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On Drinfeld modules with Rasmussen-Tamagawa type conditions: a resume

By

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Abstract

This is an announcement of the author's results [7] on a non-existence problem of Drinfeld modules defined over global function fields with some arithmetic constraints. The motivation is a conjecture of Christopher Rasmussen and Akio Tamagawa related with abelian varieties over number fields with constrained ℓ-power torsion points. In this paper, considering the arithmetic similarity between Drinfeld modules and elliptic curves, we show that a Drinfeld module analogue of the Rasmussen-Tamagawa conjecture holds in some cases. We also see that there is a counter-example of the Drinfeld module analogue of the conjecture in a special case.

§ 1. Introduction

Let k be a finite extension of $\mathbb{Q}$ and g a positive integer. For a prime number ℓ, denote by $\tilde{k}_\ell$ the maximal pro-ℓ extension of $k(\mu_\ell)$ which is unramified outside ℓ, where $\mu_\ell = \mu_\ell(\bar{k})$ is the set of ℓ-th roots of unity in $\bar{k}$. For an abelian variety X over k, write $k(X[\ell^\infty]) := k(\bigcup_{n\geq 1} X[\ell^n])$ for the field generated by all ℓ-power torsion points of X. Define $\mathscr{A}(k, g, \ell)$ to be the set of isomorphism classes $[X]$ of g-dimensional abelian varieties over k which satisfy the following equivalent conditions:

- $k(X[\ell^\infty]) \subseteq \tilde{k}_\ell$,
- X has good reduction at any finite place of k not lying above ℓ and $k(X[\ell])/k(\mu_\ell)$ is an ℓ-extension,

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- X has good reduction at any finite place of k not lying above ℓ and the mod ℓ representation $\bar{\rho}_{X,\ell} : G_k \to \mathrm{Aut}_{\mathbb{F}_\ell}(X[\ell]) \simeq \mathrm{GL}_{2g}(\mathbb{F}_\ell)$ is of the form

$$\bar{\rho}_{X,\ell} \simeq \begin{pmatrix} \chi_\ell^{i_1} & * & \cdots & * \\ & \chi_\ell^{i_2} & \ddots & \vdots \\ & & \ddots & * \\ & & & \chi_\ell^{i_{2g}} \end{pmatrix},$$

where χ_ℓ is the mod ℓ cyclotomic character and $i_1, \ldots, i_{2g}$ are positive integers.

The equivalence of these conditions follows from the Néron-Ogg-Shafarevich criterion and some group theoretic lemma. According to the Shafarevich conjecture proved by Faltings, the set $\mathscr{A}(k, g, \ell)$ is always finite. Rasmussen and Tamagawa conjectured the following.

Conjecture 1.1 ([10, Conjecture 1]). *The set $\mathscr{A}(k, g, \ell)$ is empty if ℓ is large enough.*

For example, the following cases are known:

- $k = \mathbb{Q}$ and $g = 1$ [10, Theorem 2],
- $k = \mathbb{Q}$ and $g = 2, 3$ [11, Theorem 7.1 and Theorem 7.2],
- for abelian varieties with everywhere semistable reduction [8, Corollary 4.5] if $k/\mathbb{Q}$ has odd degree or the discriminant of K is not divisible by ℓ, and [11, Theorem 3.6] in the general case,
- for abelian varieties with abelian Galois representations [9, Corollary 1.3],
- for QM abelian surfaces over certain imaginary quadratic fields [1, Theorem 9.3].

We notice that, under the assumption of the Generalized Riemann Hypothesis (GRH) for Dedekind zeta functions of number fields, the conjecture is true in general [11, Theorem 5.1]. The key tool of this proof is the effective version of the Chebotarev density theorem for number fields, which holds under GRH.

In this paper, we would like to consider a function field analogue of this conjecture. Let $\mathbb{F}_q$ be the finite field with q elements of characteristic p. Let $A = \mathbb{F}_q[t]$ be the ring of polynomials in indeterminate t and $F = \mathbb{F}_q(t)$ the fraction field of A. Instead of abelian varieties over number field, we consider Drinfeld modules defined over a finite extension K of F. There are deep arithmetic similarity between Drinfeld modules and elliptic curves. Under these analogy, for any positive integer r and monic irreducible element $\pi \in A$, we construct the set $\mathscr{D}(K, r, \pi)$ of isomorphism classes of rank-r Drinfeld modules over K satisfying Rasmussen-Tamagawa type conditions (see Proposition 2.5). We shall not distinguish between monic irreducible elements of A and finite places of F.

The followings are main results in this article.

Theorem 1.2. *Suppose that $r = p^\nu$ for some integer $\nu > 0$ and does not divide the inseparable degree $[K : F]_{\mathrm{i}}$. Then the set $\mathscr{D}(K, r, \pi)$ is empty for any π whose degree* $\deg(\pi)$ *is large enough.*

Theorem 1.3. *Suppose that $r = r_0 p^\nu$, where $r_0 > 1$ is an integer prime to p and $\nu \geq 0$. Then the set $\mathscr{D}(K, r, \pi)$ is empty for any π whose degree* $\deg(\pi)$ *is large enough.*

Therefore a Drinfeld module analogue of Conjecture 1.1 holds if r does not divide $[K : F]_{\mathrm{i}}$. Conversely, if r divides $[K : F]_{\mathrm{i}}$, then $\mathscr{D}(K, r, \pi)$ is never empty for any monic irreducible π (see §4).

§ 2. Construction of $\mathscr{D}(K, r, \pi)$

First, we give a brief introduction of Drinfeld modules (see [3], [5] and [12] for details). Let K be a finite extension of F. Denote by $K\{\tau\}$ the non-commutative polynomial ring over K in variable τ satisfying $\tau c = c^q \tau$ for any $c \in K$. Here $K\{\tau\}$ is isomorphic to the ring $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/K})$ of $\mathbb{F}_q$-linear endomorphisms of the additive group $\mathbb{G}_{a/K}$ over K. Let r be a positive integer.

Definition 2.1. A *Drinfeld A-module* (or *Drinfeld module*, for short) ϕ of rank r over K is an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to K\{\tau\}; a \mapsto \phi_a$ such that $\phi_t = t + c_1\tau + \cdots + c_r\tau^r \in K\{\tau\}$ and $c_r \neq 0$.

Remark. In general, for any field $\mathcal{F}$ equipped with an $\mathbb{F}_q$-algebra homomorphism $\iota : A \to \mathcal{F}$, a rank-$r$ Drinfeld module ϕ over $\mathcal{F}$ is defined to be an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to \mathcal{F}\{\tau\}$ such that $\phi_t = \iota(t) + c_1\tau + \cdots + c_r\tau^r$ and $c_r \neq 0$.

Definition 2.2. A *morphism* $\mu : \phi \to \psi$ between two Drinfeld modules over K is an element $\mu \in K\{\tau\}$ such that $\mu\phi_a = \psi_a\mu$ for any $a \in A$. Namely μ makes the following diagram commutative

$$\begin{tikzcd} \mathbb{G}_{a/K} \arrow[r, "\mu"] \arrow[d, "\phi_a"'] & \mathbb{G}_{a/K} \arrow[d, "\psi_a"] \\ \mathbb{G}_{a/K} \arrow[r, "\mu"'] & \mathbb{G}_{a/K} \end{tikzcd}$$

for any $a \in A$. We say that μ is an *isomorphism* if $\mu \in K^\times$.

Example 2.3. The rank-one Drinfeld module $\mathcal{C} : A \to F\{\tau\}$ determined by $\mathcal{C}_t = t + \tau$ is called the *Carlitz module*.

Let ϕ be a rank-r Drinfeld module over K and K^{sep} the separable closure of K. Then ϕ endows K^{sep} with a new A-module structure defined by $a \cdot \lambda := \phi_a(\lambda)$ for any $a \in A$ and $\lambda \in K^{\mathrm{sep}}$. For a non-zero element $a \in A$, the a-torsion points of ϕ is $\phi[a] = \{\lambda \in K^{\mathrm{sep}}; a \cdot \lambda = \phi_a(\lambda) = 0\}$. It is a free A/aA-module of rank r on which the absolute Galois group G_K of K acts. Let π be a monic irreducible element of A. Define $\mathbb{F}_\pi := A/\pi A$ and $q_\pi := \#\mathbb{F}_\pi = q^{\deg(\pi)}$. The action of G_K on $\phi[\pi]$ defines an $\mathbb{F}_\pi$-linear r-dimensional G_K-representation

$$\bar{\rho}_{\phi,\pi} : G_K \to \mathrm{Aut}_{\mathbb{F}_\pi}(\phi[\pi]) \simeq \mathrm{GL}_r(\mathbb{F}_\pi).$$

The *π-adic Tate module* of ϕ is $T_\pi(\phi) := \varprojlim \phi[\pi^n]$, which is a free $A_\pi(:= \varprojlim A/\pi^n A)$-module of rank r with continuous G_K-action.

Let v be a finite place of K, that is, v is a place lying above a monic irreducible element $\pi_0 \in A$. Denote by K_v, $\mathcal{O}_{K_v}$ and $\mathbb{F}_v$ the completion of K at v, its ring of integers and its residue field, respectively. Suppose that there exists a rank-r Drinfeld module ψ over K_v such that ϕ is isomorphic to ψ over K_v and $\psi_t = t + c'_1\tau + \cdots + c'_r\tau^r$ for some $c'_1, \ldots, c'_r \in \mathcal{O}_{K_v}$. Then we say that ϕ has *stable reduction* at v if $c'_{r_0} \in \mathcal{O}_{K_v}^\times$ for some $1 \leq r_0 \leq r$, and has *good reduction* at v if $c'_r \in \mathcal{O}_{K_v}^\times$. It is known that ϕ has good reduction at v if and only if $T_\pi(\phi)$ is unramified at v for any (in fact, for some) $\pi \neq \pi_0$, which is an analogue of the Néron-Ogg-Shafarevich criterion for abelian varieties.

Remark. It is known that ϕ has *potentially stable reduction* at v, that is, there exists a finite extension K' of K and a place w of K' lying above v at which ϕ has stable reduction as a Drinfeld module over K'. Moreover the above (K', w) can be taken as a finite separable extension and a place whose ramification index $e_{w|v}$ is a divisor of the integer $\prod_{s=1}^{r}(q^s - 1)$, so that $w|v$ is tamely ramified.

Example 2.4. The action of G_F on the π-torsion points $\mathcal{C}[\pi]$ of the Carlitz module defines the character

$$\chi_\pi : G_F \to \mathbb{F}_\pi^\times,$$

which is called the *mod π Carlitz character*. It is an analogue of the mod ℓ cyclotomic character $\chi_\ell : G_\mathbb{Q} \to \mathbb{F}_\ell^\times$. For instance, the extension $F(\mathcal{C}[\pi])/F$ is cyclic and its Galois group is isomorphic to $\mathbb{F}_\pi^\times$ via χ_π. Moreover, since $\mathcal{C}$ has good reduction at any finite place π_0 of F, the action of Frobenius element Frob_{π_0} on $\mathcal{C}[\pi]$ is well-defined if $\pi \neq \pi_0$ and then we have $\chi_\pi(\mathrm{Frob}_{\pi_0}) \equiv \pi_0 \pmod{\pi}$.

Under the above analogy, let us define the set $\mathscr{D}(K, r, \pi)$. For a rank-r Drinfeld module ϕ over K and a monic irreducible element $\pi \in A$, set $K(\phi[\pi^\infty]) := K(\cup_{n\geq 1}\phi[\pi^n])$ and consider the subfield $K_{\phi,\pi} := K(\phi[\pi]) \cap K(\mathcal{C}[\pi])$ of $K(\phi[\pi])$. Note that $K(\phi[\pi])$ may not contain $K(\mathcal{C}[\pi])$. By similar arguments in the abelian variety case, we can prove the following.

Proposition 2.5. *The following conditions are equivalent.*

- $K(\phi[\pi^\infty])/K_{\phi,\pi}$ *is a pro-p extension which is unramified at any finite place of* $K_{\phi,\pi}$ *not lying above* π,
- ϕ *has good reduction at any finite place of* K *not lying above* π *and* $K(\phi[\pi])/K_{\phi,\pi}$ *is a p-extension,*
- ϕ *has good reduction at any finite place of* K *not lying above* π *and the representation* $\bar{\rho}_{\phi,\pi}$ *is of the form*

$$\bar{\rho}_{\phi,\pi} \simeq \begin{pmatrix} \chi_\pi^{i_1} & * & \cdots & * \\ & \chi_\pi^{i_2} & \ddots & \vdots \\ & & \ddots & * \\ & & & \chi_\pi^{i_r} \end{pmatrix}.$$

Define $\mathscr{D}(K, r, \pi)$ to be the set of isomorphism classes $[\phi]$ of rank-r Drinfeld modules over K satisfying the above equivalent conditions, which is an analogue of $\mathscr{A}(k, g, \ell)$ in Section 1.

Remark. Although $\mathscr{A}(k, g, \ell)$ is always finite, the set $\mathscr{D}(K, r, \pi)$ may not be finite since the Drinfeld module analogue of the Shafarevich conjecture, which is a base of the finiteness of $\mathscr{A}(k, g, \ell)$, does not hold. For example, for any $a \in A$, consider the rank-two Drinfeld module $\phi^{(a)} : A \to F\{\tau\}$ defined by $\phi_t^{(a)} = t + a\tau + \tau^2$. Then it is easily seen that $\phi^{(a)}$ has everywhere good reduction and the set $\{[\phi^{(a)}]; a \in A\}$ of isomorphism classes is infinite [7, Example 5.11]. For this reason, the finiteness of $\mathscr{D}(K, r, \pi)$ is still unknown. In fact if $r \geq 2$ and $\pi = t$, then we can construct an infinite subset of $\mathscr{D}(K, r, t)$ [7, Proposition 5.15].

§ 3. Outline of proofs

Throughout this section, let ϕ be a rank-r Drinfeld module over K and suppose that $[\phi] \in \mathscr{D}(K, r, \pi)$ for a monic irreducible element $\pi \in A$. Using the strategy in [11] adapted to Drinfeld modules, we would like to show that there are some contradiction if $\deg(\pi)$ is sufficiently large.

§ 3.1. Sketch of the proof of Theorem 1.2

Suppose that $r = p^\nu$ for some positive integer $\nu > 0$ and r does not divide $[K : F]_{\mathrm{i}}$. We may assume that $\deg(\pi) > 1$.

Let u be a finite place of K lying above π. Recall that there exists a separable extension (K', w) of (K, u) such that ϕ has stable reduction at w and $e_{w|u}$ divides

$\prod_{s=1}^{r}(q^s-1)$. Applying the Drinfeld-Tate uniformization [3] to ϕ seen as a Drinfeld module over K'_w, we obtain an exact sequence

$$0 \to \psi[\pi] \to \phi[\pi] \to H_{\mathbb{F}_\pi} \to 0 \tag{3.1}$$

of $\mathbb{F}_\pi[G_{K'_w}]$-modules, where ψ is a Drinfeld module over K'_w with good reduction and $H_{\mathbb{F}_\pi}$ is a finite $\mathbb{F}_\pi$-vector space which $G_{K'_w}$ acts as a finite group. Denote by $\bar{\rho}_{\psi,\pi}$ and $\bar{\rho}_{H_{\mathbb{F}_\pi}}$ the $G_{K'_w}$-representations attached to $\psi[\pi]$ and $H_{\mathbb{F}_\pi}$, respectively. Then the sequence (3.1) means that the semisimplification $\bar{\rho}^{\mathrm{ss}}_{\phi,\pi}$ of $\bar{\rho}_{\phi,\pi}$ is of the form $\bar{\rho}^{\mathrm{ss}}_{\phi,\pi} = \bar{\rho}^{\mathrm{ss}}_{\psi,\pi} \oplus \bar{\rho}^{\mathrm{ss}}_{H_{\mathbb{F}_\pi}}$.

Estimating the ramification of $H_{\mathbb{F}_\pi}$, we can take a finite separable extension L of K'_w satisfying the followings:

- the action of the inertia subgroup I_L on $H_{\mathbb{F}_\pi}$ is trivial,
- for the maximal tamely ramified extension L_0 of K'_w in L, the ramification index $e(L_0/K'_w)$ divides the integer $\prod_{s=1}^{r-1}(q^s-1)$.

Write $e_{u,\phi} := e(L_0/F_\pi)$ for the absolute ramification index of L_0, so that it divides the integer $M(K,r) := [K:F](q^r-1)\prod_{s=1}^{r-1}(q^s-1)^2$ by construction. Note that $M(K,r)$ is prime to p.

Now we see that $\bar{\rho}^{\mathrm{ss}}_{\phi,\pi}(= \bar{\rho}^{\mathrm{ss}}_{\psi,\pi} \oplus \bar{\rho}^{\mathrm{ss}}_{H_{\mathbb{F}_\pi}}) = \chi_\pi^{i_1} \oplus \cdots \oplus \chi_\pi^{i_r}$ by Proposition 2.5. Let $\chi_\pi^{i_s}$ be an irreducible factor of $\bar{\rho}^{\mathrm{ss}}_{\phi,\pi}$. If it comes from $\bar{\rho}^{\mathrm{ss}}_{\psi,\pi}$, then by Theorem 2.14 of [4] there exists an integer $j_s \in [0, e_{u,\phi}] \cap \mathbb{Z}$ such that

$$\chi_\pi^{i_s}|_{I_{L_0}} = \omega_1^{j_s},$$

where $\omega_1 : I_{L_0} \to \mathbb{F}_\pi^\times$ is the level-one fundamental character. On the other hand, if $\chi_\pi^{i_s}$ is a factor of $\bar{\rho}^{\mathrm{ss}}_{H_{\mathbb{F}_\pi}}$, then we can show that $\chi_\pi^{i_s}|_{I_{L_0}} = \omega_1^0 = 1$ by Theorem 2.14 of [4] and a little computation. Since the relation $\chi_\pi|_{I_{L_0}} = \omega_1^{e_{u,\phi}}$ is also known [6, Proposition 9.4.3], for any $1 \le s \le r$, we obtain

$$i_s e_{u,\phi} \equiv j_s \pmod{q_\pi - 1} \tag{3.2}$$

for some integer $j_s \in [0, e_{u,\phi}] \cap \mathbb{Z}$.

Let v be a finite place of K above t. Since ϕ has good reduction at v, for any integer n, the characteristic polynomial $P_{v,n}(x) = \det(x - \mathrm{Frob}_v^n | T_\pi(\phi))$ of n-power of Frobenius element at v is well-defined and it has coefficients in A. By the above relation (3.2), the congruence

$$P_{v,e_{u,\phi}}(x) \equiv \prod_{s=1}^{r}(x - \chi_\pi(\mathrm{Frob}_v)^{i_s e_{u,\phi}}) \equiv \prod_{s=1}^{r}(x - \chi_\pi(\mathrm{Frob}_v)^{j_s}) \equiv \prod_{s=1}^{r}(x - t^{f_{v|t} j_s}) \pmod{\pi}$$

holds. We can check that the absolute values of all coefficients of $P_{v,e_{u,\phi}}(x)$ and $\prod_{s=1}^{r}(x-t^{f_{v|t}j_s})$ are smaller than $q^{r[K:F]M(K,r)}$. Assume that

$$\deg(\pi) > r[K : F]M(K, r). \tag{3.3}$$

Since the absolute value of π is $|\pi| = q^{\deg(\pi)}$, the above congruence implies $P_{v,e_{u,\phi}}(x) = \prod_{s=1}^{r}(x - t^{f_{v|t}j_s})$. Comparing the absolute values of roots of $P_{v,e_{u,\phi}}(x)$ and $\prod_{s=1}^{r}(x - t^{f_{v|t}j_s})$, we see that r divides $e_{u,\phi}$. In particular $r|M(K, r)$.

Apply the same arguments to any place u of K lying above π and set $e_\phi :=$ $\gcd\{e_{u,\phi}; u|\pi\}$. Denote by K_{s} the separable closure of F in K. There are only finitely many ramified places of F in K_{s}, so that we may assume that π is unramified in K_{s}. Note that if $K \neq K_{\mathrm{s}}$, then K/K_{s} is totally ramified at any places. Hence all $e_{u|\pi}$ must divide $[K : F]_{\mathrm{i}}$. Since r is a p-power, we see that $r|[K : F]_{\mathrm{i}}$. This is a contradiction.

§ 3.2. Sketch of the proof of Theorem 1.3

Next, we consider the case where r has a divisor $r_0 \geq 2$ which is prime to p, so that $r = r_0p^\nu$ for some integer $\nu \geq 0$. We also define the index e_ϕ same as the previous subsection and assume (3.3).

Now let $i_1, \ldots, i_r$ be positive integers satisfying $\bar{\rho}^{\mathrm{ss}}_{\phi,\pi} \simeq \chi_\pi^{i_1} \oplus \cdots \oplus \chi_\pi^{i_r}$. For any r-tuple $\mathbf{s} = (s_1, \ldots, s_r)$ of integers $1 \leq s_1, \ldots, s_r \leq r$, set $\varepsilon_{\mathbf{s}} := {\chi_\pi}^{i_{s_1}+\cdots+i_{s_r}-1}$ and define

$$\epsilon := (\varepsilon_{\mathbf{s}}) : G_F \to (\mathbb{F}_\pi^\times)^{\oplus r^r}.$$

Set $m_\phi := \#\epsilon(G_F)$, which is the least common multiple of the orders of $\varepsilon_{\mathbf{s}}$. Since ϵ factors through $\mathbb{F}_\pi^\times$, the integer m_ϕ divides $q_\pi - 1$. Using the assumption (3.3), we can also prove that m_ϕ divides e_ϕ, and hence $m_\phi|M(K, r)$. By the consequence of the effective Chebotrev density theorem [2, Corollary 3.4], we can prove the following.

Proposition 3.1. *For any positive integer m dividing $q_\pi - 1$, there exists a positive constant $C = C(K, m)$ depending only on K and m which satisfies the following: if $\deg(\pi) > C$, then there exist a monic irreducible element $\pi_0 \in A$ distinct to π and a place v of K above π_0 such that*

- $\epsilon(\mathrm{Frob}_{\pi_0}) = 1$,
- $\deg(\pi_0) < \deg(\pi)$,
- $f_{v|\pi_0} = 1$.

We also assume that $\deg(\pi) > \max\{C(K, m);\ m|M(K, r)\}$. Take π_0 and v as in Proposition 3.1. Recall that r is of the form $r = r_0p^\nu$. Under the above assumptions, we can prove that the existence of $[\phi] \in \mathscr{D}(K, r, \pi)$ contradicts to properties of $v|\pi_0$ as follows. For the characteristic polynomial

$$P_{v,1}(x) = a_r + a_{r-1}x + \cdots + a_{p^\nu}x^{r-p^\nu} + \cdots + x^r \in A[x]$$

of Frob_v, since $\epsilon(\mathrm{Frob}_v)=\epsilon(\mathrm{Frob}_{\pi_0})=1$, we can check that the r_0-power $(a_{p^\nu})^{r_0}$ of the coefficient of x^{r-p^ν} in $P_{v,1}(x)$ satisfies

$$(a_{p^\nu})^{r_0} \equiv (-1)^r \binom{r}{p^\nu}^{r_0} \pi_0 \pmod{\pi}. \tag{3.4}$$

Since $\binom{r}{p^\nu}$ is prime to p, the right hand side of (3.4) is not zero. We see that $|(a_{p^\nu})^{r_0}| \le q^{f_{v|\pi_0}\deg(\pi_0)} = q^{\deg(\pi_0)} = |\pi_0|$ by Proposition 3 of [12]. Then (3.4) and $\deg(\pi_0)<\deg(\pi)$ imply that $(a_{p^\nu})^{r_0}=\alpha\pi_0$ for some $\alpha\in\mathbb{F}_q^\times$, which contradicts to the assumption $r_0\ge 2$.

§ 4. Construction of a Drinfeld module contained in $\mathscr{D}(K,r,\pi)$

Suppose that r divides $[K:F]_{\mathrm{i}}$. Under this assumption, we can construct the rank-r Drinfeld module Φ over K such that $[\Phi]\in\mathscr{D}(K,r,\pi)$ for any π. If $r=1$, then the Carlitz module $\mathcal{C}$ satisfies the properties in Proposition 2.5. Hence we may assume that $r>1$.

First, we prepare some notations. Since r is a p-power, the map $A\to A; a\mapsto a^r$ is a ring homomorphism. For any element $a=\sum_{i=0}^n x_i t^i \in A$ with $x_i\in\mathbb{F}_q$, set $\hat{a} := \sum_{i=0}^n x_i^{1/r} t^i$. Then $A\to A; a\mapsto \hat{a}$ is a ring automorphism and the composite $A\to A: a\mapsto \hat{a}^r$ is an $\mathbb{F}_q$-algebra homomorphism.

By the general theory on function fields, the separable closure K_{s} of F in K coincides with the field $K^{[K:F]_{\mathrm{i}}}$ of all $[K:F]_{\mathrm{i}}$-power elements of K. Since r divides $[K:F]_{\mathrm{i}}$, the r-power root $\sqrt[r]{t}$ of $t\in F$ is contained in K. Now consider the $\mathbb{F}_q$-algebra homomorphism $\iota: A\to K$ given by $\iota(t)=\sqrt[r]{t}$ and define the rank-one Drinfeld module $\mathcal{C}'$ over (K,ι) by $\mathcal{C}'_t=\sqrt[r]{t}+\tau$. Then we can relate $\mathcal{C}'$ with the Carlitz module $\mathcal{C}$ as follows.

Lemma 4.1. *For any element $\lambda\in\mathcal{C}'[\pi]$, there exists a unique $\delta\in\mathcal{C}[\pi]$ such that $\lambda=\delta^{1/r}$.*

Since $\iota(\hat{a}^r)=a$ for any $a\in A$ by construction, the composite

$$\Phi: A \xrightarrow{\hat{(\cdot)}} A \xrightarrow{(\cdot)^r} A \xrightarrow{\mathcal{C}'} K\{\tau\}$$

is an $\mathbb{F}_q$-algebra homomorphism and

$$\Phi_t = \mathcal{C}'_{t^r} = (\sqrt[r]{t}+\tau)^r = t+\cdots+\tau^r.$$

Therefore Φ is a rank-r Drinfeld module over K and has good reduction at any finite place v of K since $\sqrt[r]{t}\in\mathcal{O}_{K_v}$. Note that if $r=1$, then Φ is the Carlitz module $\mathcal{C}$. Computing the G_K-action on $\Phi[\pi]$ by using Lemma 4.1, we obtain the following.

Proposition 4.2. *Let i be a positive integer satisfying $ir \equiv 1 \pmod{q_\pi - 1}$. Suppose that r divides $[K:F]_{\mathrm{i}}$. Then the representation $\bar{\rho}_{\Phi,\pi}$ is of the form*

$$\bar{\rho}_{\Phi,\pi} \simeq \begin{pmatrix} \chi_\pi^i & & * \\ & \ddots & \\ & & \chi_\pi^i \end{pmatrix}.$$

In particular $\mathscr{D}(K, r, \pi)$ is never empty for any π.

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