \in \mathbf{N}, \lambda_i \geq 0, p_i \in B^n(0, \rho_2), \right. \\ &\quad \left. \sum_{i=1}^k \lambda_i = 1, \sum_{i=1}^k \lambda_i p_i = p \right\}. \end{aligned}$$

Hence we have

$$\widehat{H}(x, p) = \widehat{H}_{\rho_2}(x, p) \quad \text{for } (x, p) \in \mathbf{R}^n \times B^n(0, R).$$

Thus H satisfies (4)–(6). $\square$

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