数理解析研究所講究録 1364 巻 2004 年 1-12

Strong Unique Continuation Property of Two-dimensional Dirac Equations and Schrödinger Equations with Aharonov-Bohm Fields

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1 Introduction

It is well known that, if any harmonic function $u(x)$ in a domain $\Omega\subset\mathbf{R}^{n}$ satisfies

$$\partial_{x}^{\alpha}u(x_{0})=0$$

for all multi-indices α at a point $x_{0}\in\Omega$, then $u(x)$ vanishes identically in Ω. Recently, it is shown by Grammatico [3] that, if Ω contains the origin and $u\in W_{\mathrm{loc}}^{2,2}(\Omega)$ (Sobolev space) satisfies

$$|\Delta u|\leq\frac{M}{|x|^{2}}|u(x)|+\frac{C}{|x|}|\nabla u| \tag{1}$$

(a.e. on Ω) with $M>0$ and $0<C<1/\sqrt{2}$, and

$$\lim_{\varepsilon\rightarrow+0}\varepsilon^{-\ell}\int_{|x|<\varepsilon}|u|^{2}dx=0, \tag{2}$$

then $u(x)$ vanishes identically in Ω (one can see some related works in the References of Grammatico [3]). Then we say that the inequality (1) has the strong unique continuation property. If $u(x)$ satisfies (2), $u(x)$ is said to vanish of infinite order at the origin, or to be flat at the origin. We can not expect the strong unique continuation property for every $C>0$. For Alinhac-Baouendi [1] shows that, if $C>1$, there is a non-trivial complex-valued function $v\in C^{\infty}(\mathbf{R}^{2})$, which is flat at the origin satisfying $\operatorname{supp}v=\mathbf{R}^{2}$ and (1) with $M=0$ (see also Pan-Wolff [7]).

For corresponding problems to the Dirac operator

$$L_{0}=\sum_{j=1}^{n}\alpha_{j}p_{j}\quad\left(p_{j}=\frac{1}{i}\frac{\partial}{\partial x_{j}},\quad n\geq 2\right),$$

where α_{j} are $N\times N$ Hermitian matrices satisfying $\alpha_{j}\alpha_{k}+\alpha_{k}\alpha_{j}=2\delta_{jk}I_{N}$ ($N=2^{[(n+1)/2]}$), De Carli-Ōkaji [2] shows that, if a positive constant $C<1/2$, then the inequality

$$|L_{0}u|\leq\frac{C}{|x|}|u|\ \text{ a.e. on }\Omega\quad(u\in W_{\mathrm{loc}}^{1,2}(\Omega)^{N}) \tag{3}$$

has the strong unique continuation property, where $|u| = \sqrt{|u_1|^2 + |u_2|^2}$ (see also Kalf–Yamada [5] and Ōkaji [6]). The restriction on $C < 1/2$ is needed to treat the angular momentum term (spin–orbit term) but the radial part of L_0. As is also pointed out by De Carli–Ōkaji [2], the counter example by Alinhac–Baouendi [1] implies that a certain restriction on the constant C in (3) is also necessary. In fact, if we set

$$u_1 := \partial\, u = (\partial_1 - i\partial_2)v, \quad u_2 := \bar{\partial}\, u = (\partial_1 + i\partial_2)v,$$

then we can see that u_1 and $u_2 \not\equiv 0$ are flat at the origin satisfying (1) with the same constant $C > 1$ (cf. Corollary below). It is an open problem what happens for $1/2 \leq C \leq 1$. In this note we investigate the strong unique continuation property for 2-dimensional Dirac operators with Aharonov–Bohm effect, which is one of singular magnetic fields at the origin, and give a perturbation to the spin–orbit term. Our proof is given along the same line as in De Carli–Ōkaji [2] and Kalf–Yamada [5].

2 The Result

Let us consider 2–dimensional Dirac operators with Aharonov–Bohm fields

$$L_\beta := \sigma \cdot D = \sigma_1 D_1 + \sigma_2 D_2,$$

where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$D_j := p_j - b_j(x) = -i\frac{\partial}{\partial x_j} - b_j(x),$$
$$b_1(x) := -\beta\,\frac{x_2}{|x|^2}, \quad b_2(x) := \beta\,\frac{x_1}{|x|^2},$$

and β is a real number. Such a magnetic field has a delicate singularity at the origin in spectral theory (see, e.g., Tamura [8]).

Put $\tilde{\beta} := \beta - [\beta]$, where $[\cdot]$ is Gauss's symbol.

Theorem 1. Let Ω be a connected open set in $\mathbf{R}^2$ containing the origin. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega)^2$ is flat at the origin and

$$|L_\beta\, u| \leq \frac{C_0}{|x|}\,|u| \tag{4}$$

a.e. on Ω for a positive constant $C_0 < \gamma(\beta)$ with

$$\gamma(\beta) := \begin{cases} \dfrac{1-2\tilde{\beta}}{2} & \left(0 \le \tilde{\beta} < \dfrac{1}{4}\right), \\ \tilde{\beta} & \left(\dfrac{1}{4} \le \tilde{\beta} < \dfrac{1}{2}\right), \\ 1-\tilde{\beta} & \left(\dfrac{1}{2} \le \tilde{\beta} < \dfrac{3}{4}\right), \\ \dfrac{2\tilde{\beta}-1}{2} & \left(\dfrac{3}{4} \le \tilde{\beta} < 1\right), \end{cases}$$

then u vanishes identically on Ω.

Corollary. Let $S_\beta := D_1^2 + D_2^2$ be the Schrödinger operator. Let Ω be an open set containing the origin. If $v \in W^{2,2}_{\mathrm{loc}}(\Omega)$ is flat at the origin satisfying

$$|S_\beta v| \le \frac{C_0}{|x|}|Dv| \tag{5}$$

a.e. on Ω for a positive constant $C_0 < \gamma(\beta)$, then v vanishes identically on Ω, where $|Dv| := \sqrt{|D_1 v|^2 + |D_2 v|^2}$.

For the proof of Corollary, let us put $u_1 := (D_1 - iD_2)v$ and $u_2 := (D_1 + iD_2)v$. Since v is flat at the origin, we can show that $D_1 v$ and $D_2 v$ are flat at the origin by using (5). Therefore, u_1 and u_2 are flat at the origin and satisfy

$$D_1 v = \frac{u_1 + u_2}{2}, \quad D_2 v = -\frac{u_1 - u_2}{2i},$$
$$D_1 D_2 v = D_2 D_1 v.$$

Moreover, we have

$$\begin{aligned} |L_\beta u| &= \sqrt{2}\,|(D_1^2 + D_2^2)v| \le \frac{\sqrt{2}\,C_0}{|x|}|Dv| \\ &= \frac{C_0}{\sqrt{2}\,|x|}\sqrt{|u_1 - u_2|^2 + |u_1 + u_2|^2} \\ &= \frac{C_0}{|x|}|u|, \end{aligned}$$

which gives from Theorem 1 that $u_1 = u_2 \equiv 0$ and $\dfrac{\partial v}{\partial r} \equiv 0$ in Ω. Since v is flat at the origin, we have $v \equiv 0$.

Moreover, applying the proof of Grammatico [3], we can prove the above property even if $C_0 < \sqrt{2}\,\gamma(\beta)$. In fact, we can see the following result:

Theorem 2. If $v \in W^{2,2}_{\text{loc}}(\Omega)$ is flat at the origin satisfying

$$|S_\beta v|^2 \le \frac{M^2}{|x|^4}|v|^2 + \frac{A^2}{|x|^2}|\partial_r v|^2 + \frac{B^2}{|x|^4}|(\partial_\theta - i\beta)v|^2 \tag{6}$$

a.e. on Ω, with positive constants M, A, B such that $A^2 + B^2 < 4\gamma(\beta)^2$, then v vanishes identically on Ω, where $(r,\, \theta)$ is the polar coordinate and $\partial_r = \partial/\partial r$, $\partial_\theta = \partial/\partial\theta$.

Therefore, if $v \in W^{2,2}_{\text{loc}}(\Omega)$ is flat at the origin satisfying

$$|S_\beta v| \le \frac{C_0}{|x|}|Dv|$$

a.e. on Ω for a positive constant $C_0 < \sqrt{2}\,\gamma(\beta)$, then v vanishes identically on Ω, by setting $A = B$ and $M = 0$ in (6).

3 Proof of Theorem 1

Here we introduce some notations. Let

$$D_r := \sum_{j=1}^{2} \frac{x_j}{r} D_j, \qquad \sigma_r = \sum_{j=1}^{2} \frac{x_j}{r}\sigma_j,$$

$$\begin{aligned} S \ &:= \ \frac{1}{2} - i\sigma_1\sigma_2(x_1 D_2 - x_2 D_1) \\ &= \ \frac{1}{2} + \sigma_3(x_1 p_2 - x_2 p_1 - \beta), \end{aligned}$$

where

$$\sigma_3 := -i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin–orbit operator S is written by polar coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ as

$$S = \begin{pmatrix} \frac{1}{2} - \beta - i\frac{\partial}{\partial\theta} & 0 \\ 0 & \frac{1}{2} + \beta + i\frac{\partial}{\partial\theta} \end{pmatrix}, \tag{7}$$

which can be regarded as a self-adjoint operator on $L^2(S^1)^2$. Then we have

$$\sigma \cdot D = \sigma_r\left(D_r + \frac{i}{r}S\right), \qquad \sigma_r^2 = I, \tag{8}$$

$$\sigma_r D_r = D_r\sigma_r, \quad \sigma_r S = -S\sigma_r, \quad D_r S = S D_r, \tag{9}$$

$$D_r^2 \ge \frac{1}{4r^2} \tag{10}$$

on $C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$. The last inequality can be shown by a commutator relation

$$\left[D_r, \frac{1}{r}\right] = \frac{i}{r^2}.$$

Lemma 2. For a real number m we put

$$A := \sigma \cdot D - i\frac{m}{r}\sigma_r.$$

Then we have

$$A^*A \geq \frac{1}{r^2}\left(S - m - \frac{1}{2}\right)^2 \tag{11}$$

on $C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$, and the spectrum $\sigma(S)$ consists of discrete eigenvalues

$$\left\{n + \frac{1}{2} \pm \beta \mid n \in \mathbf{Z}\right\}. \tag{12}$$

Proof. The properties (8), (9) and (10) give

$$\begin{aligned} A^*A &= \left[\sigma_r\left(D_r + \frac{i}{r}S\right) + \frac{im}{r}\sigma_r\right] \\ &\quad \cdot \left[\sigma_r\left(D_r + \frac{i}{r}S\right) - \frac{im}{r}\sigma_r\right] \\ &= \left[D_r - \frac{i}{r}(S-m)\right]\left[D_r + \frac{i}{r}(S-m)\right] \\ &= D_r^2 - \frac{1}{4r^2} + \frac{1}{r^2}\left(S - m - \frac{1}{2}\right)^2 \\ &\geq \frac{1}{r^2}\left(S - m - \frac{1}{2}\right)^2, \end{aligned}$$

which shows (11). Since S has a complete orthonormal eigenfunctions in $L^2(S^1)^2$,

$$\frac{1}{\sqrt{2\pi}}\begin{pmatrix} e^{in\theta} \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2\pi}}\begin{pmatrix} 0 \\ e^{-in\theta} \end{pmatrix} \quad (n \in \mathbf{Z}),$$

we obtain (12).

Lemma 3. There exists a sequence of positive numbers m_j $(j = 1, 2, \cdots)$ with $m_j \to \infty$ as $j \to \infty$ such that

$$\|r^{-m_j}(\sigma \cdot D)u\| \geq \gamma(\beta)\|r^{-m_j-1}u\|$$

for any $u \in W^{1,2}(\mathbf{R}^2)^2$ whose support does not include a neighborhood of the origin, where $\gamma(\beta)$ is what is defined in Theorem 1.

Proof. Let $\varphi \in C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$. In view of lemma 2 we have

$$\begin{aligned} &\int_{\mathbf{R}^2} r^{-2m}|\sigma \cdot D\varphi|^2 dx \\ &= \int_{\mathbf{R}^2} \left|A\left(r^{-m}\varphi\right)\right|^2 dx \\ &\geq \min_{n \in \mathbf{Z}} |n \pm \beta - m|^2 \int_{\mathbf{R}^2} r^{-2m-2}|\varphi|^2 dx \end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$ and $m \in \mathbf{R}$. Seeing the definition of $\gamma(\beta)$ in Theorem 1, we can find a sequence $m_j \to \infty$ such that

$$\min_{n \in \mathbf{Z}} |n \pm \beta - m_j|^2 = \gamma(\beta).$$

For a given $u \in W^{1,2}(\mathbf{R}^2)^2$ whose support does not include a neighborhood of the origin, there exists a sequence $\{\varphi_j\}_{j=1,2,\cdots} \subset C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$ such that $\varphi_j \to u$ in $W^{1,2}(\mathbf{R}^2)$ $(j \to \infty)$, which completes the proof.

Lemma 3 yields the following

Lemma 4. Suppose that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)^2$ is flat at the origin with (4). Let $B_{R_0} := \{x \in \mathbf{R}^2 \mid |x| < R_0\} \subset \Omega$. For any $R_1 < R_0$ there exists a positive constant $C_1 = C_1(R_0, R_1)$ independent of m_j such that

$$\begin{aligned} & [\gamma(\beta)^2 - C_0^2] \int_{B_{R_1}} r^{-2m_j-2}|u|^2\, dx \\ \leq\ & 2C_0^2 \int_{R_1<|x|<R_0} r^{-2m_j-2}|u|^2\, dx \\ & +C_1 \int_{R_1<|x|<R_0} r^{-2m_j}|u|^2\, dx, \end{aligned} \tag{13}$$

where m_j is the one given in Lemma 3.

Proof. Fix $0 < R_1 < R_0$ and take $\delta > 0$ and a smooth function $\chi_\delta \in C_0^\infty(0, R_0)$ such that

$$\chi_\delta(r) = \begin{cases} 1 & (\delta \leq r \leq R_1) \\ 0 & (r \leq \delta/2) \end{cases}$$

and

$$|\chi_\delta'(r)| \leq \begin{cases} C_2\delta^{-1} & (\delta/2 \leq r \leq \delta) \\ C_2 & (R_1 \leq r \leq R_0) \end{cases}$$

for a positive constant C. Then Lemma 3 and the condition (4) yield

$$\begin{aligned} & \gamma(\beta)^2 \int_{\delta \leq r \leq R_1} r^{-2m_j-2}|u|^2\, dx \\ & \leq \gamma(\beta)^2 \int r^{-2m_j-2}|\chi_\delta u|^2\, dx \\ & \leq \int |r^{-2m_j}(\sigma \cdot D)(\chi_\delta u)|^2\, dx \\ & \leq 2 \int_{\delta/2 \leq r \leq \delta} r^{-2m_j} \left[C_2^2\delta^{-2} + C_0^2 r^{-2}\right] |u|^2\, dx \\ & \quad +C_0^2 \int_{\delta \leq r \leq R_1} r^{-2m_j-2}|u|^2\, dx \\ & \quad +2 \int_{R_1 \leq r \leq R_2} r^{-2m_j} \left[C_2^2 + C_0^2 r^{-2}\right] |u|^2\, dx. \end{aligned} \tag{14}$$

Since u is flat at the origin, the last three integrals tend to zero if $\delta \to 0$. Therefore we have (13) with $C_1 = 2C_2^2$.

Proof of Theorem 1. Let $B_{R_0} \subset \Omega$ and take $0 < R_2 < R_1 < R_0$. In view of (13) we have

$$\begin{aligned}
&[\gamma(\beta)^2 - C_0^2]\left(\frac{R_1}{R_2}\right)^{2m_j} \int_{B_{R_2}} \frac{|u|^2}{r^2}\, dx \\
\le\ & [\gamma(\beta)^2 - C_0^2] R_1^{2m_j} \int_{B_{R_1}} r^{-2m_j-2}|u|^2\, dx \\
\le\ & 2C_0^2 R_1^{2m_j} \int_{R_1<|x|<R_0} r^{-2m_j-2}|u|^2\, dx \\
& + C_1 R_1^{2m_j} \int_{R_1<|x|<R_0} r^{-2m_j}|u|^2\, dx \\
\le\ & 2C_0^2 \int_{R_1<|x|<R_0} \frac{|u|^2}{r^2}\, dx \\
& + C_1 \int_{R_1<|x|<R_0} |u|^2\, dx.
\end{aligned}$$

Making $m_j \to \infty$, we have $u \equiv 0$ in B_{R_2}. Since R_1 and R_2 are arbitrary, we have $u \equiv 0$ in B_{R_0}.

Assume that there is $x_0 \in \Omega$ with $|x_0| = R_0$. The condition (3) yields

$$|L_0 u| \le \frac{C_0 + |\beta|}{|x|}|u| \quad \text{in } \Omega.$$

Set $x_\varepsilon = (1-\varepsilon)x_0$ for $0 < \varepsilon < R_0$. If

$$0 < \rho < \frac{R_0 - \varepsilon}{1 + 2(C_0 + |\beta|)},$$

then we can find a positive constant $C' < 1/2$ such that

$$|L_0 u| \le \frac{C'}{|x - x_\varepsilon|}|u| \ \text{ in } \ \Omega \cap B_\rho(x_\varepsilon),$$

where $B_\rho(x_\varepsilon)$ is the open ball with radius ρ and center x_ε. This fact implies, by De Carli–Ōkaji [2],

$$u \equiv 0 \ \text{ in } \ \Omega \cap B_{R_1},$$

where $R_1 := R_0[1 + \{2(C_0 + |\beta|) + 1\}^{-1}]$. By repeating this procedure we have $u \equiv 0$ in Ω.

4 Proof of Theorem 2

We shall apply the method developed in Grammatico[3] to (6). The spectrum $\gamma(\Delta'_\theta)$ coincides of eigenvalues $\{(k-\beta)^2 \mid k \in \mathbf{Z}\}$ with the coresponding eigenfunction $\varphi_k(\theta) = (1/\sqrt{2\pi})e^{ik\theta}$.

We introduce the coordnates $(T,\theta) \in \mathbf{R} \times S^1$ wiht $T = \log r$.

For $V \in C_0^\infty(\mathbf{R} \times S^1)$ we write

$$V(T,\theta) = \sum_{k\in\mathbf{Z}} f_k(T)\varphi_k(\theta).$$

We note that

$$\int\int |V(T,\theta)|^2 dTd\theta = \sum_{k\in\mathbf{Z}} \int |f_k(T)|^2 dT,$$

since

$$\|V(T,\theta)\|^2_{L^2(S^1)} = \sum_{k\in\mathbf{Z}} |f_k(T)|^2,$$

where $\|\cdot\|$ denotes the $L^2(S^1)$–norm. Set

$$Q = r^2\, S_\beta$$

and

$$Q_\tau = e^{-\tau T}(Qe^{\tau T}V),$$

where τ is a real parameter.

We can see directly

$$Q_\tau V = -(\partial_T^2 + 2\tau\partial_T + \tau^2 + \Delta'_\theta)V.$$

Hence we have

$$\begin{aligned}\int \|Q_\tau V(T,\cdot)\|^2 dT &= \int \|\partial_T^2 V(T,\cdot)\|^2 + 2\int \langle \partial_T^2 V, \Delta'_\theta V\rangle dT + 2\tau^2 \int \|\partial_T V(T,\cdot)\|^2 dT \\ &\quad + \tau^4 \int \|V(T,\cdot)\|^2 dT + 2\tau^2 \int \langle V, \Delta'_\theta V\rangle dT + \int \|\Delta'_\theta V(T,\cdot)\|^2 dT.\end{aligned}$$

Since we obtain

$$\int \langle \partial_T^2 V, \Delta'_\theta V\rangle dT = \int dT \int |\partial_T \Omega_\beta V|^2 d\theta \geq 0$$

by using $\Delta'_\theta = \Omega_\beta^* \Omega_\beta$, we have

$$\begin{aligned}\int \|Q_\tau V(T,\cdot)\|^2 dT &\geq 2\tau^2 \int \|\partial_T V(T,\cdot)\|^2 dT + \tau^4 \int \|V(T,\cdot)\|^2 dT \\ &\quad + 2\tau^2 \int \langle V, \Delta'_\theta V\rangle dT + \int \|\Delta'_\theta V(T,\cdot)\|^2 dT\end{aligned}$$

and consequently

$$\begin{aligned}\int \|Q_\tau V(T,\cdot)\|^2 dT &\geq \tau^4 \sum_{k\in\mathbf{Z}} \int |f_k(T)|^2 dT - 2\tau^2 \sum_{k\in\mathbf{Z}} (k-\beta)^2 \int |f_k(T)|^2 dT \\ &\quad + \sum_{k\in\mathbf{Z}} (k-\beta)^4 \int |f_k(T)|^2 dT + 2\tau^2 \int \|\partial_T V(T,\cdot)\|^2 dT.\end{aligned}$$

The later inequality can be written as

$$\int \|Q_\tau V(T,\cdot)\|^2 dT \geq \sum_{k\in\mathbf{Z}} (\tau^2 - (k-\beta)^2)^2 \int |f_k(T)|^2 dT + 2\tau^2 \int \|\partial_T v(T,\cdot)\|^2 dT. \tag{15}$$

Seeing the definition of $\gamma(\beta)$ in Theorem 1, we can find a sequence $\tau_j \to \infty$ such that

$$\min_{k\in\mathbf{Z}} \frac{(\tau_j^2 - (k-\beta)^2)^2}{(k-\beta)^2} = C_\beta, \tag{16}$$

where $C_\beta = 4\gamma(\beta)^2$. Then we obtain from (15)

$$\int \|Q_\tau V(T,\cdot)\|^2 dT \geq C_\beta \sum_{k\in\mathbf{Z}} (k-\beta)^2 \int |f_k(T)|^2 dT = C_\beta \int \|\Omega_\beta V(T,\cdot)\|^2 dT. \tag{17}$$

Setting $U = e^{\tau T} V$, the above inequality can be written as

$$\int e^{-2\tau T} \|QU\|^2 dT \geq C_\beta \int e^{-2\tau T} \|\Omega_\beta U\|^2 dT. \tag{18}$$

For any $C'_\beta < C_\beta$ we can find a sufficiently large τ_0 such that

$$(\tau^2 - (k-\beta)^2)^2 \geq C'_\beta \geq \tau^2$$

for any $\tau \geq \tau_0$ satisfying (16). Then, in view of (15) we have

$$\begin{aligned} \int \|Q_\tau V(T,\cdot)\|^2 dT &\geq C'_\beta \tau^2 \sum_{k\in\mathbf{Z}} \int |f_k(T)|^2 dT + 2\tau^2 \int \|\partial_T V(T,\cdot)\|^2 dT \\ &\geq C'_\beta \left(\tau^2 \int \|V(T,\cdot)\|^2 dT + \int \|\partial_T V(T,\cdot)\|^2 dT \right). \end{aligned} \tag{19}$$

From now on, we consider $\tau \geq \tau_0$ satisfying (16). We recall $U = e^{\tau T} V$ so that

$$\int e^{-2\tau T} \|QU\|^2 dT \geq C'_\beta \int e^{-2\tau T} \|\partial_T U\|^2 dT. \tag{20}$$

For any $\alpha \in [0,1]$ we have from (18) and (20)

$$\begin{aligned} \int e^{-2\tau T} \|QU\|^2 dT &\geq \alpha\, C'_\beta \int e^{-2\tau T} \|\partial_T U\|^2 dT \\ &\quad + (1-\alpha)\, C_\beta \int e^{-2\tau T} \|\Omega_\beta U\|^2 dT. \end{aligned} \tag{21}$$

On the other hand, we obtain from (19)

$$\int e^{-2\tau T} \|QU\|^2 dT \geq C'_\beta \tau^2 \int e^{-2\tau T} \|U\|^2 dT. \tag{22}$$

Therefore, (21) and (22) give

$$\begin{aligned} (1 + \frac{1}{\tau'}) \int e^{-2\tau T} \|QU\|^2 dT &\geq \tau' \int e^{-2\tau T} \|U\|^2 dT + \alpha\, C'_\beta \int e^{-2\tau T} \|\partial_T U\|^2 dT \\ &\quad + (1-\alpha)\, C_\beta \int e^{-2\tau T} \|\Omega_\beta U\|^2 dT, \end{aligned} \tag{23}$$

where $\tau' = C_\beta'^{\frac{1}{2}}\,\tau$.

Now let us set $W_c^{2,2}(\mathbf{R}^2) = \{v \mid v \in W^{2,2}(\mathbf{R}^2)$ has a compact support$\}$. The Inequality (23) still holds for $v \in W_c^{2,2}(\mathbf{R}^2)$, since the fact follows from the denseness of $C_0^\infty(\mathbf{R}^2)$ in $W_c^{2,2}(\mathbf{R}^2)$.

We set $B_R = \{x \in \mathbf{R}^2 \mid |x| < R\} \subset \Omega$ and choose $\chi(T) \in C^\infty(\mathbf{R})$ such that $0 \le \chi \le 1$ and

$$\chi(T) = \begin{cases} 1, & T < T_0 \\ 0, & T > \log R, \end{cases}$$

where $e^{T_0} < R$. Let $\phi \in C^\infty(\mathbf{R}^2)$ such that

$$\phi(T) = \begin{cases} 0, & |x| < \dfrac{1}{2} \\ \\ 1, & |x| > 1. \end{cases}$$

and $\phi_j(x) = \phi(jx)$ $(j \in \mathbf{N})$.

Let $u \in W_{loc}^{2,2}(\mathbf{R}^2)$ be flat at the origin for which (6) holds. Then the functions $\phi_j \chi u \in W_c^{2,2}(\mathbf{R}^2)$ satisfy (23). If we take thr limit as $j \to \infty$, we see that χu also satisfies (23).

By (T, θ) coordinates (6) becomes

$$|Qu|^2 = |e^{2\tau T} S_\beta u|^2 \le M^2 |u|^2 + A^2 |\partial_T u|^2 + B^2 |\Omega_\beta u|^2 \tag{24}$$

for $T < \log R$.

By applying (23) to χu we have for τ big enough

$$\begin{aligned} (1 + \frac{1}{\tau'}) \Big(\int_{-\infty}^{T_0} e^{-2\tau T} \|Qu(T,\cdot)\|^2 dT + \int_{T_0}^{+\infty} e^{-2\tau T} \|Q(\chi u)(T,\cdot)\|^2 dT \Big) \\ \ge \tau' \int_{-\infty}^{T_0} e^{-2\tau T} \|u(T,\cdot)\|^2 dT + \alpha\, C_\beta' \int_{-\infty}^{T_0} e^{-2\tau T} |\partial_T u(T,\cdot)|^2 dT \\ + (1-\alpha)\, C_\beta \int_{-\infty}^{T_0} e^{-2\tau T} \|\Omega_\beta u(T,\cdot)\|^2 dT. \end{aligned} \tag{25}$$

If we set

$$\psi(T) = M^2 \|u(T,\cdot)\|^2 + A^2 \|\partial_T u(T,\cdot)\|^2 + B^2 \|\Omega_\beta u(T,\cdot)\|^2,$$

then we obtain from (24) and (25)

$$\begin{aligned} (1 + \frac{1}{\tau'}) \Big(\int_{-\infty}^{T_0} e^{-2\tau T} \psi(T) dT + \int_{T_0}^{+\infty} e^{-2\tau T} \|Q(\chi u)(T,\cdot)\|^2 dT \Big) \\ \ge \tau' \int_{-\infty}^{T_0} e^{-2\tau T} \|u(T,\cdot)\|^2 dT + \alpha\, C_\beta' \int_{-\infty}^{T_0} e^{-2\tau T} |\partial_T u(T,\cdot)|^2 dT \\ + (1-\alpha)\, C_\beta \int_{-\infty}^{T_0} e^{-2\tau T} \|\Omega_\beta u(T,\cdot)\|^2 dT, \end{aligned}$$

that is,

$$
\begin{aligned}
(1+\frac{1}{\tau'})\int_{T_0}^{+\infty} & e^{-2\tau T}\|Q(\chi u)(T,\cdot)\|^2 dT \\
\geq & \left(\tau' - M^2\left(1+\frac{1}{\tau'}\right)\right)\int_{-\infty}^{T_0} e^{-2\tau T}\|u(T,\cdot)\|^2 dT \\
& + \left(\alpha\, C'_\beta - A^2\left(1+\frac{1}{\tau'}\right)\right)\int_{-\infty}^{T_0} e^{-2\tau T}|\partial_T u(T,\cdot)|^2 dT \\
& + \left((1-\alpha)\, C_\beta - B^2\left(1+\frac{1}{\tau'}\right)\right)\int_{-\infty}^{T_0} e^{-2\tau T}\|\Omega_\beta u(T,\cdot)\|^2 dT.
\end{aligned}
$$

Now, if $A^2 + B^2 < C_\beta$ and τ is big enough, we can choose any $C'_\beta < C_\beta$ and $\alpha \in [0,1]$ such that

$$
\alpha C'_\beta - A^2\left(1+\left(\frac{1}{\tau'}\right)\right) > 0\,, \quad (1-\alpha)C_\beta - B^2\left(1+\left(\frac{1}{\tau'}\right)\right) > 0.
$$

Thus, we have

$$
\begin{aligned}
e^{-2\tau T_0}(1+\frac{1}{\tau'})\int_{T_0}^{+\infty} & \|Q(\chi u)(T,\cdot)\|^2 dT \\
\geq & \ (1+\frac{1}{\tau'})\int_{T_0}^{+\infty} e^{-2\tau T}\|Q(\chi u)(T,\cdot)\|^2 dT \\
\geq & \left(\tau' - M^2\left(1+\frac{1}{\tau'}\right)\right)\int_{-\infty}^{T_0} e^{-2\tau T}\|u(T,\cdot)\|^2 dT \\
\geq & \ e^{-2\tau T_0}\left(\tau' - M^2\left(1+\frac{1}{\tau'}\right)\right)\int_{-\infty}^{T_0} \|u(T,\cdot)\|^2 dT.
\end{aligned}
$$

Making $\tau = \tau_j \to \infty$, we have $u \equiv 0$ in $\{x \in \mathbf{R}^2 \mid |x| < e^{T_0}\}$, and therefore $u \equiv 0$ in B_R. With the similar argument in Theorem 1, we have $u \equiv 0$ in Ω.

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