

# Resplendent models of $\omega$ -minimal expansions of RCOF

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## Abstract

In this paper, the author gives a characterization of resplendent models of the axioms, formulated by van den Dries, of restricted analytic real fields.

## 1 Introduction.

In classical model theory, we usually investigate properties of first order theories  $T$ , using their models. The properties that we are interested in are, for example, those concerning the existence of special types of models of  $T$ , such as prime models, saturated models and compact models, and so on. Of course, such models as listed above do not always exist. A saturated model of a complete  $T$  exists under the assumption of G.C.H., but it does not exist in general without such assumptions. However, if we replace the definition of saturation by a weaker version, we can sometimes show its existence without set theoretic assumptions. Especially, every theory has a recursively saturated model. J.P. Ressayre shows the following important fact on recursive saturation, which states that resplendence and recursive saturation coincide for countable structures.

**1 Fact. (J.P. Ressayre(1972)[5])** For each countable structure  $M$  of finite language,  $M$  is resplendent if and only if  $M$  is recursively saturated.

It is not hard to show the existence of a recursively saturated model. From the fact above, we know that a resplendent model also exists for any countable theory. Resplendence seems a useful property to be studied. In Ressayre's proof of the only if part of the fact above, he finds some consistent sentence  $\varphi(P)$  with a new unary predicate  $P$  such that if a structure has a solution of  $P$ , then the structure is recursively saturated. There are some works aiming to get a more concrete  $\varphi(P)$ , when the axioms are specified. For example, P. D' Aquino, J.F. Knight and S. Starchenko find a characterization of recursively saturated model in the theory of real closed field([1]). Moreover, the author and A. Tsuboi found a characterization of recursive saturation in an  $\omega$ -minimal effectively model complete theory of real closed fields with a finite number of functions. This can be applied to A. J. Wilkie's exponential fields([8]). However, we cannot apply this result to van den Dries's restricted analytic field because the restricted analytic field is not a constructive object. The author considered a constructive fragment of theories for restricted analytic fields and find a characterization of recursive saturation for models of such theories.

## 2 Preliminaries and basic facts.

Let  $L$  be a finite language,  $M$  an  $L$ -structure,  $T$  an  $L$ -theory(not necessarily complete). Let  $L_{or}$  be the language  $\{+, \cdot, 0, 1, <\}$  of ordered rings,  $RCOF$  the theory of real closed fields,  $PA$  the theory of first order arithmetic. Let  $Th(M) := \{\phi : \phi \text{ is an } L\text{-sentence, } M \models \phi\}$  be a theory of  $M$ ,  $Diag_{el}(M) := \{\phi : \phi \text{ is an } L(M)\text{-sentence, } M \models \phi\}$  an elementary diagram of  $M$ .

**2 Definition.** (1). We say that  $M$  is **resplendent** if for any new relational symbol  $R \notin L$  and any  $L(M) \cup \{R\}$ -sentence  $\phi(R)$  if  $Diag_{el}(M) \cup \{\phi(R)\}$  is consistent, then there is an interpretation  $R^M$  on  $M$  such that  $(M, R^M) \models \phi(R)$ .

(2). We say that  $M$  is **recursively saturated** if every recursive type(with finite parameters) is realized in  $M$ .

**3 Fact. (J.P. Ressayre(1972)[5])** For each countable structure  $M$  of finite language,  $M$  is resplendent if and only if  $M$  is recursively saturated.

In Ressayre's proof of the only if part of the fact above, he finds some consistent sentence  $\varphi(P)$  with a new unary predicate  $P$  such that if a structure has a solution of  $P$ , then the structure is recursively saturated. By the meaning of  $\varphi(P)$  in Ressayre's proof, we can construct a model of arithmetic from a solution of  $\varphi(P)$ .

**4 Question.** If a theory  $T$  naturally involves some arithmetic structure, then  $\varphi(P)$  can be taken as a natural form under  $T$ .

Next fact is an answer in the case of  $T = RCOF$  for this question.

**5 Definition.** Let  $K$  be an ordered field. We call an ordered subring  $Z \subset K$  an **integer part** if it satisfies  $\forall x \in K, \exists! n \in Z \text{ s.t. } n \leq x < n + 1$ .

**6 Fact. (P. D' Aquino, J.F. Knight and S. Starchenko (2010)[1])** For a countable ordered field  $K$ , the followings are equivalent:

- $K$  is a recursively saturated model of  $RCOF$ ;
- $K$  has a non-archimedean integer part whose the non-negative part satisfies  $PA$ .

## 3 Background.

In this section, we introduce the previous investigation(A. Tsuboi and T.(2013)[8]). Firstly, we show a characterization of recursively saturated model of  $\sigma$ -minimal expansion of the theory  $RCOF$  as like Fact 6. Secondly, we will construct recursively saturated models by using nonstandard analysis.

### 3.1 $o$ -minimal analogue

In the proof of Fact 6, we use  $o$ -minimality and quantifier elimination of the theory  $RCOF$ .

**7 Question.** Are there any analogue for  $o$ -minimal expansion of  $RCOF$ ?

To answer the question above, we introduce definitions of  $o$ -minimality and weak form of quantifier elimination.

**8 Definition. ( $o$ -minimal)** We say that a theory  $T$  is  **$o$ -minimal** if for any model  $M$  of  $T$  and any definable set  $A \subset M$  (with parameters from  $M$ ),  $A$  can be described some finite union of open intervals and points.

**9 Example.** The following theories are  $o$ -minimal.

- The theory of real closed field:  $RCOF$ .
- $T_{exp} = Th(\mathbb{R}, +, \cdot, 0, 1, <, \exp)$ .
- $T_{an} = Th(\mathbb{R}, +, \cdot, 0, 1, <, (f_i)_i)$ .  
Where  $(f_i)_i$  is an enumeration of all analytic functions defined on closed box.

Next definition is a weak form of quantifier elimination.

**10 Definition.** We say that a theory  $T$  is **model complete** if every  $L$ -formula  $\phi(\bar{x})$  is equivalent to some existential  $L$ -formula  $\psi(\bar{x})$  modulo  $T$ :

$$\forall \phi(\bar{x}) \exists \psi(\bar{x}), T \models \forall \bar{x} (\phi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

**11 Example.**  $RCOF$ ,  $T_{exp}$  and  $T_{an}$  are model complete.

This definition is not sufficient to prove Fact. 6. We need an effective version of model completeness. Since  $RCOF$  is recursively axiomatized, we can effectively obtain an equivalent existential formula  $\psi(\bar{x})$  for above setting. In general, a decidable and model complete theory has same property.

**12 Definition.** We say that a theory  $T$  is **effectively model complete** if there is a effective procedure finding an existential  $L$ -formula  $\psi(\bar{x})$  which equivalent to any given  $L$ -formula  $\phi(\bar{x})$  modulo  $T$ .

A. Macintyre and A. J. Wilkie defined the effectively model completeness for finding a decidability result of  $T_{exp}$ .

**13 Fact. (A. Macintyre and A. J. Wilkie (1996)[4])**  $T_{exp}$  is effectively model complete.

Lastly we will define a notion of definably approximation which means a relevance of an integer part and additional functions, e.g. an exponential function.

**14 Definition.** Let  $R$  be a real closed ordered field with an integer part  $Z$  and let  $Q \subset R$  be the quotient field of  $Z$ . Suppose that  $N$  (the nonnegative part of  $Z$ ) satisfies  $PA$ . Finally, let  $E : R^n \rightarrow R$  be a continuous function. We say that  $E$  is  **$Z$ -definably approximated** if there exists a continuous function  $F : N \times Q^n \rightarrow Q$  such that

- $F$  is definable in the ordered field  $Q$ ;
- $\{F(m, \bar{x}) : m \in N\}$  converges uniformly to  $E(\bar{x})$  on closed bounded subsets of  $Q$ . More precisely, for all closed bounded boxes  $B \subset Q^n$  and  $\varepsilon > 0$ , there exists  $n_0 \in N$  such that, for all  $n \in N$  with  $n \geq n_0$  and all  $\bar{x} \in B$ ,  $R \models |E(\bar{x}) - F(n, \bar{x})| < \varepsilon$ .

Then we can state an answer of the question above.

**15 Theorem.** (A. Tsuboi(2013)[8]) Let  $L$  be a language  $L_{or} \cup \{f_1, \dots, f_k\}$ ,  $T$  an  $\omega$ -minimal and effectively model complete  $L$ -theory extended from  $RCOF$ . Let  $R$  be a model of  $T$ .  $R$  is a recursively saturated if there is an integer part  $Z \subset R$  such that

- the non-negative part of  $Z$  satisfies  $PA$ ,  $Z \neq \mathbb{Z}$  and
- each  $f_i$  is  $Z$ -definably approximated.

**16 Corollary.** Let  $R$  be a countable model of  $T_{exp}$ .  $R$  is recursively saturated if and only if there is an integer part  $Z \subset R$  such that

- the non-negative part of  $Z$  satisfies  $PA$ ,  $Z \neq \mathbb{Z}$  and
- $\exp(x)$  is  $Z$ -definably approximated.

Since  $T_{an}$  is a non-constructive object, we can not consider effective model completeness of  $T_{an}$ . For application, we need to consider a constructive sub-theory of  $T_{an}$ .

### 3.2 natural construction of recursively saturated real closed fields

In previous arguments, we give a characterization of recursively saturated model of a fixed theory. We do not consider applications of a given characterization. In this subsection, we will construct a recursively saturated models by using nonstandard analysis. We can easily construct a recursively saturated model by adding ideal elements, but our construction, showed below, is adding elements simultaneously.

**17 Question.** Is there a "natural" construction of recursively saturated model of  $RCOF$ ?

Next theorem is an answer of the question above.

**18 Definition.** Let  $K$  be an ordered field and  $K^*$  an elementary extension of  $K$ . We call following sets **finite part** and **infinitesimal part** respectively:

- $F_K := \{x \in K^* : \exists q \in K \text{ s.t. } |x| < |q|\}$
- $I_K := \{x \in K^* : \forall q \in K^\times \text{ s.t. } |x| < |q|\}$ .

**19 Theorem.** (A. Tsuboi and T.(2013)[8]) Let  $K$  be an ordered field with an integer part  $Z$  satisfying  $PA$ . If  $F_K \neq K^*$ , the quotient field  $R := F_K/I_K$  satisfies  $RCOF$ . Moreover, if  $Z \neq \mathbb{Z}$ , then  $R$  is recursively saturated.

Similarly, we can construct a recursively saturated model of  $T_{exp}$ . Let  $\mathbb{Q}^*$  be an  $\omega_1$ -saturated elementary extension of  $\mathbb{Q}$ . Let  $(Q^*, Q) \equiv (\mathbb{Q}^*, \mathbb{Q})$  where  $Q \neq \mathbb{Q}$ . Then  $\mathbb{R} \cong F_{\mathbb{Q}}/I_{\mathbb{Q}} \equiv F_Q/I_Q$ . Let  $\phi_{\mathbb{Z}}(x)$  be a defining formula of  $\mathbb{Z}$  in  $\mathbb{Q}$ . (by J. Robinson) Let  $Z := \phi_{\mathbb{Z}}(Q)$  and  $Z^* := \phi_{\mathbb{Z}}(Q^*)$ . Fix  $n^* \in Z^* - Z$  and define  $e(x) := \sum_{k=0}^{n^*} \frac{1}{k!} x^k$ . Define  $\exp^* : F_Q/I_Q \rightarrow F_Q/I_Q$  by  $\exp^*(x + I_Q) := e(x) + I_Q$ . Then  $(\mathbb{R}, \exp) \cong (F_{\mathbb{Q}}/I_{\mathbb{Q}}, \exp^*) \equiv (F_Q/I_Q, \exp^*)$  holds. In  $(F_Q/I_Q, \exp^*)$ ,  $\exp^*$  is approximated in its integer part  $\cong \mathbb{Z}$ .

**20 Example.**  $(F_Q/I_Q, \exp^*)$  is a recursively saturated model of  $T_{exp}$ .

## 4 Results.

We will review a definition of the restricted analytic field.

**21 Definition.** Let  $L_{an} = L_{or} \cup \{f_i\}_i$  where  $f_i$  is a function symbol,  $\mathbb{R}_{an} = (\mathbb{R}, +, \cdot, 0, 1, <, (f_i)_i)$  where  $(f_i)_i$  is an enumeration of all analytic functions defined on closed box, and  $T_{an} = Th(\mathbb{R}_{an})$ .

**22 Theorem.**  $T_{an}$  is model complete and  $o$ -minimal.

For application of our theorem15, we need a good fragment of  $T_{an}$ . Let  $F$  be a class of restricted analytic functions. Then  $L_{an}|F$  is  $L_{or} \cup F$  and  $T_{an}|F$  is restriction of  $T_{an}$  to  $L_{an}|F$ . It is easy to show that every complete subtheory of  $o$ -minimal theory is  $o$ -minimal, i.e.  $T_{an}|F$  is  $o$ -minimal (for any  $F$ ). For a subtheory of  $T_{an}$ , A. Gabrièlov finds a condition of  $F$  whether  $T_{an}|F$  is model complete.

**23 Theorem. (A. Gabrièlov(1996)[3])** Let  $F$  be a class of restricted analytic functions closed under derivation. Then  $T_{an}|F$  is model complete.

This proof is not prefer an effective version because it is a geometric. Since a proof of J. Denef and L. van den Dries (1988)[2] is algorithmic, we based on it. This proof of the model completeness of  $T_{an}$  depends on following two basic facts for analytic functions.

- Wierstrass's preparation theorem,
- van den Dries's preparation theorem

In the first subsection, we will give an outline of effective proofs. We will give a coding of restricted analytic functions and statements of an effective form of facts above. Moreover, we give a condition of a set  $F$  such that  $T_{an}|F$  is eventually effective model complete. In the second subsection, we will give a characterization of recursively saturated model of  $T_{an}|F$  for some  $F$  and a construction of recursively saturated model of it.

### 4.1 effective proof of basic facts

We fix notations.

- $O_n$  : a ring of  $n$ -ary analytic functions on neighborhood of 0;

- $R[Y]$ : a polynomial ring of a new variable  $Y$  with coefficients from a ring  $R$ ;
- We use multi-index notations: if  $\vec{i} = (i_1, \dots, i_n)$ , then  $\bar{x}^{\vec{i}} = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ ;
- For a function  $f(\bar{x}) = \sum_{\vec{i}} a_{\vec{i}} \bar{x}^{\vec{i}} \in O_n$  and a tuple of positive reals  $\bar{e}$ ,

$$\|f\|_{\bar{e}} := \begin{cases} \sum_{\vec{i}} |a_{\vec{i}} \bar{e}^{\vec{i}}| & \text{if it converges} \\ \infty & \text{otherwise} \end{cases};$$

- $|\bar{x}| \leq |\bar{e}|$  means  $\wedge_i |x_i| \leq |e_i|$ .

We will define a coding of restricted analytic functions to prove effective results.

**24 Definition. (coding of real)** Let  $(a^n)^n$  be a recursive sequence of rational numbers. We say that a real  $\alpha \in \mathbb{R}$  is **coded** by  $(a^n)^n$  if  $\forall n, |\alpha - a^n| < 2^{-n}$ .

**25 Definition. (coding of restricted analytic function)** Let  $(a_i^n)_i^n$  be a recursive multi-indexed sequence of rational numbers and  $\bar{e}, b, M$  are positive rational numbers. We say that a restricted analytic function  $f(\bar{x}) = \sum_{\vec{i}} \alpha_{\vec{i}} \bar{x}^{\vec{i}} \in O_n$  is **coded** by a code  $C = ((a_i^n)_i^n; \bar{e}, b, M)$  if  $\|f\|_{b\bar{e}} < M$ ,  $\alpha_{\vec{i}}$  is coded by  $(a_i^n)_i^n$ ,  $b > 1$  and  $\text{dom}(f) = \{\bar{x} : |\bar{x}| \leq |\bar{e}|\}$ .

For a code  $C = ((a_i^n)_i^n; \bar{e}, b, M)$ , let  $a_i^n, (C)\bar{e}(C), b(C)$  and  $M(C)$  denote components  $a_i^n, \bar{e}, b$  and  $M$  of  $C$  respectively.

**26 Example.** Let  $\pi_n$  be  $n$  decimal digits of  $\pi$  and  $M$  a sufficiently large positive number. Then the restricted sine function  $\sin(\pi x)|[-1, 1]$  can be coded by  $((\frac{1-(-1)^{i+1}}{2 \cdot (2i+1)!} \cdot \pi_n^i)_i^n, 1, 2, M)$ .

Remark: Let  $f \in O_n$  and  $g_1, \dots, g_n \in O_m$  be coded by  $C, D_1, \dots, D_n$  respectively. If  $M(D_i) \leq \bar{e}(C)_i (i < n)$ , then  $f(g_1, \dots, g_n)$  can be coded by some  $G = C_{\text{com}}(C, D_1, \dots, D_n)$ .

To state the Wierstarss's preparation, we define the regularity of an analytic function.

**27 Definition. (regularity)** We say that a restricted analytic function  $f(x_1, \dots, x_n) \in O_n$  is **regular of order  $p$  with respect to  $x_n$**  if  $f(0, 0, \dots, x_n) = c \cdot x_n^p + o(x_n^p)$  where  $c \neq 0$ .

**28 Fact. (Wierstarss's preparation)** Let  $\Phi \in O_n$  be regular of order  $p$  with respect to  $x_n$ . There exists unique unit  $Q \in O_n$  and unique  $R \in O_{n-1}[x_n]$  regular of order  $p$  with respect to  $x_n$  such that  $R = \Phi Q$ .

**29 Lemma. (Effective Wierstarss's preparation)** There exist recursive functions  $C_{WQ}(C, n), C_{WR}(C, n)$  which map from pairs of a code and a natural number to codes such that the followings holds: for any given  $\Phi \in O_n$  which is regular of order  $p$  with respect to  $x_n$  and coded by  $C, Q \in O_n$  and  $R \in O_{n-1}[x_n]$  are obtained by the Wierstarss's preparation; then for any sufficiently large  $n \in \mathbb{N}$ ,  $Q, R$  are coded by  $C_{WQ}(C, n), C_{WR}(C, n)$  respectively.

Unfortunately, there is no effective procedure finding sufficiently large  $n$ . This problem deduce to check  $\forall X, R(X) = \Phi(X)Q(X)$ . Next, we will state the van den Dries's preparation and an effective form of this.

**30 Fact. (van den Dries's preparation)** Let  $X = (X_1, \dots, X_n)$ ,  $Y = (Y_1, \dots, Y_m)$ ,  $m > 0$  and  $\Phi(X, Y) \in O_{n+m}$ . There exist  $d \in \mathbb{N}$ ,  $a_{\bar{i}}(X) \in O_n$  and units  $u_{\bar{i}}(X, Y) \in O_{n+m}$  ( $|\bar{i}| < d$ ) such that:

$$\Phi(X, Y) = \sum_{|\bar{i}| < d} a_{\bar{i}}(X) Y^{\bar{i}} u_{\bar{i}}(X, Y).$$

**31 Lemma. (Effective van den Dries's preparation)** Let  $X = (X_1, \dots, X_n)$ ,  $Y = (Y_1, \dots, Y_m)$ ,  $m > 0$ . There exist recursive functions  $C_{vA}(C, d, n, \bar{i})$ ,  $C_{vU}(C, d, n, \bar{i})$  such that the followings holds: for any given  $\Phi(X, Y) \in O_{n+m}$  be coded by  $C$ , for any sufficiently large  $d \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that  $\Phi(X, Y) = \sum_{|\bar{i}| < d} a_{\bar{i}}(X) Y^{\bar{i}} u_{\bar{i}}(X, Y)$ , where  $a_{\bar{i}}(X)$ ,  $u_{\bar{i}}(X, Y)$  are coded by  $C_{vA}(C, d, n, \bar{i})$ ,  $C_{vU}(C, d, n, \bar{i})$  respectively and each  $u_{\bar{i}}$  is a unit.

There is a problem how to find  $d, n$  effectively. This problem deduce to check  $\forall XY, \Phi(X, Y) = \sum_{|\bar{i}| < d} a_{\bar{i}}(X) Y^{\bar{i}} u_{\bar{i}}(X, Y)$ . Then we will give a condition of a set  $F$  such that  $T_{an}|F$  is eventually effective model complete and a definition of eventually effective model complete.

**32 Definition.** We say that a set  $S$  of codes **closed** if it is closed under  $C_{com}$ ,  $C_{vA}$ ,  $C_{vU}$ ,  $C_{WQ}$ ,  $C_{WR}$  and contains codes of bounded polynomial functions. Let  $F_S = \{f \in \cup_n O_n : f \text{ is coded by some element of } S\}$ .

**33 Definition.** We say that an  $L$ -theory  $T$  is **eventually effectively nearly model complete** if there is an effective procedure, for any given formula  $L$ -formula  $\phi(x)$ , finding recursive enumeration of boolean combinations of existential  $L$ -formulas  $\{\psi_n(x)\}_{n \in \omega}$  such that  $T \models \phi(x) \rightarrow \psi_m(x)$  for any  $m$  and  $T \models \phi(x) \leftarrow \bigwedge_{m < n} \psi_m(x)$  for any sufficiently large  $n$ .

We obtain a weak form of the effective model completeness for some fragment of  $T_{an}$ .

**34 Theorem. (T. 2013)** Let  $S$  be a r.e. closed set of codes,  $L = L_{an}|F_S$ . Then  $T_{an}|F_S = Th(\mathbb{R}_{an}|F_S)$  is eventually effectively nearly model complete.

## 4.2 main results

Similarly to a proof of Theorem 15, we will show the main theorem.

**35 Theorem. (revisited A. Tsuboi(2013) : modified by T.)** Let  $L$  be a language  $L_{or} \cup \{f_i\}_{i \in \mathbb{N}}$ ,  $T$  an  $\mathcal{o}$ -minimal and eventually effectively nearly model complete  $L$ -theory extended from  $RCOF$ . Let  $R$  be a model of  $T$ . Then  $R$  is a recursively saturated if there is an integer part  $Z \subset R$  such that:

- the non-negative part of  $Z$  satisfies  $PA$ ,  $Z \neq \mathbb{Z}$  and
- each  $f_i$  is  $Z$ -definably approximated by a  $\Sigma_{k_0}$ -formula where  $k_0$  does not depend on  $i$ .

We fix  $L$ ,  $T$ ,  $R$  and  $Z$  as in Theorem 35, and prove a series of lemmas before proving the theorem. Let  $N$  be the non-negative part of  $Z$ ,  $Q$  the quotient field of  $Z$  in  $R$ . Choose  $k_0$  such that every  $f_i(\bar{x})(i \in \omega)$  is  $Z$ -definably approximated by a  $\Sigma_{k_0}$ -formula. To prove Theorem 35, we need following lemmas proved in [8].

**36 Lemma.** ([8]) Every  $L$ -term (i.e., every term constructed from  $+$ ,  $\cdot$  and the  $f_i$ 's) is  $Z$ -definably approximated by  $\Sigma_{k_0}$ -formulas.

**37 Lemma.** ([8] modified by T.) Let  $\varphi(\bar{x})$  be a boolean combination of existential  $L$ -formulas. Then we can effectively find an  $L$ -formula  $\varphi_0(\bar{x})$  and an  $L_{or}$ -formula  $\varphi'(\bar{x})$  such that

- $R \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \varphi_0(\bar{x}));$
- $R \models \varphi_0(\bar{b}) \iff Q \models \varphi'(\bar{b}),$  for all  $\bar{b} \in Q.$

The formula  $\varphi'$  obtained in Lemma 37 is a  $\Sigma_{k_0+5}$ -formula.

**38 Lemma.** ([8] modified by T.) Let  $\varphi(\bar{x})$  and  $\psi(\bar{x})$  be boolean combinations of existential  $L$ -formulas such that  $R \models \forall \bar{x}(\varphi \rightarrow \psi)$ . Let  $\varphi'$  and  $\psi'$  be the formulas obtained in Lemma 37. Then  $Q \models \forall \bar{x}(\varphi' \rightarrow \psi')$ .

**39 Lemma.** ([8]) For any  $\bar{a} \in R$ ,  $\text{dcl}(\bar{a})$  is a bounded subset of  $R$ .

**Proof. (Proof of Theorem35)** Let  $\Sigma(x, \bar{a}) = \{\varphi_i(x, \bar{a}) : i \in \omega\}$  be a (non-algebraic) recursive type with  $\bar{a} \in R$ . We can assume that  $\varphi_{i+1}(x, \bar{a}) \rightarrow \varphi_i(x, \bar{a})$  holds in  $R$ . Since other cases can be treated similarly, we assume that  $\bar{a} \in R \setminus \text{dcl}(\emptyset)$  and that elements in  $\bar{a}$  are mutually non-algebraic. For each  $\varphi_i(x, \bar{a}) \in \Sigma$ , let  $\theta_i(u_0, u_1, \bar{v}_0, \bar{v}_1) = \theta_i(u_0, u_1, v_{0,0}, \dots, v_{0,k-1}, v_{1,k}, \dots, v_{1,k-1})$  be the formula

$$\forall x \bar{y} \left( u_0 < x < u_1 \wedge \bigwedge_{j < k} v_{0j} < y_j < v_{1j} \rightarrow \varphi_i(x, \bar{y}) \right),$$

where  $k$  is the length of  $\bar{a}$ . Notice that  $\exists u_0 u_1 (u_0 < u_1 \wedge \bigwedge_{j < k} v_{0j} < a_j < v_{1j} \wedge \theta_i(u_0, u_1, \bar{v}_0, \bar{v}_1))$  is satisfiable in  $R$ . (We can use the cell decomposition theorem to see this.) We can assume that  $\theta_i$  is an boolean combination of existential formulas by the eventually effective nearly model completeness assumption.

By the  $\sigma$ -minimality, there exist minimum  $\bar{b}_0$  and maximum  $\bar{b}_1$  (in the lexicographic ordering) such that  $\exists u_0 u_1 (u_0 < u_1 \wedge \bigwedge_{j < k} b_{0j} < a_j < b_{1j} \wedge \theta_i(u_0, u_1, \bar{b}_0, \bar{b}_1))$  holds in  $R$ . Therefore,  $\bar{b}_0, \bar{b}_1 \in \text{dcl}(\bar{a}) \cup \{\pm\infty\}$ . Using Lemma 39, choose a sufficiently large integer  $n^*$  such that  $\text{dcl}(\bar{a}) < n^*$ . We can choose  $\bar{c}_0, \bar{c}_1 \in Q$  with  $\sum_{j < k} |c_{0j} - c_{1j}| < 1/n^*$  such that  $b_{0j} < c_{0j} < a_j < c_{1j} < b_{1j}$  ( $j < k$ ). Then  $\exists u_0 u_1 (u_0 < u_1 \wedge \theta_i(u_0, u_1, \bar{c}_0, \bar{c}_1))$  holds in  $R$  regardless of the choice of  $i \in \omega$ .

For each  $\theta_i$ , choose a formula  $\theta'_i$  having the property described in Lemma 37. Namely, choose  $\theta'_i$  such that

1.  $R \models \forall u_0 u_1 \bar{v} (\theta_i \leftrightarrow \theta_{i,0});$
2.  $R \models \theta_{i,0}(q_0, q_1, \bar{r}, \bar{s}) \iff Q \models \theta'_i(q_0, q_1, \bar{r}, \bar{s}),$  for any  $q_0, q_1, \bar{r}, \bar{s} \in Q.$

In the present situation,  $\exists u_0 u_1 (u_0 < u_1 \wedge \theta_{i,0}(u_0, u_1, \bar{c}_0, \bar{c}_1))$  holds in  $R$ . Since  $u_0, u_1$  can be chosen from  $Q$ ,  $\exists u_0 u_1 (u_0 < u_1 \wedge \theta'_i(u_0, u_1, \bar{c}_0, \bar{c}_1))$  holds in  $Q$ . Then, by Lemma 38,  $\{\theta'_i(u_0, u_1, \bar{c}_0, \bar{c}_1) : i \in \omega\}$  is a recursive  $\Sigma_{k_0+5}$ -type in  $Q$ . So, by the  $\Sigma_{k_0+5}$ -recursive saturation of  $Q$ , there exists  $(d_1, d_2) \in Q^2$  such that  $Q \models \bigwedge_{i \in \omega} \theta'_i(d_0, d_1, \bar{c}_0, \bar{c}_1)$ . Hence,  $\Sigma(x, \bar{a})$  is realized in  $R$  by any  $e$  between  $d_0$  and  $d_1$ .  $\square$

**40 Example.** Let  $F_{sin}$  be a closed r.e. set contains a code of  $\sin(\pi x)|[-1, 1]$ . Let  $T_{sin} = T_{an}|F_{sin}$ .

**41 Corollary.** Let  $R$  be a model of  $T_{sin}$ .  $R$  is a recursively saturated if there is an integer part  $Z \subset R$  such that :

- the non negative part of  $Z$  satisfies  $PA$ ,  $Z \neq \mathbb{Z}$  and
- each  $f \in F_{sin}$  is  $Z$ -definably approximated by a  $\Sigma_{k_0}$ -formula where  $k_0$  does not depend on  $f$ .

Finally, we will construct a recursive saturated model of  $T_{sin}$  by using nonstandard analysis. Let  $\mathbb{Q}, \mathbb{Q}^*, Q, Q^*, Z, Z^*, n^*$  be in a construction of Example 20. For any  $f \in F_{sin}$  coded by  $((a_i^n)_i^n; \bar{e}, b; M)$ , define  $f^* : F_Q/I_Q \rightarrow F_Q/I_Q$  by  $f^*(x + I_Q) := \sum_{|\bar{i}| < n^*} (\lim_n a_i^n) x^{\bar{i}} + I_Q$ . Then  $(\mathbb{R}, F_{sin}) \cong (F_Q/I_Q, \{f^* : f \in F_{sin}\}) \equiv (F_Q/I_Q, \{f^* : f \in F_{sin}\})$ . In  $(F_Q/I_Q, \{f^* : f \in F_{sin}\})$ ,  $f^*$  is approximated in its integer part  $\cong Z$ .

**42 Example.**  $(F_Q/I_Q, \{f^* : f \in F_{sin}\})$  is a recursively saturated model of  $T_{sin}$ .

## References

- [1] P. D'Aquino, J. F. Knight and S. Starchenko, Real closed fields and models of Peano arithmetic, J. Symb. Log. **75**(1), 1-11(2010).
- [2] J. Denef, L. van den Dries, P-adic and real subanalytic sets. The Annals of Math., **128**(1), 79-138(1988).
- [3] A. Gabrièlov, Complements of subanalytic sets and existential formulas for analytic functions. Inventiones mathematicae, **125**(1), 1-12(1996).
- [4] A. Macintyre and A. J. Wilkie, On the decidability of the real exponential field, Kreiseliana: About and around Georg Kreisel (AK Peters, Wellesley, 1996), p. 441-467.
- [5] B. Poizat, A course in model theory: an introduction to contemporary mathematical logic. Springer(2000).
- [6] J. Robinson, Definability and decision problems in arithmetic, J. Symb. Log. **14**(2), 98-114(1949).
- [7] A. Tarski, A decision method for elementary algebra and geometry (Springer, Vienna, 1988), p. 24-84.
- [8] A. Tsuboi and Y. Tanaka, A construction of real closed fields, preprint(2013).
- [9] A. J. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, J. Am. Math. Soc. **9**(4), 1051-1094(1996).